DOCUMENT RESUME

ED 076 388 SE 015 896

| | |
|---|---|
| AUTHOR | Berger, Emil J., Ed. |
| TITLE | Instructional Aids in Mathematics. National Council of Teachers of Mathematics Yearbook 34. |
| INSTITUTION | National Council of Teachers of Mathematics, Inc., Washington, D.C. |
| PUB DATE | 73 |
| NOTE | 442p. |
| AVAILABLE FROM | National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Virginia 22091 ($12.00, $11.00 for NCTM members) |
| EDRS PRICE | MF-$0.65 HC Not Available from EDRS. |
| DESCRIPTORS | *Activity Learning; Audiovisual Aids; Computers; Elementary School Mathematics; *Instruction; Instructional Aids; *Instructional Materials; Laboratory Procedures; Manipulative Materials; *Mathematical Enrichment; *Mathematics Education; Secondary School Mathematics; Teaching Techniques; Textbooks; Yearbooks |


ABSTRACT

Information about available instructional aids, suggestions for selecting and evaluating these materials, and guidance in using them are provided in this yearbook. Chapters cover the use of instructional space; textboods and other printed materials; programed instruction and teaching machines; calculating devices and computers; projection devices; manipulative aids; projects, exhibits, games, puzzles, and contests; and the teacher's role. The chapters on computers, manipulative devices, and projects include extensive bibliographies. The appendix lists names and addresses of producers and distributors of instructional aids. (DT)

The National Council of Teachers of Mathematics

# INSTRUCTIONAL AIDS IN MATHEMATICS

Thirty-fourth Yearbook

National Council of Teachers of Mathematics

*Thirty-fourth Yearbook*

# INSTRUCTIONAL AIDS IN MATHEMATICS

*Thirty-fourth Yearbook*

National Council of Teachers of Mathematics

1973

Correspondence relating to and orders for additional copies of the Thirty-fourth Yearbook and earlier yearbooks should be addressed to:

The National Council of Teachers of Mathematics
1201 Sixteenth Street, NW
Washington, D.C. 20036



 Library of Congress Catalog Card Number: 76-189323.
Printed in the United States of America.

The development and use of instructional aids in the teaching and learning of mathematics have expanded tremendously in the past quarter century. Ushering in that era of change was a 1945 report to the Board of Directors of the National Council of Teachers of Mathematics by the Committee on Multi-Sensory Aids. The Council published this report as its Eighteenth Yearbook under the title *Multi-Sensory Aids in the Teaching of Mathematics*. The introduction to the Eighteenth Yearbook contains a prophetic statement anticipating the expansion that is recorded in the present volume: "It is hoped that the report will be followed in a few years by one showing improvements and changes which have kept pace with the progress of such aids in the world about us."

Since World War II the school mathematics curriculum has been completely overhauled. Increased activity in mathematical research, advances in automation, and invention of the electronic computer had caused a revolution in mathematics, creating pressure for curriculum reform.

Efforts to incorporate new materials into the mainstream of school mathematics precipitated the writing of "new mathematics" textbooks and pamphlets containing changed content and new pedagogical strategies. Production of supporting films, filmstrips, overhead visuals, models, manipulative devices, and pertinent professional publications soon followed.

Beginning around the mid-fifties, a speedup occurred in the trend to individualize instruction in the schools. This evolvement prompted significant new advances in instructional aids in all subject fields, mathematics included.

The combined effects of compulsory attendance laws and the post–World War II population boom had brought about burgeoning school enrollments that included a large subpopulation of students for whom the established pattern was inadequate. The new population of students presented a wider range of ability than formerly and a greater variety of personal and socioeconomic backgrounds.

Coupled with the swelling enrollments and changing traits of school youth was society's deepening concern for the individual—a concern that looked to the schools for expression.

To stimulate planning to improve the quality of education for all students—the bright, the average, and the slow—J. Lloyd Trump published the pamphlet *Images of the Future*.

Trump proposed that schools be organized around three kinds of activity—large group instruction, individual study, and small group discussion. He believed this scheme would result in improved instruction and make more effective provision for

individual differences. While Trump's proposal was directed primarily at the secondary level, the effects of his recommendation were also felt at the elementary level.

Trump's recommendation served as a catalyst that spurred architects to design buildings that included group spaces of various sizes, and encouraged flexible scheduling, team teaching, and the use of a variety of instructional aids.

Skinner's and Crowder's work with teaching machines and programed instruction marked another step forward in the enlivened move to individualize instruction. These two men devised concrete ways of adapting instruction to the individual learner. The vast potential of the computer was also felt as a positive force. Computer-based teaching machines have the capability of completely individualizing instruction. To date, however, this particular use of computers has been prohibitively expensive. Presently the most popular use of computers in the schools is as a tool in problem solving and flexible scheduling.

Government support also helped extend the development and use of instructional aids in mathematics. Enactment by Congress of the National Defense Education Act in 1958 and the Elementary and Secondary Education Act in 1965 increased the financial ability of schools to procure instructional aids, thereby placing them in the hands of a great many teachers and students who had not previously used them.

Business and industry saw federal aid to education as an opportunity to expand their enterprises into the education market. Passage of the 1958 and 1965 laws was followed by a rapid proliferation in the design and function of instructional aids in mathematics and a buildup of the inventories of such aids.

The events and trends identified above, because of their synergetic nature, will undoubtedly induce continuing expansion of both the development of instructional aids and their use. This makes it impossible for today's teachers of mathematics to keep track of new developments, make appraisals with confidence, and incorporate new materials in their teaching without reliable, organized assistance.

To help teachers cope with this perplexing situation, the National Council of Teachers of Mathematics challenged a group of authors to produce a yearbook containing information about the growing spectrum of available instructional aids, suggestions for selecting and evaluating these materials, and guidance in using them. From their diverse backgrounds—representing experience in the schools, universities, business, and industry—the authors who accepted the Council's challenge attempted to produce a yearbook that can stand the test of time and change. That some instructional aids will be obsolete as this yearbook goes to press is inevitable; that others will be developed and perhaps also pass into obsolescence is not peculiar to mathematics instruction alone. But the reader of this yearbook will not go unarmed.

EMIL J. BERGER
*Editor*

# ACKNOWLEDGEMENTS

Grateful acknowledgement is hereby expressed to the persons listed below and to others who contributed ideas, references, suggestions, and editorial assistance

Paul Berrisford, University of Minnesota Library, Minneapolis, Minnesota
Aaron Bisberg, Image Publishing Corporation, White Plains, New York
John H. Bolstad, Lawrence Radiation Laboratories, Livermore, California
James D. Bristol, Shaker Heights Senior High School, Shaker Heights, Ohio
Lee J. Brockway, Shaver and Company, Architects, Michigan City, Indiana
Emalou Brumfield, Shaker Heights City School District, Shaker Heights, Ohio
William W. Caudill, Architect, Houston, Texas
Louis S. Cohen, Bloomington Public Schools, Bloomington, Minnesota
Eleanor Duckworth, Educational Services, Inc., Watertown, Massachusetts
John C. Egsgard, Saint Michael's College School, Toronto, Ontario
Robert Ferguson, Diamond Path School, Rosemount, Minnesota
W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts
Ourania Forbes, Flossmoor, Illinois
Alexandra Forsythe, Gunn High School, Palo Alto, California
Emil Greishaber, Minnesota Mining and Manufacturing Company, Saint Paul, Minnesota
Ken Heggen, Minnetonka High School, Minnetonka, Minnesota
Gregory W. Johnson, Stillwater High School, Stillwater, Minnesota
Randall E. Johnson, University of Minnesota, Minneapolis, Minnesota
Pamela Katzman, Annandale, Minnesota
Donald Thomas King, Saint Paul Public Schools, Saint Paul, Minnesota
Joan Kirkpatrick, University of Alberta, Edmonton, Alberta
Muriel Lange, Johnson High School, Saint Paul, Minnesota
Jack Lown, Minnesota School Facilities, Minneapolis, Minnesota
Lawrence Marks, Byron Junior High School, Shaker Heights, Ohio
Sister Mary Peter Marthaler, Sauk Rapids, Minnesota
Sol Mastbaum, Murray High School, Saint Paul, Minnesota
LaRue West Miller, Fridley, Minnesota
Joanne Moses, Saint Paul Public Schools, Saint Paul, Minnesota
Anthony Nogales, Seattle Public Schools, Seattle, Washington
Bea Ogrens, Fridley, Minnesota
William Overman, Las Cruces Public Schools, Las Cruces, New Mexico
G. Franklin Padgett, D.C. Public Schools, Washington, D.C.
Robert Pecha, Churchill Elementary School, Cloquet, Minnesota
Eugene F. Peckman, Wexford, Pennsylvania
Daniel Pedoe, University of Minnesota, Minneapolis, Minnesota

Charles M. Proctor, Jr., Montgomery County Schools, Rockville, Maryland
Diane M. Ptak, Oakland County Schools, Pontiac, Michigan
Seymour Schuster, Carleton College, Northfield, Minnesota
Pamela Schwandt, Normandale State Junior College, Bloomington, Minnesota
Judy Schwarzkopf, Saint Louis Park, Minnesota
Marge Senninger, San Jose, California
Joseph Senta, Murray High School, Saint Paul, Minnesota
Marla Shapiro, Cambridge, Massachusetts
Carol Shingles, Harding High School, Saint Paul, Minnesota
Fern Singer, Mounds Park Junior High School, Saint Paul, Minnesota
Henry Slotnick, National Assessment of Educational Progress, Denver, Colorado
James Thueson, University of Minnesota Library, Minneapolis, Minnesota
John Wagner, Michigan State University, East Lansing, Michigan
Robert R. Willson, Seattle Public Schools, Seattle, Washington
Clista M. Wood, Harding High School, Saint Paul, Minnesota
Lauren G. Woodby, Michigan State University, East Lansing, Michigan

The following students contributed ideas:

Mark Deziel, West Saint Paul, Minnesota
Wendy Johnson, Dassel, Minnesota
Malcolm Ritter, Saint Paul, Minnesota
Steven Saliterman, Saint Louis Park, Minnesota

Students who posed for division-page pictures are:

Daniel Brown, Saint Paul, Minnesota
Robert Lewis, Minneapolis, Minnesota
Bruce McNamara, Minneapolis, Minnesota
Lynne Nelson, Minneapolis, Minnesota
Denise Riley, Minneapolis, Minnesota

Deserved praise and gratitude are extended to the following specialists in graphic arts:

*Photographer*

Charles Hoffman, North Saint Paul, Minnesota

*Artists*

Marshall Kathan, Washington, D.C.
Judy Norman, Minneapolis, Minnesota
Noel Thorpe, Saint Paul, Minnesota

Also acknowledged are the generosity and cooperation of producers and distributors of instructional aids who contributed pictures or gave permission to print pictures of specialized instructional aids.

Finally, recognition and appreciation are extended to three generous, loyal, and dedicated people who gave expert assistance and timely advice in resolving a myriad of special problems relating to the production of this yearbook:

Jack E. Forbes, who served in the capacity of assistant editor but refused appointment.

Carroll S. Brown, who served as the editor's chief consultant in planning and producing the artwork and selecting pictures.

Mary Kaiser, the editor's secretary, who handled reams of correspondence flawlessly and whose meticulous attention to bibliographical detail contributed immeasurably to the quality of this publication.

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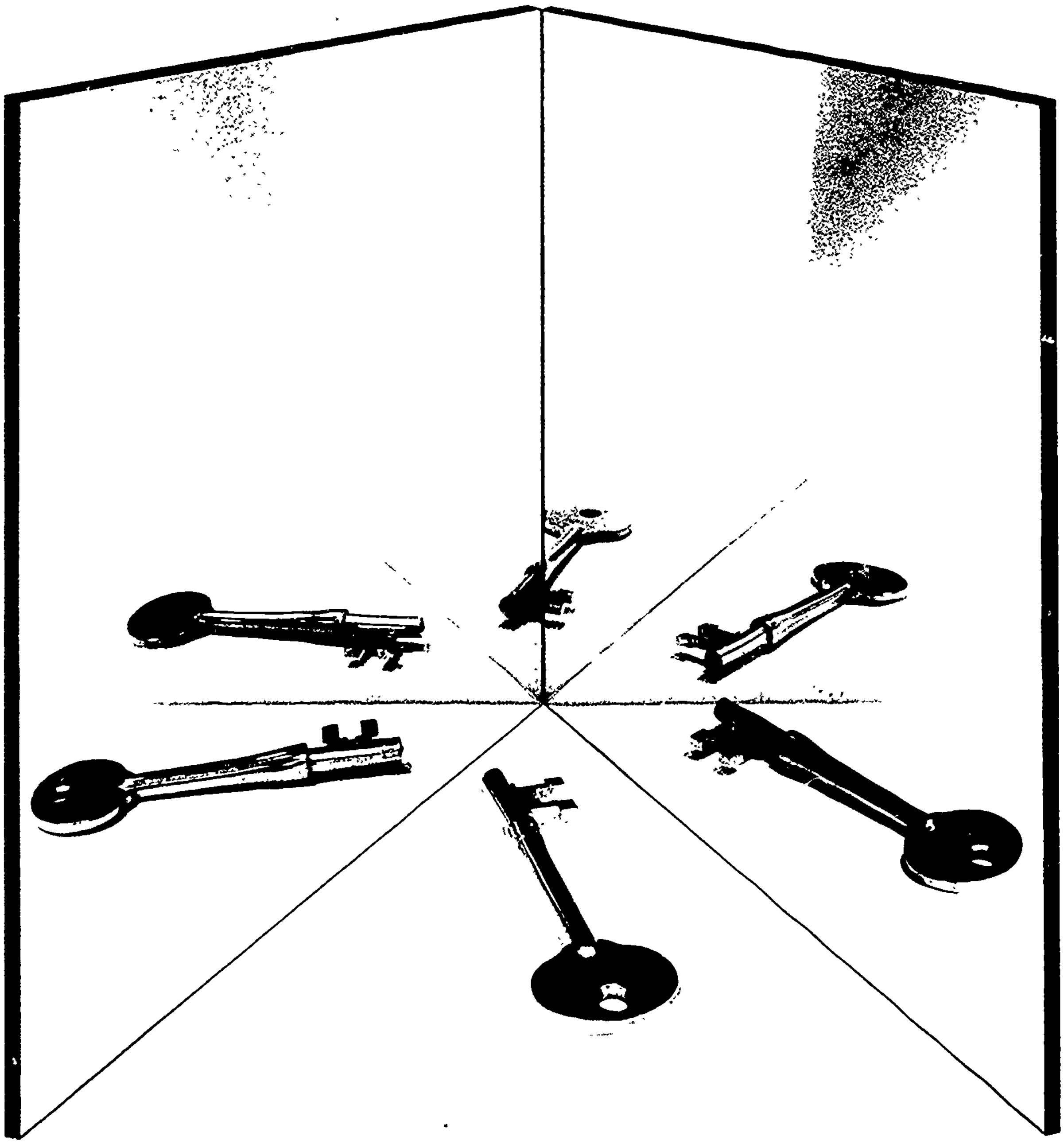
THE DIHEDRAL KALEIDOSCOPE is an instructional aid that is remarkable for its simplicity and for the sophistication of the mathematical idea it depicts. It consists of two plane mirrors joined to form a dihedral angle of $180^\circ/n$ for $n = 1, 2, 3, \ldots$, and physically exhibits a dihedral group of order $2n$. The mirrors in the picture are inclined at an angle of $60^\circ$, exhibiting a group of order 6. The key between the mirrors yields 6 visible images, including the key itself.



CHAPTER 1

# PERSPECTIVE, PURPOSE, AND PROFILE

*by*
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The opening chapter serves as an introduction to the Thirty-fourth Yearbook. The significance of the title "Instructional Aids in Mathematics" is illuminated with a capsulation of two affinitive ideas—learning in mathematics and instructional functions related to instruction in mathematics. The working definition of instructional aids in mathematics that was used to give direction and organization to this yearbook accents specifically the "things" that can be used to promote learning. The chapter elaborates on the four major objectives of this book and presents a summary of the chapter-to-chapter themes.

# 1. PERSPECTIVE, PURPOSE, AND PROFILE

Learning among human beings manifests itself as changes in the behavior of individuals.

"*When I know something new about the world, I'm different. When I know something new about myself, the world is different.*"[1]

Some changes in behavior considered indicative of learning in mathematics are an increased facility in recognizing and using mathematical terminology, performance indicative of a growing comprehension of concepts, and an expanded versatility in applying mathematical principles. Other changes in behavior regarded as learning in mathematics include a stepped-up proficiency in solving problems, an increased resourcefulness in analyzing relationships, and an advance in the level of creative ideas generated.

The educational process is characterized by an instructional activity that is designed to meet the requirements of getting students to learn. What instruction means can be clarified by examining some of its functions. Among the functions that pertain to mathematics instruction are the following:

1. Motivating the student to begin learning
2. Presenting a sequence of stimuli that is structured to keep the student engaged in learning
3. Giving the student directions to do something, look at something, or start some activity
4. Describing or portraying the kind of performance expected of the student when the learning period ends
5. Guiding the student in discovery learning by giving appropriate prompts
6. Communicating the structure of the subject (i.e., relating concepts, definitions, assumptions, and theorems)
7. Immediately reinforcing learning
8. Providing opportunities for practice
9. Presenting variations of previously learned ideas
10. Presenting problems in application
11. Monitoring the student's progress
12. Evaluating terminal behavior in accordance with accepted criteria.

To some degree, the teacher can perform all these functions through oral communication. At appropriate times the teacher's performance can be supported and the learner's experiences enlarged with the aid of textbooks, teaching machines, projection devices, computers, models, manipulative devices, television, and suitably designed instructional spaces.

With this much introduction, the working definition used to give direction and organization to the production of this yearbook is presented:

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*An instructional aid in mathematics is viewed as any object, piece of equipment, model, device, instrument, publication, picture, chart, or any facility that serves the purpose of enhancing or effecting the learning of mathematics.*

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This definition refers specifically to the set of "things" that can be used to promote the learning of mathematics. It refers to everything in the learner's environment that is external to the learner, exclusive of the teacher. The teacher is the human side of the learner's resources and is responsible for helping the learner regulate his environment.

1. Attributed to Helen Linden in *VIP: A Guidebook for Volunteers in Participation*, Office of Teacheraide and Volunteer Services Publication no. 352 (St. Paul, Minn.: St. Paul Public Schools, 1970).

## PURPOSE OF THIS BOOK

The production of this volume was inspired by the premise that the thoughtful use of instructional aids by teachers of mathematics will serve to improve mathematics education now and in the future. Specifically, this yearbook was planned to meet the following objectives:

1. To give teachers reliable organized information regarding the range of possibilities of acquisition and utilization of instructional aids in mathematics
2. To provide teachers with a basis for evaluating the quality and utility of instructional aids
3. To provide teachers with suggestions for making effective use of instructional aids
4. To stimulate teachers to create instructional aids.

To meet the first objective, this book has been organized according to the classification scheme suggested by the chapter titles that follow. This scheme is intended to help the reader internalize a feeling of organization for the variety of static, graphic, mechanical, electronic, and environmental phenomena considered to be instructional aids in mathematics. The examples included in the different chapters should serve to inform the reader regarding the many kinds of instructional aids that can be acquired and their manifold uses.

The second objective deals with the problem of evaluation. The examples of instructional aids presented in the different chapters embody acceptable standards of quality and utility. By giving thoughtful consideration to these examples, the reader should gain a sense of discrimination in selecting instructional aids suited to his purposes.

The third objective is related to the second. The examples of instructional aids included in the various chapters were chosen to illustrate particular instructional functions and techniques of use. There are suggestions for teaching modern concepts with age-old symbols of craftsmanship, as well as suggestions for teaching classical concepts with devices that are brand new.

The fourth objective has been met by incorporating illustrations of original teacher- and student-made devices that invite imitation. The authors of the various chapters are aware that creativity in teaching really comes to flower in the area of instructional aids.

## ORGANIZATION OF THIS BOOK

In keeping with the broad point of view regarding the nature of instructional aids that forms the basis of this yearbook, Chapter 2 directs attention to the setting in which instruction in mathematics takes place in today's schools. The chapter seeks to indicate how different kinds of instructional spaces can effect or enhance the learning of mathematics. It is proper to think of space as an instructional aid because the kind of spaces that are available dictate to some extent the kinds of instructional functions that are possible. In the organizational strategy of Chapter 2, different instructional arrangements are first described, and then suitable spaces for each are identified.

Chapter 3 concentrates on the most revered of all instructional aids, the textbook. The main concern of the chapter is with the problem of selection. The scheme proposed for selecting textbooks involves decision-making that is based on behavioral objectives.

In the not-too-distant past the mathematics textbook for a particular subject or grade tended to be *the* curriculum, but this is not as true as it used to be. Today there are available a large number of monographs on special topics, charts, workbooks, drill and practice kits, pamphlets, pictures, newsletters, periodicals, mathematics library books, and books that are referred to as supplementary references. All these are discussed in Chapter 4 under the title "Other Printed Materials."

Chapter 5 contains a brief account of the history and development of programed instruction

and teaching machines. Guidelines for selecting and evaluating programed instruction materials are sketched. Also included is a frank discussion of the uses and abuses of programed instruction materials.

Chapter 6 is concerned with calculating devices and computers. Included are a historical résumé highlighting major developments in computer science and descriptions of calculating devices and computers that can be used as instructional aids in mathematics. A major part of the chapter is devoted to a discussion of current practices and probable future uses of computers and computational devices in the schools.

Chapter 7 gives an overview of the various kinds of projection devices that have proved their effectiveness over the years, as well as those innovations that teachers of mathematics have begun using only recently. Specific instructional aids treated include films, filmstrips, slides, overhead and opaque projectors, and television.

Chapter 8 deals with the use of models. The chapter develops theory associated with the kinds of instructional aids to which much of the Council's Eighteenth Yearbook is devoted, gives a description of the major types of models, presents an illustrated discussion of various uses of models, identifies promising practices, and contains an extensive listing of models that are available from commercial sources.

Chapter 9 focuses on the subject of manipulative devices. The treatment presented concerns the kinds of devices that are useful in helping elementary school students learn about the nature of number, counting, systems of numeration, properties of operations with numbers, fractions, and spatial relationships. Considerable space is devoted to a discussion of the use of manipulative devices as concrete referents in promoting discovery of concepts and improving computational skills.

Chapter 10 is devoted to projects, exhibits, fairs, games, puzzles, and contests. Usually these aspects of school mathematics are thought of as individual or group activities. However, since these activities normally involve the use or production of materials that have the same attributes as instructional aids, a chapter dealing with these aspects of school mathematics is included in this yearbook.

The general availability and wide variety of instructional aids discussed in Chapters 2 through 10 have enlarged the teacher's role in selecting and utilizing instructional aids. The last chapter—that is, Chapter 11—gives an in-depth discussion of the expanded role of the teacher as manager of an instructional system containing many components that contribute to student learning. The terms *instructional system, systems approach to instruction, systems thinking,* and so on, all describe an orderly process for making decisions concerning what is to be taught, to whom, and using which materials. The chapter includes an explanation of the systems approach to instruction and an analysis of various instructional aids as components of an overall system. There is a detailed description of the teacher's role as systems manager as well as comments on possible contributions of computers to the discharge of management functions. Finally, the chapter contains an extensive example of the application of systems thinking to the determination of objectives and the selection of materials of instruction for a popular topic in mathematics.

Since Chapter 11 presents a plan for the orderly selection of instructional aids, it is as close as one can come to a summary of the diverse contents of the preceding chapters. In a sense it is both a summary chapter, which provides a structure for the information acquired in the first reading of previous chapters, and an introductory chapter for a second, in-depth, reading of these chapters.

*To quickly locate topics that are of immediate interest or concern, the reader is referred to the table of contents on pages xi–xvi. This table includes both the primary and secondary subheads for each chapter.*

STUDY CARREL showing built-in sound equipment in use

MATHEMATICS INTEREST CENTER, Creek Valley Elementary School, Edina, Minnesota ▼

CHAPTER 2

# INSTRUCTIONAL SPACES

*by*
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Chapter 2 treats the kinds of instructional arrangements currently being used for mathematics instruction and the types of spaces needed to accommodate these arrangements. The chapter begins with a look at the origins and major sources of influence that helped shape the various instructional arrangements and then describes in detail each of six different arrangements: the self-contained classroom, large group instruction, small group instruction, the mathematics laboratory, independent study, and tutorial. The open space school is viewed as a school building design concept that encourages flexible grouping patterns and individualization of instruction.

# 2. INSTRUCTIONAL SPACES

Instructional spaces serve as instructional aids by accommodating other components of instruction, that is, students, teachers, time, textbooks, teaching machines, computers, projectors, models, and manipulative materials.

The purpose of this chapter is to indicate how different kinds of instructional spaces can enhance and effect the learning of mathematics. If at times the discussion seems to have a more general reference than just to mathematics, it is because instructional spaces have not usually been designed specifically to accommodate the teaching and learning of mathematics.

Six different arrangements of instructional components are recognized: self-contained classroom, large group, small group, laboratory, independent study, and tutorial. These arrangements are not mutually exclusive; and it is natural that they overlap, for the last five—large group, small group, laboratory, independent study, and tutorial—emerged through separation, accentuation, and expansion of instructional functions originally performed in the self-contained classroom. This differentiation of instructional functions and provision for spaces to house them came about through American society's continually enlarging demands for more and better education of its youth.

The historical résumé that follows recounts some of the major influences that helped shape the different instructional arrangements. The next section treats the self-contained classroom, and the ensuing section treats the other five instructional arrangements. Then there are two sections that deal respectively with team teaching and scheduling. The final section contains illustrations of efforts to individualize instruction by coordinating form and function. Included is an extensive discussion of open space schools.

## ORIGINS AND MAJOR INFLUENCES

Until quite recently, the most pervasive influence in the design and use of instructional spaces has been the centuries-old belief that the way to teach is for one teacher to get together with twenty-five or so students. The student-teacher ratio of twenty-five to one can be traced back at least as far as the fourth century in the Hebrew *Babylonian Talmud:*

> The number of pupils to be assigned to each teacher is twenty-five. If there are fifty, we appoint two teachers. [Baba Bathra 21a-21b]

The one-room schoolhouse of the colonial and early national periods of America was based on this credo. These schools were built to accommodate from fifteen to twenty-five students ranging in age from six to sixteen, and they were largely ungraded. The teacher dealt with each student individually. The usual procedure was for students to come up to the teacher's desk, one by one, to recite whatever they had been assigned to memorize. The exchange between teacher and student was largely a mechanical one, involving almost no technique on the part of the teacher. His time was taken up by hearing lessons, assigning new tasks, setting copies, making quill pens, and dictating sums.

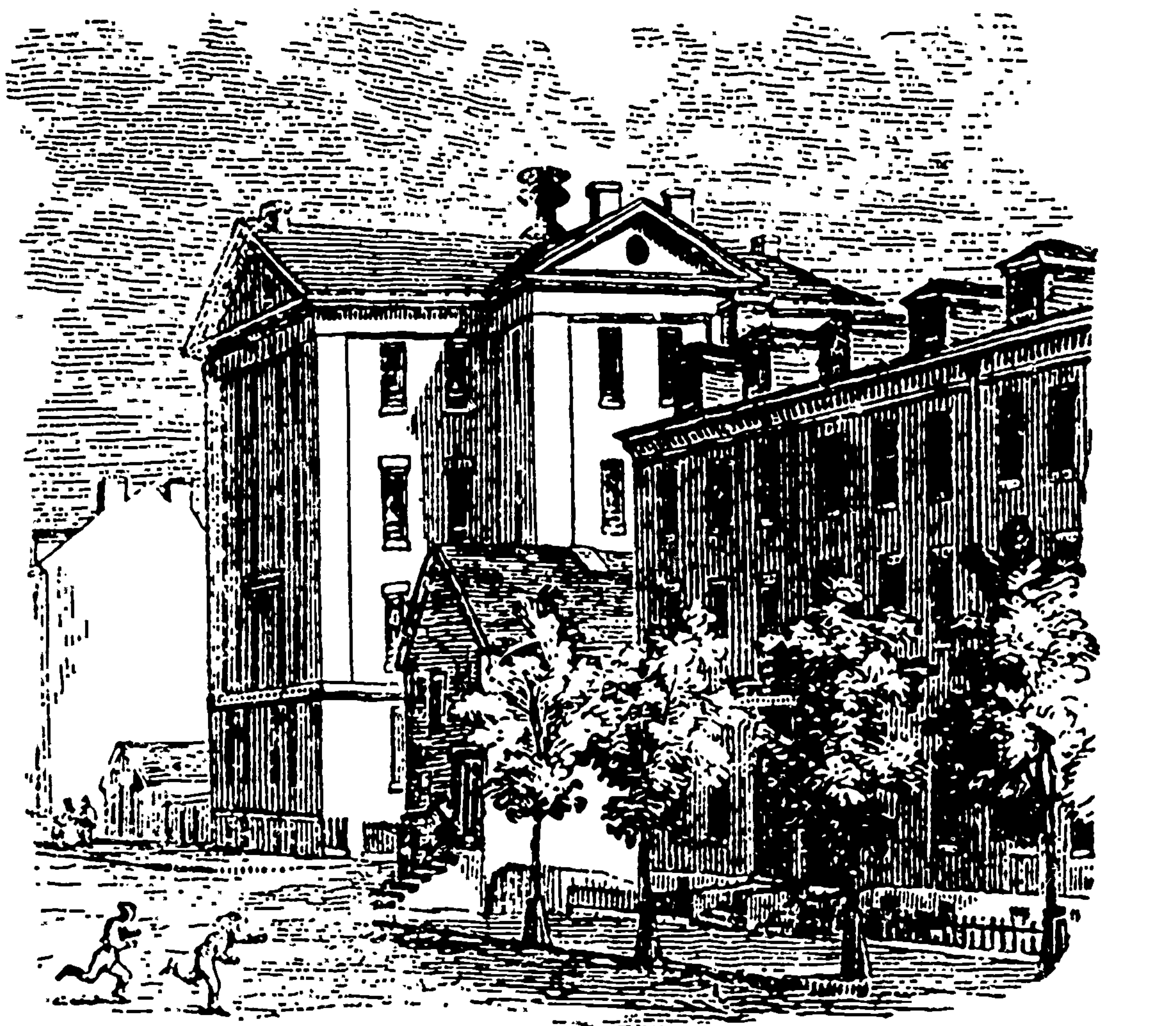
The opening of the Quincy School in Boston in 1847 inaugurated a new kind of school organization and a new kind of school building, both of which were to become standard in America until the last few decades. This was the first school in America in which pupils were divided according to age into groups that would stay together for a year's work. Each grade met with a single teacher in a separate room and was given assignments deemed appropriate to that age. To accommodate this new scheme of organization the one-room schoolhouse was multiplied under

one tool and stacked up egg-crate fashion. The *Boston Almanac* in 1819 offered this description of the building:

> This school house, being the last erected in the city, contains most of the modern improvements. . . . It is four stories high, and contains twelve school rooms, each of which accommodates 56 scholars, and a hall, furnished with settees, which will seat 700 pupils. It also has 6 small recitation rooms. Its greatest improvements consist in having a separate room for each teacher, and a separate desk for each scholar. [37, p. 72]

Everything was supposed to happen in this standard instructional cell—content presentations, student recitations, student-teacher conferences, independent study (usually doing assignments), and taking and correcting tests. Because one teacher and one group of students performed all these functions in one room, this arrangement came to be known as the self-contained classroom.

Until almost 1940, school buildings were constructed more or less on the plan of the Quincy School, with rectangular classroom boxes arranged on either side of corridors. The architect



Courtesy of Educational Facilities Laboratories

FIGURE 2.1. *Quincy School*

William Caudill describes the complete acceptance of this classroom box:

> At the turn of the century nearly every state in the United States passed laws which specified what this box should look like. Laws stated the size, shape, fenestration, and even the orientation. It wasn't long before the classroom box and its 25-to-one pupil-teacher ratio became a "SACRED COW" to educators and architects all over this nation and the Western World. [16, p. 8]

After World War II the construction of school buildings was influenced by the residential rambler and picture-window fad that accompanied the dash to the suburbs. The result was sprawling schoolhouses with larger classrooms and large areas of window space. But in spite of this subtle influence for openness, these sprawling schools functioned like the egg-crate schools. The idea of the self-contained classroom continued to dictate instructional functions. Elementary students remained in the same box with the same teacher for a year, and secondary students changed boxes and teachers every hour. The box maintained its legal status, rigidity, and indestructibility, and the halo on the Sacred Cow remained untarnished.

Only during the past two decades have educators and architects become noticeably sensitive to the idea that school buildings and facilities should be designed to accommodate and encourage desirable instructional functions. This has led to innovations in the design of new buildings and facilities and modifications in the use of older ones. What has motivated these innovations? In the case of mathematics, some of the impetus has come from the drive to improve school mathematics that started after World War II. Changes affecting education generally have come from a steadily growing zeal for the goal of individualizing instruction in all subject areas. The two influences have not been independent, and they are not yet history.

The drive to improve school mathematics that was launched in 1952 and was given a boost by Sputnik in 1957 resulted in a complete overhaul of the school mathematics curriculum and led

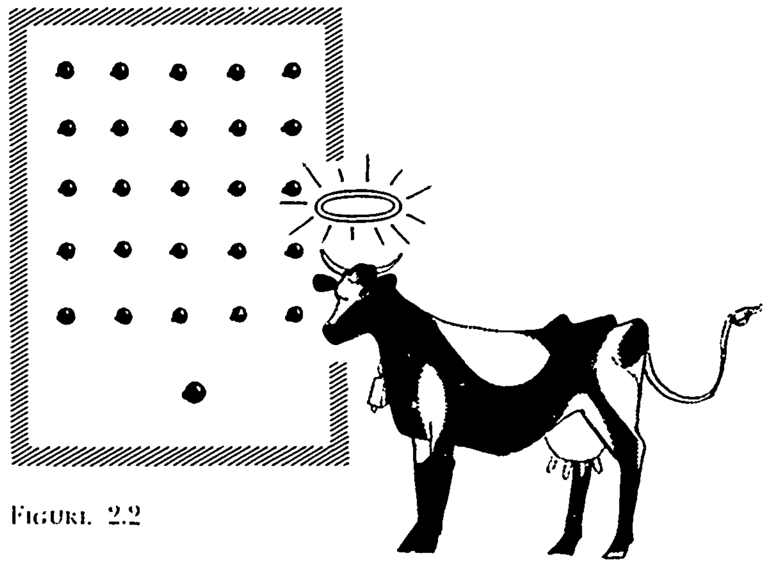


FIGURE 2.2

*Reproduced from* Shells and the Educating Process *by permission of Caudill, Rowlett, and Scott*

almost immediately to several kinds of changes in the use and design of spaces for mathematics instruction.

In the high school one change took the direction of specializing existing standard-sized classrooms as "mathematics classrooms" by moving in special furnishings and bringing in a variety of instructional aids for mathematics. Such specialization was followed by recommissioning of small unused spaces as adjunct spaces (e.g., mathematics libraries and mathematics laboratories). In new construction some of the special furnishings were built in as fixtures, and adjunct spaces and standard-sized classrooms were put next to each other to form mathematics "suites." In the elementary school a modest trend developed toward semidepartmentalization (e.g., mathematics and science) and related specialization of standard-sized classrooms. However, all these changes acknowledged the accepted status and constraints of the self-contained classroom arrangement.

The main impetus for architectural change came from a different direction. In 1959 J. Lloyd Trump opined that the challenge for quantity in American education had largely been met (73). To stimulate planning to improve the quality of education he proposed that schools be organized

around three kinds of activities: large group instruction, individual study, and small group discussion. Trump's plan assumed team teaching and implied flexible scheduling. He believed this scheme would result in improved instruction and make more effective provision for individual differences. While Trump's proposal was directed primarily at improving the quality of education at the secondary level, the effects of his recommendation were also felt at the elementary level.

With the development of instructional arrangements based on Trump's proposal there rose a need for more flexible spaces. Some schools tried to meet this need by varying the locations of permanent partition walls—that is, removing walls between rooms and hallways to create large group spaces and building new walls within rooms to create small group spaces. An alternate approach was to put up and take down demountable walls to make available different sized spaces as needed. Unfortunately, the continuing fixation over the necessity of having separate enclosed spaces always involved compromises, and changing partitions often delayed changes in teaching and learning rather than accommodating them. Operable and folding walls came next and provided a partial solution. But now the demand turned to instant flexibility, and someone got the idea of leaving out interior walls entirely. The result was the open space school. A forerunner of this concept in schoolhousing design was an elementary school in Carson City, Michigan. In 1957, the school planners in that city wanted a nongraded, team-taught elementary school and gained nationwide attention by fusing four standard-sized classrooms into a jumbo-sized rectangular box by leaving out interior partition walls.

Another major influence on the use and design of instructional spaces, and the one that will undoubtedly have the greatest impact in the future, is the evolutionary trend to individualize instruction.

This trend involves two ideologies. One follows the belief that the way to individualize instruction is to make effective provision for individual differences by grouping together students with similar instructional needs or aims. This notion of individualization of instruction has had a long history in this country and has many adherents.

The organization of the Quincy School, in which students were divided according to age, was really an attempt to individualize instruction by applying the assumption that grouping students according to age would increase the prospect of individualizing teaching and learning within groups.

The advent of compulsory education in the high school following the turn of the century resulted in burgeoning enrollments. This, coupled with popularization of educational testing after World War I, resulted in students being grouped in one of three curricula according to their vocational and educational plans—manual training, commercial, or college preparatory.

The 1940s marked the beginning of homogeneous ability grouping in the high schools with accelerated classes for the bright and remedial classes for the less able. This was followed in the 1950s with ability grouping by subjects (14). In the 1960s the secondary schools switched to grouping patterns based on instructional functions to be performed, that is, large group, small group, and so forth. This is basically the Trump plan and has already been discussed. Recent efforts to individualize instruction at the elementary level have taken the direction of performance level grouping, nongrading, and semidepartmentalization in the larger schools.

Nongrading and the latest refinements in grouping practices, especially performance level grouping in different subjects, point to the second view of individualization of instruction, a view espoused by Mitzel. He interprets individualization of instruction to mean "tailoring subject matter presentations to fit the special requirements and capabilities of each learner" (45, p. 436). This conception of individualization of instruction is usually referred to as adaptive education. According to Mitzel's view, the learner should be allowed to proceed through content at a self-determined rate, to begin studying a

subject (or topic) at a point appropriate to his past performance, and to have a variety of instructional aids at his disposal.

An early form of individualized instruction that leans toward the view held by Mitzel was developed by Search around 1894. Search instituted an individualized high school program in which the student was placed as an individual, in which he worked as an individual, progressed as an individual, and graduated as an individual (66, p. 154).

Between 1911 and 1920 Helen Parkhurst devised an individualized self-paced laboratory-type plan (called the Dalton Plan) in which the teacher and student formed a contract with the student agreeing to undertake assignments for several weeks duration in his various subjects. The student was quite free to work at his own rate, except that he could receive no new contract until he finished the one on hand (59).

The work of Skinner and Crowder with teaching machines and programed instruction in the mid-1950s and early 1960s also gave impetus to the adaptive point of view of individualization of instruction (69).

Still more recently, a technique called "individually prescribed instruction" or IPI has been developed in five elementary school subjects out of the work of the Learning Research and Development Center at the University of Pittsburgh (38; 63). This technique consists of prescribing for each student a program of studies tailored to his learning needs. The curriculum for each subject area is specified by a carefully sequenced set of behavioral objectives, and students are placed in the program at various levels in accordance with their performance on pretests. The mathematics curriculum is based on more than four hundred specific objectives. Students work individually on a precisely ordered set of materials built on these objectives (63).

The two views of individualization of instruction that have been described converge on assumptions such as the following:

1. Students of different abilities, backgrounds, or competence levels should have curricula appropriate to their needs and purposes.
2. Instructional strategies should be based on well-defined behavioral objectives.
3. Performance grouping and periodic regrouping should be based on diagnostic and achievement testing.
4. Small groups should be organized to meet particular instructional purposes.
5. Students should have opportunities to study independently.
6. Students should have access to a variety of instructional aids.
7. Assignments should be varied to meet individual students' needs and interests.
8. Every student needs dialogue with a teacher—this includes tutoring.

Both views of individualization of instruction have had an impact on the use and design of instructional spaces. Up until now the influence of the grouping doctrine has been paramount. This is reflected in the choice of content and the organization given this chapter. At the same time, the influence stemming from the adaptive theory has not been neglected.

## SELF-CONTAINED CLASSROOM ARRANGEMENT

Unlike much of today's writing about the self-contained classroom, the treatment given here is one of respect. The dominant position of the self-contained classroom as an instructional arrangement is recognized. While its limitations are acknowledged, its possibilities for enhancing the teaching and learning of mathematics are emphasized. Even though a variety of innovative instructional space designs has been incorporated in new construction in recent years, the economics of schoolhouse financing are such that the self-contained classroom arrangement will not soon be abandoned. Instructional functions performed under this arrangement developed within constraints imposed by the size of the room and by the accompanying school organization.

In the elementary school this means having one teacher meet with twenty-five to thirty-five

students in one room, six to seven hours per day, five days per week. Grouping is usually by age, and this is also the basis for regulating student progress. Mathematics is taught from thirty to forty-five minutes each day by the same teacher who teaches all subjects except perhaps the fine arts, industrial arts, and physical education.

The secondary school organization in which one teacher teaches one subject to one group of twenty-five to thirty-five students in one room, for one fifty-five minute period, four or five times per week is a variant of the self-contained classroom organization of the elementary school. Grouping is usually by ability level, by subject, or both, and promotion depends on whether or not the student does the minimum amount and quality of work required for credit.

The instructional space associated with the self-contained classroom arrangement is usually referred to as a standard-sized classroom.

A typical unsophisticated standard-sized classroom has from 700 to 1000 square feet of floor space and is rectangular in shape, with window openings along one side wall and chalkboard mounted along the front wall and perhaps also along the other side wall. The back wall may have a bulletin board panel. Furniture for students consists of individual deskchairs that are arranged in rows parallel to the side walls.

The teacher usually has an executive desk and a large chair mounted on a swivel. In addition, there is usually a modest sized bookcase, a storage cabinet, a filing cabinet, a shelf or table for models, and a closet for the teacher's belongings.

At the elementary level the space facility described may have some specialized features for different subjects, although usually not many. In the high school this room, when used for mathematics instruction, may be equipped with a demonstration slide rule and a graph chart. In small communities the rooms used by elementary school students and secondary school students are often of the same design, located under the same roof, and a casual observer could not distinguish one from the other except for the size of students' deskchairs.

In an instructional space such as the one described the teacher performs most of the instructional functions listed below:

1. Motivating students to learn new topics and concepts
2. Presenting basic content using the chalkboard, overhead projector, charts, and demonstration models
3. Clarifying statements in students' textbooks
4. Stimulating discussion among students by directing questions to the whole class or to individual members
5. Assigning practice work (by handing out worksheets or assigning problems from a textbook) chosen to give students of different abilities opportunities to study independently
6. Supervising seatwork (practice work or problem solving), giving students in need of help brief (sometimes momentary) amounts of tutorial assistance
7. Administering and correcting tests, and reteaching content with which there is widespread difficulty.

This list of instructional functions is, of course, not delimiting and will vary with the individual teacher. In addition the teacher performs many instructional backup functions before and after school and at other times when he has the room to himself. He reads, studies, plans the curriculum, prepares presentations, duplicates worksheets and laboratory guides, writes tests, plans assignments, evaluates textbooks and other instructional aids, evaluates student performance, keeps records, and holds parent conferences.

What students do is for the most part generated by what the teacher does. Students listen to and watch lecture demonstrations in which the teacher uses the chalkboard, overhead projector, and demonstration models and devices. Students take notes, ask and answer questions, do assignments, take tests, and give demonstrations on the chalkboard or overhead projector when asked to do so.

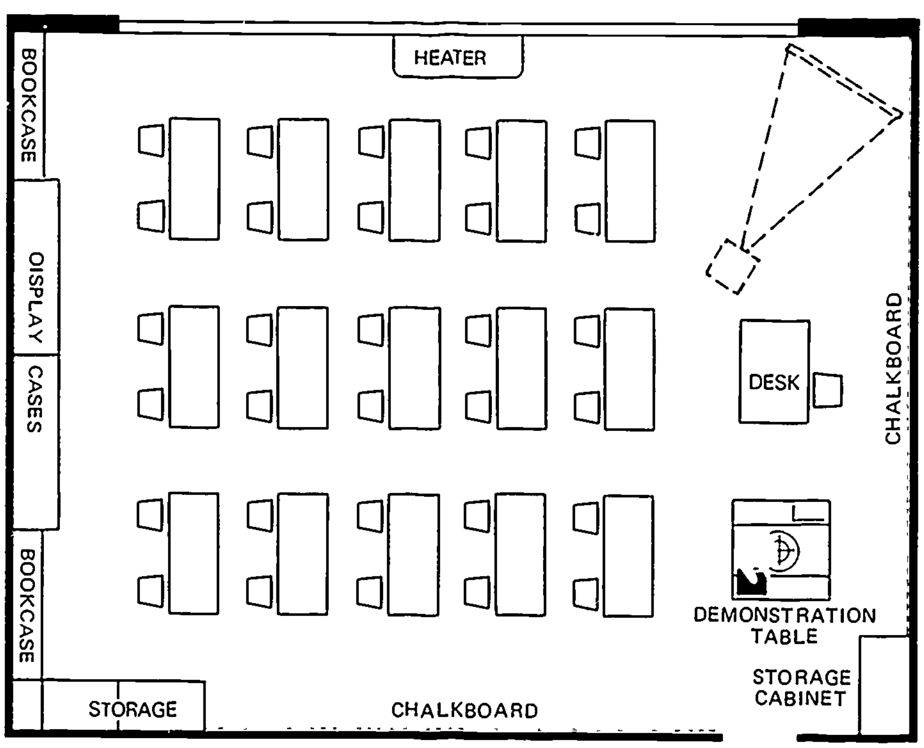
Various efforts have been made to specialize

the standard-sized classroom for mathematics instruction (Figure 2.3). In some instances this has been accomplished by introducing a variety of instructional aids such as models, charts, manipulative devices, measuring and drawing instruments, supplementary textbooks, student journals, and teaching machines; by moving in a glass-enclosed display case to store valuable models and instruments; by providing an open display case to display students' projects and exhibits; and by supplying a magazine rack, book display stand, and book truck to display and make printed materials easily accessible. In addition, a demonstration table with sink and drain has been installed (or a demonstration cart moved in), gas and water have been made available, electrical outlets and shelves to support TV receivers have been put in, and a permanent screen for projecting films and filmstrips has been mounted using wall or ceiling brackets.

If the room is large, lower room dividers have been placed to partition off a corner of about 100 square feet for a mathematics library and a separate space of from 100 to 300 square feet for a mathematics laboratory. The room dividers are equipped with bulletin boards on one side and shelves on the other, or they may consist of hinged panels of chalkboard. Sometimes tall movable bookcases are used instead of lower room dividers. Individual deskchair units are replaced by two-chair tables in order to encourage joint exploration of mathematical topics by two or more students and to make possible seating arrangements for small groups.

Specializing the standard-sized classroom for mathematics instruction by moving in furniture and equipment and installing appropriate fixtures encouraged the expansion of instructional

FIGURE 2.3. *A standard-sized classroom specialized for mathematics instruction by use of two-chair tables, a mobile demonstration work center, bookcases, display cases, storage cabinets, overhead projector, and viewing screen.*




*Adapted from drawings by E. H. Sheldon Equipment Company*

functions with one teacher in charge, and concomitantly served to improve the quality of mathematics education possible under one teacher. The desire to expand instructional functions in the standard-sized classroom by setting aside portions of it for library and laboratory spaces led to incorporation of adjunct spaces for such purposes in new building designs and to the creation of such spaces through remodeling in older buildings.

Attempts to individualize instruction in the self-contained classroom arrangement have taken the path of making provision for individual differences by varying levels of procedure, by varying assignments, by grouping students within the classroom according to their performance, by providing opportunities for students to attack a topic in a variety of ways, and by conducting a limited style of tutoring.

To vary levels of procedure the teacher usually works with the entire group. For example, to introduce the topic of carrying in addition, the teacher can explain methods appropriate for different ability levels by using concrete objects and counting, by using ten as the key, by using a number line, and by using the computation algorithm.

Strategies for varying assignments include making open-ended assignments that challenge the able but do not discourage the less able, specializing assignments for individual students, and making two-part assignments following a presentation. The first part of a two-part assignment may deal directly with the basic content of a lesson and will be identical for all students. The second part can be varied depending on students' needs and abilities, assigning challenging problems to the better students and novel practice exercises for the less able.

Regrouping students into subgroups is a third way of providing for individual differences in a self-contained classroom. Such regrouping ideally is tentative and is based on a diagnosis of strengths and difficulties to determine who has mastered a skill or concept and who has not. Write-in textbooks should be available for students who need extra help, and rapid learners should be encouraged to start special projects by perusing a file of suggestions for projects that is available for this purpose.

Sometimes the teacher can follow a strategy of interrelating total group instruction with small group instruction. For example, a teacher may present the basic part of a new unit to the entire class, give a test, and form small groups based on the results. Then he can tutor the slow learners and let rapid learners enlarge on the basic idea by doing problems in supplementary textbooks. The total class is reassembled for the next unit.

As another strategy, the teacher can present basic content to the total group, set students to work individually, and mentally tabulate the membership of three groups—the most progressive, the slowest, and the group that is holding its own. Members of the progressive group can serve as tutors for members of the slowest group. As the teacher spots widespread difficulties, he brings the group back together.

Observations of instructional procedures carried on in the self-contained classroom indicate that effective individualization of instruction under this arrangement is difficult. Teachers who attempt to individualize instruction in a self-contained classroom end up by dealing with individuals, not as individuals, but as members of a group. One teacher cannot work individually with each of twenty-five to thirty-five students often enough to make the encounter worthwhile. Neglecting students of high ability because the needs of low ability students are greater is not a satisfactory form of individualization. Every student has a right to dialogue. Small group discussion, although manageable, is generally confined to brief exchanges between the teacher and a small fraction of the total group. Laboratory work is possible only for a small range of concepts, and learning by discovery is usually reserved for the few students who make discoveries initially. Independent study does occur but is limited because of the close proximity of students to each other and a lack of the variety of instructional aids needed to encourage it.

## VARIED-SIZED GROUP ARRANGEMENTS

This section concentrates on the five instructional arrangements—large group, small group, laboratory, independent study, and tutorial. Characteristic instructional functions for each are described, and appropriate spaces for each are identified. When used in combination, these arrangements offer the possibility of providing individualized instruction for every student. This does not mean providing one teacher for every student. It means customizing a program for every student, that is, giving each student an opportunity to participate in the mixture of arrangements considered optimal for him. Some schools include the standard-sized classroom as a medium-sized group space in this medley of arrangements. In such instances there are six different varied-sized group arrangements.

Suggestions presented in this section are intended to be provocative but realistic. No imperatives are set down. Instead, the exposition inclines toward options that the teacher can use to meet particular objectives, using the instructional spaces that are available to him.

### *Large Group Instruction*

The goal of customizing instruction for every student has turned into a quest for ways of providing appropriate spaces and time for small group meetings, independent study, laboratory work, and tutoring. To save teacher time so that instruction can be provided for groups smaller than medium-sized and for individual students, it is necessary to balance such instruction with large group instruction.

Since it has been estimated that about one-third of all instruction that might be given to a medium-sized group is typically one-way, large group instruction saves teacher time by reducing duplication. It is an efficient arrangement for reaching at one time all students who are studying the same subject. Besides this rather obvious practical consideration, large group instruction can be employed to effect specific instructional functions. Experienced teachers of mathematics report success with the following:

1. Motivating a new topic by presenting historical background
2. Motivating a new topic by presenting practical problems involving applications
3. Motivating a new topic by relating it to topics studied previously
4. Introducing a new topic by presenting test items similar to those that will be used to evaluate achievement
5. Developing proofs of key theorems
6. Demonstrating techniques for problem solving
7. Presenting demonstrations involving large dynamic devices and mathematical models
8. Giving presentations requiring the use of scarce or expensive equipment (showing a sine curve with an oscilloscope)
9. Presenting a film or filmstrip and a summarizing commentary
10. Presenting information not available through other sources—for example, notes from conventions, trips, field trips, or relatively inaccessible periodicals or books
11. Presenting enrichment materials by detailing the contents of books and pamphlets in order to interest students in reading them—for example, working a problem from *Mathematical Challenges* (18)
12. Presenting information on professional and vocational opportunities in mathematics
13. Engaging students in recreational activities such as "mathematical bingo"
14. Developing a concept, proving a theorem, or outlining the solution of a problem with which there is widespread difficulty
15. Summarizing a unit by developing a topical outline and posing sample test questions
16. Administering a test
17. Returning a test and correcting errors made by students
18. Making assignments
19. Familiarizing students with materials and resources available to them

20. Presenting live telephone conferences with great living mathematicians using a special phone hookup with students being able to ask questions
21. Moderating a panel discussion.

Besides being typically one-way, large group instruction is teacher centered. The teacher lectures (talks), uses the overhead projector, writes on the chalkboard, shows films and slides, gives demonstrations, and plays tapes and records. Sometimes the teacher is a guest lecturer, or the lesson is presented over television. In the latter instance the "large group" can be decentralized. That is, students can receive instruction in small rooms viewing a television receiver, or in independent study carrels using small screen receivers.

The student's role in large group instruction is usually restricted to viewing visuals, observing demonstrations, listening, and taking notes. Occasionally the teacher may ask an individual student to present a paper he has prepared, give a solution to a problem, develop a proof, or take part in a sociodrama that is planned to motivate some facet of mathematical inquiry. However, such direct involvement by a student is at the discretion of the teacher who is in charge of the large group, and, in general, the student's role is one of giving attention and being receptive rather than participating actively.

Preparing for a large group instruction session demands more creativity and ingenuity than does preparing for smaller groups, where the give and take of discussion can hold teacher and students together.

In a large group presentation the teacher should rely on broad effects using simple, penetrating explanations: easy, offbeat examples; drawings, pictures, and cartoons; and thought-provoking questions (perhaps introduced as potential test questions) shown on prepared visuals and handouts. The opening and critical parts of handouts should be displayed with visuals.

A large group presentation should have a readily discernible structure, because the student will gain much of his stimulation from his comprehension of this structure. Teachers should make a careful estimate of what students can be expected to grasp in a large group presentation and should resist the temptation to cover a topic completely. A large group is better used to stimulate interest in a topic or to explain a small portion of it than to give it exhaustive treatment. The meeting of the large group should terminate with an open-ended assignment, offering students a sense of direction for further study.

Below are several principles to observe in preparing for a large group presentation.

1. More materials should be prepared than are expected to be used, in case the presentation takes less time than is anticipated.
2. The number of major ideas should be limited, perhaps to as few as one or two.
3. A substitute presentation should be readied to be available in case of emergency.
4. Several illustrative examples should be collected for each idea to be developed. Problem solutions should be handled from several points of view. It is a rare problem that cannot be looked at or done in several different ways.

Following a large group session students should be given opportunity to reexamine new material in small groups and to confront the subtleties, qualifications, and exemplifications of the new material in independent study.

The optimal length of a large group session is unknown, but in any case should last no longer than mental alertness lasts. That time will vary, but forty minutes is a good average length.

What is the maximum number of students that can be instructed effectively in a large group? This depends partly on the age and maturity of the students. At the secondary level 150 to 200 students is the largest group advisable, but at the elementary level not more than 100 should be given instruction at one time. The actual number making up any large group will depend ultimately on the number of students who are ready for the experience and also on the size of the space that is available; students will learn best if they are comfortable and uncrowded.

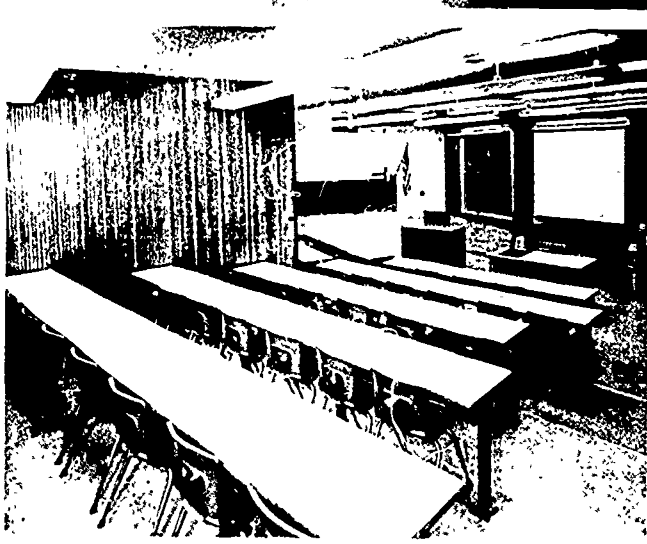


FIGURE 2.4. *Large group space used for mathematics instruction in White Bear Senior High School, White Bear Lake, Minnesota. This space is equipped with an operable wall that can be used to form two medium-sized group spaces by extending the operable wall to the front of the room (see inset).*

The success of a large group meeting depends not only on the careful choice and arrangement of content, but also on efficient management. It is essential that meetings of large numbers of students be run smoothly. To leave the teacher free to concentrate on the presentation itself, assistants should take care of clerical and mechanical tasks such as taking attendance, collecting papers, distributing handouts, moving demonstration carts, adjusting projection screens, room-darkening, operating film projectors, adjusting sound amplifiers, and the like.

All equipment and mechanical devices must be in good working order and in the right place. The intended effect of a carefully prepared slide presentation can be spoiled by awkward pauses taken by the teacher while he gropes for a plug-in or removes from the projector slides shown upside down or in the wrong order.

A large group instruction space should be arranged so that every student can sit comfortably, hear well, see well, and have a smooth writing surface. Rows of deskchairs or two-chair formica-topped tables fastened down on terraced platforms with the rear rows elevated is an arrangement that meets most of these requirements. A space equipped as described is especially suited to instruction of large groups in mathematics (Figure 2.4). Another possibility is a school auditorium with the floor sloping downward to the front and theater-style seating with each seat equipped with a retractable tablet arm writing

surface. A large group space like this is often used by teachers of various subjects. This means that specialized features needed for instruction of large groups in mathematics may be lacking. Large group spaces should be air-conditioned and windowless, and sound amplification should be designed so that every student can hear. A general purpose gymnasium-auditorium is not suitable.

Because projectors of different types are used frequently in large group instruction, the effectiveness of different projectors as teaching tools should be accurately assessed. An indispensable requirement for good viewing is a large screen suspended from the ceiling. Each type of projector should be located in a handy place, and there should be a sufficient number of easily accessible electrical outlets. Motion picture and slide projectors should be placed at the back of the room to minimize distraction. Since the overhead projector is probably the most used of all projectors, it should be positioned so as to be immediately available to the teacher without obstructing student viewing.

Sound amplifiers should be part of the school's total audio system, and there should be provision for receiving television programs. To be optimally functional, a large group space should have a media transmission center where a microphone and overhead projector are located, along with remote controls for raising and lowering the viewing screen; starting film projectors, tape players, and disc players; and turning on television receivers.

Teachers of mathematics usually make liberal use of chalkboards, so the amount of chalkboard space must be adequate and appropriately placed. One desirable arrangement is to have at least two large panels mounted and counterbalanced so that they can be raised or lowered at finger touch. Multiple panels can also be mounted to be moved sideways (Figure 2.5). Either arrangement increases the total amount of chalkboard space available for teacher use.

### *Small Group Instruction*

Small group instruction affords students certain unique opportunities for learning mathematics that are not present in other instructional arrangements. The most advantageous of these is that the structure of a small group encourages students to become actively involved in the learning process through interaction with each other and with the teacher.

Ideally, a small group should have from seven to fifteen members. A group of this size will give each student a chance to present and defend his ideas, ask and answer questions, reflect on and criticize the statements of others, check hunches, and form conclusions. Increasing the group be-

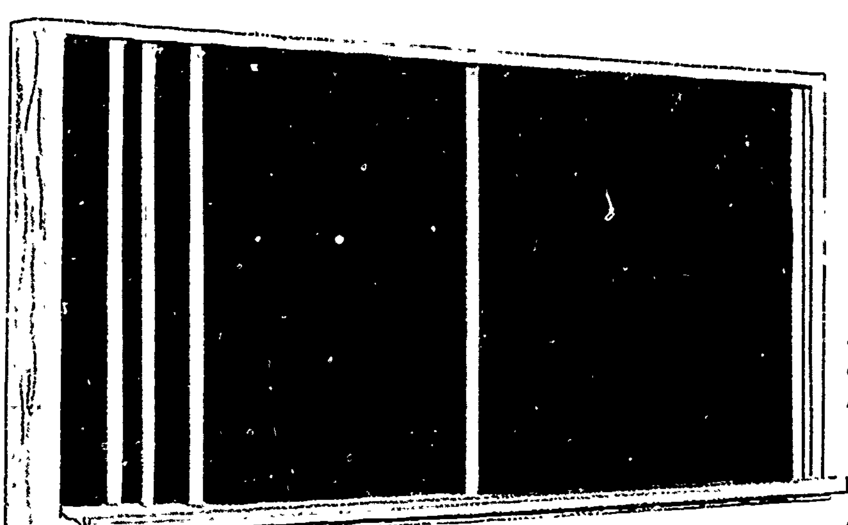


FIGURE 2.5
*Multiple-paneled chalkboard with each panel resting on a separate horizontal track*

*Courtesy of E. H. Sheldon Equipment Company*

yond fifteen will deprive members of adequate opportunity to take part in discussions. Reducing the group below seven will reduce the friction and stimulation needed to generate discussion. A group of twelve is probably the optimal size. As Manlove and Beggs suggest, "Perhaps there is good reason why no more than twelve are on a jury and why there were only twelve apostles" (12, p. 21).

As a general procedure small groups should not have a fixed registration but should be organized to meet particular purposes. This means that if a small group serves its purpose, membership in it will keep changing as individual needs arise and are met. At any given time membership in a small group will be based on some instructional need, progress level, common interest, or assigned task.

Small group meetings may be convened in response to requests by students; they can be conducted on a regular basis in conjunction with large group instruction; or they can be organized at any time to handle particular instructional needs perceived by the teacher when monitoring independent study (e.g., to reteach an algorithm to a small group of students who are having difficulty).

Small groups can be organized for the purpose of teaching basic concepts and skills. For instance, in the primary grades a small group in mathematics can serve much the same purpose as the familiar reading group, that is, to give each student sufficient opportunity to learn basic skills through direct participation. Such participation can be encouraged by means of competitive skill-based games and the use of drill and practice cards. If the student is learning concepts, his participation can be encouraged by means of a guided discovery approach that relies on the use of manipulative materials, display charts, and chalkboard demonstrations.

A related purpose for organizing a small group is to reinforce concepts and skills with the aid of audio drill tapes. Students listen to and write responses to prerecorded exercises and then correct each other's work. (With certain commercially prepared tapes students can correct their own work.) After this, students pair off and assist each other with exercises they were not able to complete alone. The number of students who can work together under this arrangement is limited by the number of headsets accommodated by the equipment that is used. Usually the maximum number is eight.

FIGURE 2.6. *Small group using audio drill tapes*

*Courtesy of Minnesota Mining and Manufacturing Company, Inc.*

A major purpose for holding small group sessions is as a follow-up for large group instruction. The specific functions of such small group sessions are to amplify, clarify, and give additional instruction on content presented in large group sessions. Students can ask and answer questions and share interpretations. They can also, through group discussion and suggestions from the small group teacher, gain insight into their own individual learning processes. This can help them become more receptive to the information given in the large group presentation. The teacher should point up concepts and correct errors of fact in thinking from the sidelines, if possible, rather than from the chalkboard. By listening to students' questions and hearing their reactions to the large group presentation, the teacher can obtain feedback to pass along to the large group presenter and identify students who need specific tutorial assistance in order to make progress.

A small group can be organized to meet before as well as after a large group session. When a new topic is to be developed in a large group session the small group may be divided into subgroups, with each subgroup asked to take careful notes on one aspect of the large group presentation. Each subgroup then prepares an in-depth report and presents it to the small group—one perhaps on the proofs or techniques developed in the large group session, one on applications of the mathematical principles presented, and one relating the new ideas to previously learned material.

Following a large group session, a small group may gather spontaneously and ask the teacher to hold a problem solving session. Through collaboration in working problems students can gain the confidence needed to find solutions. Problem solving sessions have particularly high motivational value just before quizzes or tests. The quizzes and tests can be administered orally to the small group; students often find the competition stimulating and learn a great deal from each other.

In the case of a particularly puzzling problem, a brainstorming session may be in order to obtain suggestions for attacking the problem. Brainstorming can lead to inventive, innovative solutions that no student would have thought of by himself. Brainstorming is also valuable to gather ideas for applications of skills, concepts, and techniques and to formulate generalizations. Criticism and evaluation by either the teacher or students are out of place in sessions like these. The purpose is to generate a quantity of ideas including all manner of variations, combinations, and recombinations.

In a similar way a small group of students with a common interest in a topic related to but not part of the regular curriculum may be brought together to work on projects.

Still another purpose for organizing small groups is to bring together students who wish to carry out an experiment, plan a field trip, prepare a mathematics exhibit, or get ready for a mathematics contest.

Finally, at nearly every level there are small groups of slow learners who need to be brought together to go over more slowly and deliberately material that was presented earlier in large group sessions.

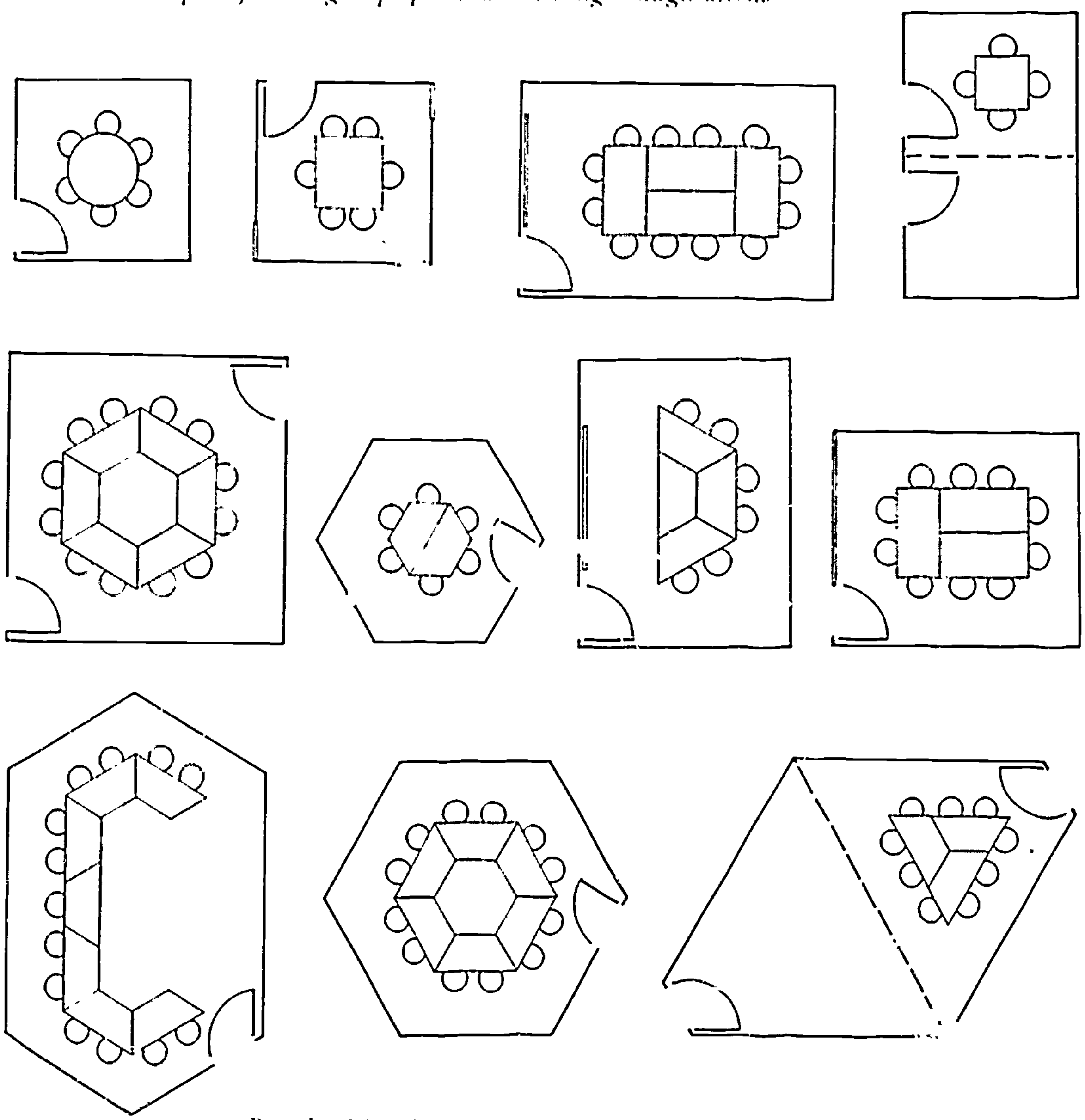
The role of the teacher who has responsibility for a small group will vary with the ability and maturity of the group and the objectives for which the group is organized. The teacher can be in charge; he can participate as an active member of the group; he can be an observer; he can work with half the group and let students in the other half direct themselves; or he can serve as a consultant. With younger students the teacher will usually need to give form to the work undertaken by the group by providing partially completed materials, selecting and making available manipulative devices, and asking questions that stimulate thinking. Sometimes the teacher and a few good students can serve as tutors for members of a small group. And sometimes the teacher serves a small group best by being absent. With no authority present to settle differences students are forced to rely on themselves in checking solutions to problems.

formulating generalizations, determining the validity of proofs, and so on.

The frequency with which a small group should meet will vary with the ability and maturity of students and with the purpose for which the group is organized. In the primary grades, small groups organized to learn concepts and skills may need to meet every day for twenty minutes. In the high school each student should have the opportunity to participate in a small group about twice each week in sessions lasting at least forty minutes.

An effort should be made to convene small groups in appropriate spaces. Some possible

FIGURE 2.7. *Shapes of small group spaces and seating configurations*



*Reproduced from* The School Library *by permission of the publisher, Educational Facilities Laboratories*

areas are teachers' offices (when not in use), a table with chairs in a corner of the cafeteria or on a patio, an alcove in a standard-sized classroom or library, classroom adjunct spaces, corridor benches, or small areas meant for relaxation. Students should be able to sit facing one another, in order to enhance interaction and exchange of ideas. Ideally, of course, small group spaces should be in proximity to the mathematics resource center so that students can have easy access to textbooks and other printed materials, films, tapes, and models. A specialized small group space should be furnished with tables, a mobile mathematics unit or cart, and enough chairs for twelve to fifteen students, the teacher, and possibly a guest or two. There should be plenty of chalkboard space, either portable or fixed. Small group spaces should measure about 200 to 250 square feet and should be reasonably soundproofed, well lighted, and ventilated for student comfort. A standard-sized classroom can be recommissioned to accommodate two or three small groups by installing sliding or folding partitions.

### *Mathematics Laboratory*

The mathematics laboratory developed as a separate instructional arrangement through differentiation of instructional functions originally performed in the self-contained classroom, and these specialized laboratory functions have since proliferated so that it is no longer possible to accommodate them all in a single space. In view of this situation, several types of mathematics laboratory arrangements are described, along with appropriate spaces for each type.

The English Nuffield Project stresses laboratory learning in the primary grades through informal exploration (52; 53; 54). Listed below are some of the specific investigative procedures that characterize the project. Engaging in these can help the young student develop an interest in and an awareness of mathematical concepts.

1. Sorting and grouping objects
2. Pairing off the members of two sets of objects to determine if the sets match
3. Checking the fourness of four, the fiveness of five, and so on
4. Putting puzzles together
5. Measuring all sorts of objects
6. Filling spaces, pouring water or sand from a cup into various containers
7. Ordering containers according to capacity
8. Ordering children according to height or weight
9. Using a balance made of a coat hanger and two little pails to compare weights
10. Making graphs with blocks, buttons, chips, or picture repetitions to show comparisons
11. Experimenting with objects to determine relations such as which objects in a set are larger than, smaller than, or the same size as a given object
12. Using relations to establish sets of ordered pairs

    For example, the relation "will float on" determines the ordered pairs (ball, water) and (block, water), but for pairs like (nail, water) the relation is "will not float on."
13. Examining the properties of a relation

    Does it work if you say it backwards? This can lead to discussions of reflexive, symmetric, and transitive properties of relations.
14. Combining and partitioning sets of objects and relating these actions to addition and subtraction
15. Tiling a surface or making floor patterns using squares, rectangles, pentominoes, or other shapes
16. Measuring distances with a ruler or trundle wheel
17. Finding areas of surfaces by covering them with small blocks or tiles
18. Creating or discovering symmetry by folding napkins, paper, leaves, and so on
19. Playing mathematical games.

All these explorations are intended to be unstructured, and informal and to be carried on in an atmosphere that is reasonably free of direction. As students advance to the intermediate grades laboratory learning should be encouraged

to follow this sequence: carrying out investigations with models and dynamic devices, formulating hypotheses based on observations or measured data, testing the hypotheses by completing additional trials, and communicating the results. Because these learning experiences are often highlighted by exciting discussions among students and teachers, they should be carried on in small groups.

The facilities for a mathematics exploration laboratory should include adequate cupboard and shelf space to store an abundant supply of variously colored objects of many shapes and sizes, models and manipulative devices, simple measuring instruments, boxes, containers, games, and puzzles. All these materials should be readily accessible when needed and out of the way when not. There should be sinks with nearby table space, floor areas where the students can gather in small groups, a puzzle and game center, and an area for cutting and pasting.

A teacher at a midwestern university laboratory school wanted high school students to have an opportunity to design and make mathematical models, make and solve mathematical puzzles, organize collections of realia, prepare demonstrations for presentation before large and medium-sized groups, prepare exhibits for mathematics fairs, and do independent research projects. The teacher designed a separate facility

FIGURE 2.8. *A group of students at Mirror Lake Elementary School, Broward County, Florida, exploring number relationships using manipulative materials*

that included a craftsman's workbench: waist-high work counters with water, gas, and electrical outlets: carpenters' and machinists' hand tools and a cabinet to store them: and a large paper cutter. The total floor space of this facility is about half the size of a standard-sized classroom and will accommodate from seven to ten students at one time. The facility has the appearance of an arts and crafts shop and could be converted into one by adding metal and wood lathes, a solder-and-glue bench, and a paint hood.

A third kind of mathematics laboratory is one in which students learn to use measuring and drawing instruments. Laboratories of this kind involve instructional functions that are common to both mathematics and industrial arts. Some of the specific skills a student can learn in a measuring and drawing laboratory are listed below:

1. Measuring distances with a Rolatape, carpenter's rule, and engineer's tape
2. Reading electric, water, and gas meters
3. Using vernier calipers to obtain inside and outside measurements
4. Measuring the thickness of wire with a micrometer and with a wire gauge
5. Measuring weights using spring scales and balance scales
6. Estimating weights, volumes, and distances using an appropriate unit
7. Measuring angles with transits, sextants, and protractors
8. Calculating areas, volumes, inaccessible distances, and production costs based on actual measurements
9. Enlarging a drawing using a pantograph, a proportional divider, or printer's proportion slide rule or by the method of radiation
10. Using a percentage protractor to make circle graphs
11. Finding the areas of irregular shapes by counting squares
12. Cutting paper stock, lumber, and fabric to get the maximum usage

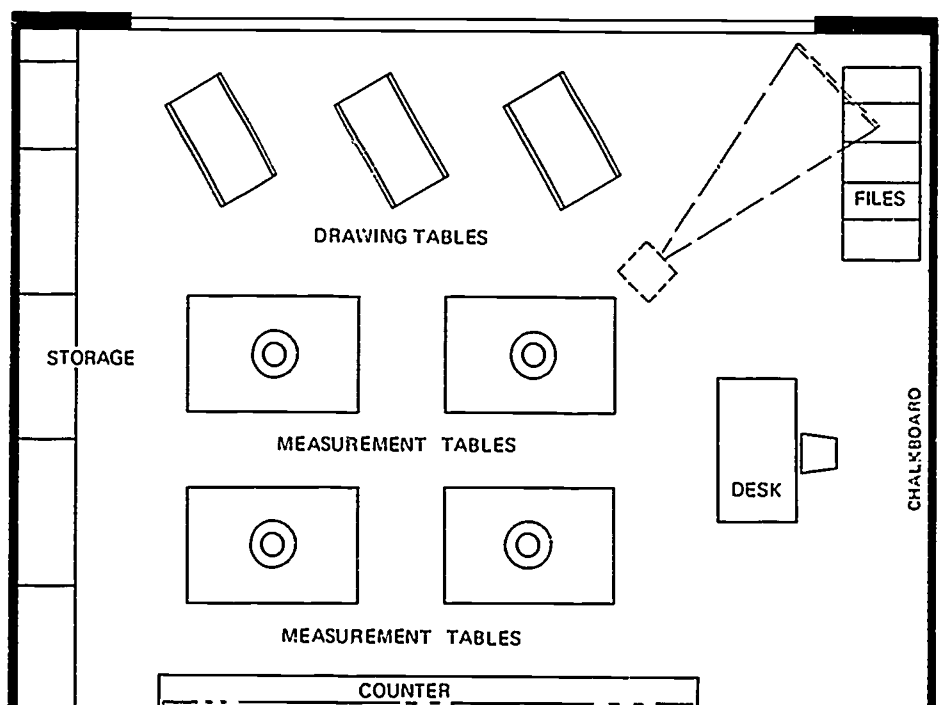



FIGURE 2.9. *Measuring and drawing laboratory*

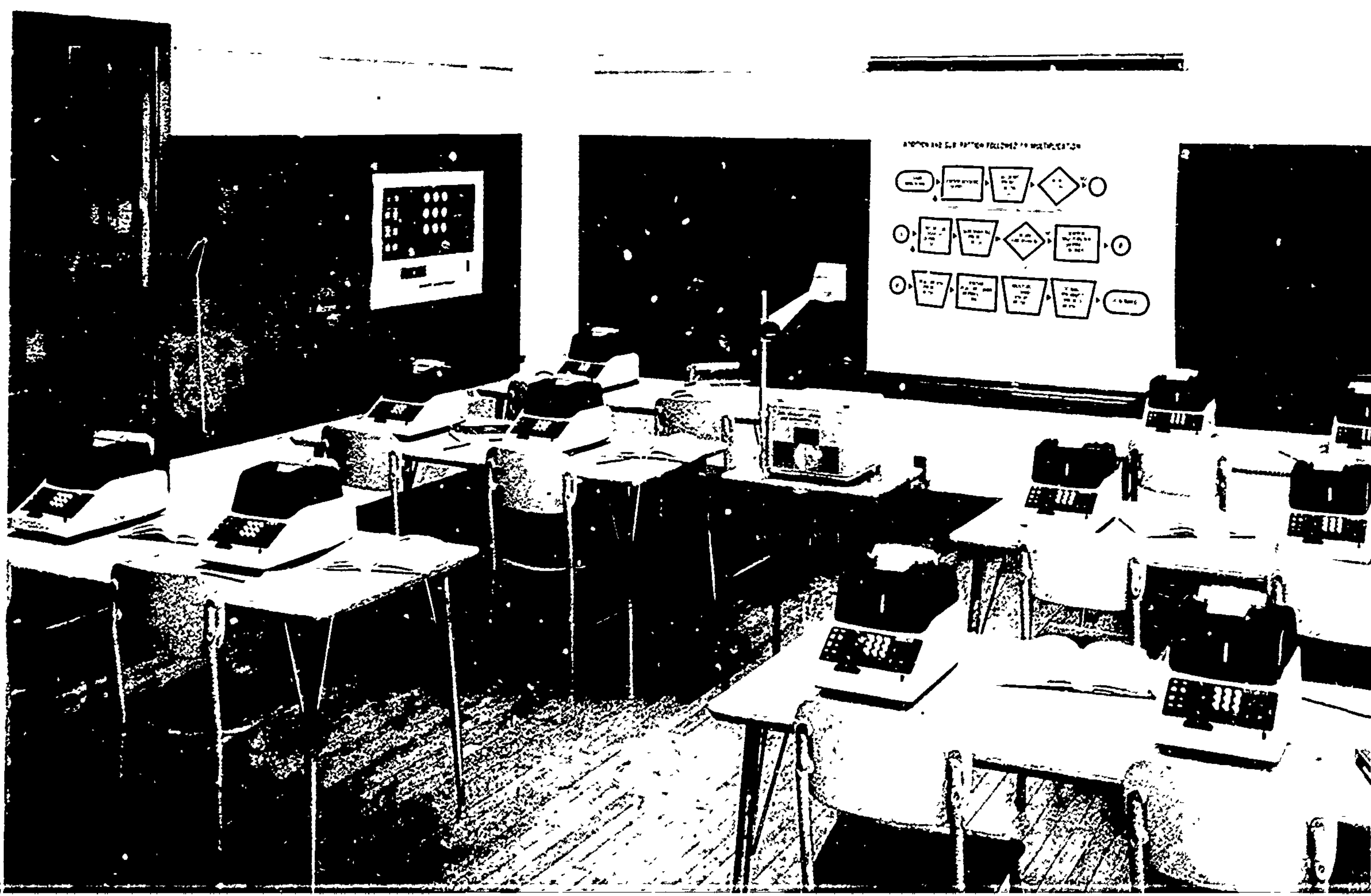
*Adapted from drawings by E. H. Sheldon Equipment Company* 24' x 32'

13. Using a scale ruler to make charts, graphs, diagrams, floor plans, and nomographs
14. Using try squares, miter squares, and sliding T bevels to lay out angles.

The size of a measuring and drawing laboratory depends on the number of students to be accommodated. A standard-sized classroom is large enough to enable twenty students to learn and to practice the skills indicated. The furnishings of the room should include floor-to-ceiling storage cabinets capable of holding all necessary instruments and field equipment, large tables for making drawings, a waist-high counter along one side wall for measuring and calculating, and a chalkboard along the length of another wall.

In recent years another kind of mathematics laboratory has appeared in some schools. General, remedial, consumer, business, and shop mathematics curricula in the secondary schools and the upper elementary grades have been restructured to include instructional objectives involving the use of ten-key electric calculators. In this type of laboratory the students make flow charts showing steps in addition, subtraction, multiplication, and division algorithms. They learn to calculate with decimals using calculators, and they check paper and pencil solutions to problems using the calculators. A calculator laboratory can be housed in a standard-sized classroom or equivalent space. Enough calculators should be available so that each student, or every two students, can have immediate access to a calculator. One version of a mathematics laboratory of this type is shown in Figure 2.10. Occasionally calculator laboratories are combined with measuring and drawing laboratories.

FIGURE 2.10. *Calculator laboratory*

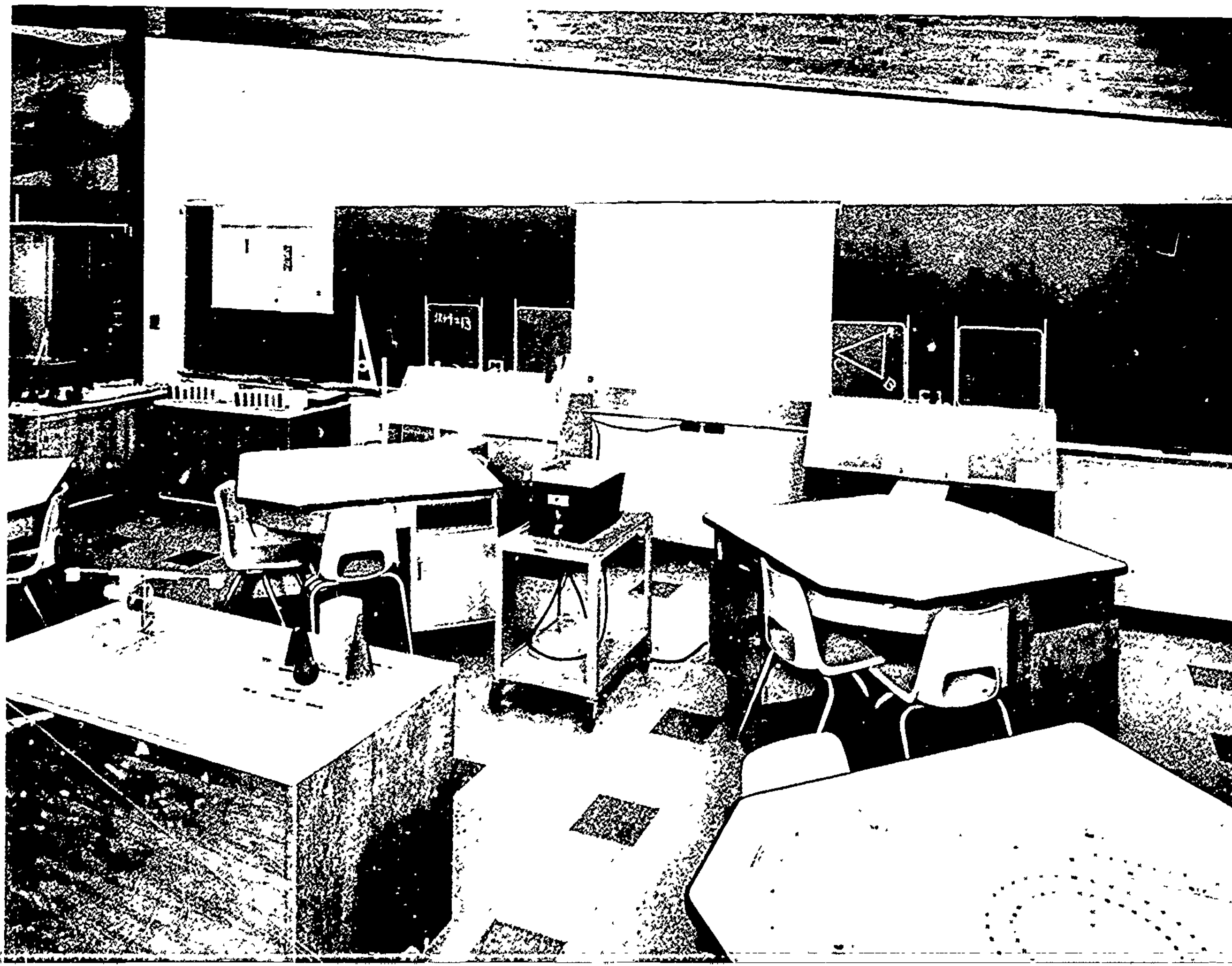


Perhaps the most prestigious and exciting mathematics laboratory is the school computer center. Emphasis is placed on having students solve problems and on reinforcing concepts learned earlier, using the computer. In addition the computer can be used for drill and practice: simulation of social, economic, and political systems: and games of strategy. Computer-facilitated instruction is treated in detail in Chapter 6. The school computer center is usually a separate enclosed space set aside to house one or more remote computer terminals and peripheral equipment. Some school computer centers also include desk-top computers. The size of the space that is needed for a computer center depends on the number of terminals to be housed. In most schools a space large enough to accommodate a small group is adequate.

This section on mathematics laboratories would not be complete without an excursus on a combination mathematics-science laboratory, which some advocate for both elementary and secondary schools.[1] It is argued that the integration of mathematics with science contributes to integration within the study of mathematics itself and that learning mathematics through its relationship to natural phenomena fulfills the two-fold educational need of most students to know mathematics and to know the real world. It is further argued that since the study of science depends on measurement and the theory of measurement is essentially mathematical, teachers of mathematics should not confine the study

1. The rationale and description of the mathematics and science laboratory presented in this section are adapted from an article by Eugene F. Peckman (60).

FIGURE 2.11. *Mathematics-science laboratory in the Golden Valley Middle School, Golden Valley, Minnesota. This laboratory is planned for use by fifth- and sixth-grade students.*



of measurement to space and time and the counting of money but should extend it to the measurement of mechanical, thermal, electrical, and chemical energy. Furthermore, formulas from science should not simply be brought in as facts to be juggled. Instead opportunities for dealing with quantity in science should follow this path—observation, measuring (and counting), manipulating, reasoning, generalizing, and finally expressing a relationship by a formula. When this occurs the mathematics laboratory and the science laboratory are one. According to this point of view experiences with science engender ideas; ideas are next verbalized and finally expressed as formulas. The following list suggests some of the steps students in a mathematics-science laboratory might pursue as part of this progression:

1. Obtaining data by counting and measuring
2. Organizing data in a systematic way
3. Formulating hypotheses based on recorded data
4. Testing hypotheses by performing actual laboratory tests (e.g., do the forces balance; does the metal expand as hypothesized?)
5. Making graphs to solve problems that cannot be solved experimentally (e.g., composition and resolution of forces and velocities)
6. Observing and making measurements of phenomena involving growth functions
7. Expressing relationships among physical phenomena by formulas.

Any standard-sized classroom is large enough for a mathematics-science laboratory and so are spaces half this size. A room used for this purpose should have work-top spaces, each capable of accommodating from one to five students working on a project, adequate storage for all equipment and apparatus that may be used, chalkboard, a large demonstration table for experiments involving large materials, bulletin board, cabinets to store unfinished projects, a map rail to hang charts, glass display cases for models, and security cabinets for expensive instruments and tools. All work-top spaces should be equipped


*Courtesy of E. H. Sheldon Equipment Company*

FIGURE 2.12
*A small group of elementary school students clustered in a mathematics-science laboratory*

with electrical outlets, gas and water, sink, and disposal for refuse.

## *Independent Study*

It is important that a student of mathematics have opportunities to study independently—to work by himself, at his own pace, in his own way. Mathematics is not a spectator sport. To learn it, one must play the game. In independent study the student has an opportunity to search for alternate ways of solving a problem and to check his solutions, undisturbed by the kibitzing of others. He has an opportunity to digest ideas presented to large groups or to pursue the ramifications of concepts, problems, and proofs discussed in a small group. He has a chance to investigate a topic in which he has a particular interest or to undertake a creative project that will last over an extended period of time. In independent study he has the opportunity to do

the delving, imagining, mulling, intuiting, and guessing that he alone can do for himself. Sometimes it will frustrate him. More often it will give him satisfaction. Occasionally it will exhilarate him.

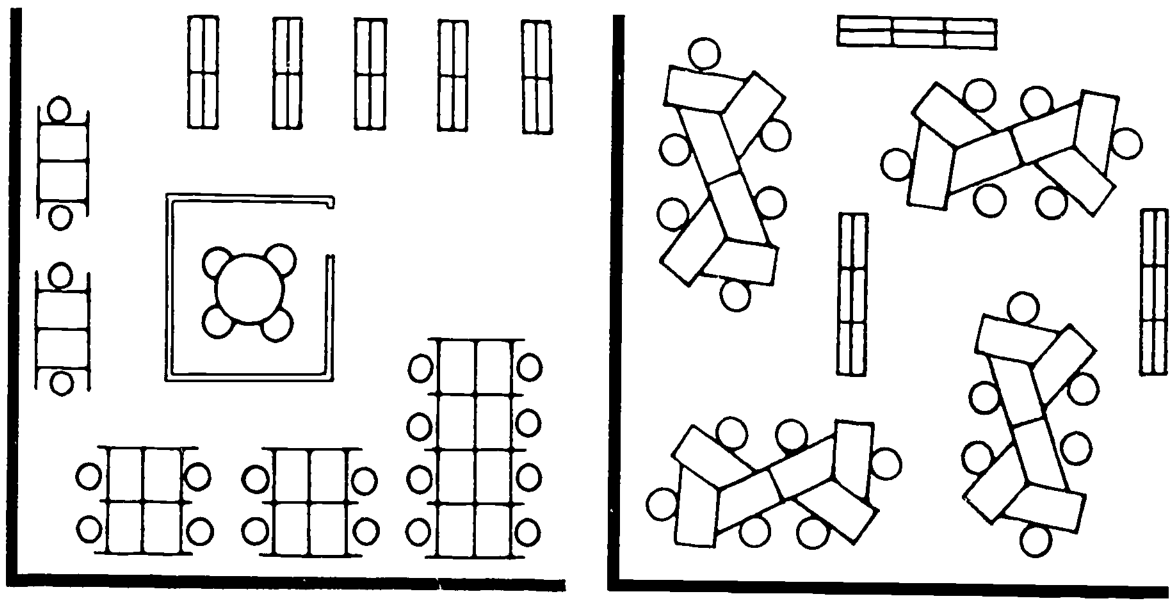
The student can study independently in a number of ways. He might read a textbook, pamphlet, or student journal. He might use audio cassette drill tapes, drill and practice filmstrips and film loops, drill and practice kits, and programs presented on electronic computer terminals to reinforce concepts learned earlier and to master computational skills. He can pursue a programed unit, either in a printed textbook or on a learning machine, or use an individual filmstrip viewer to learn new concepts or to work ahead of other students. He can use models, manipulative materials, and drawing and measuring instruments to do experiments, to determine relationships and to substantiate proofs. He can write a research paper, prepare a report on a great mathematician, or make a mathematical model or display.

For most students, independent study time should be incorporated in their individual programs. However, it should also be possible to arrange extra time for students who request it or who have been assigned a project requiring it. The amount of time that a student should devote to independent study will vary with his needs and interests. The elementary school student in need of drill and practice should probably not spend more than twenty minutes a day at independent study. On the other hand, a bright student may find it profitable to spend from one-third to one-half of the time he devotes to mathematics studying by himself. Successful students at all levels can be expected to spend progressively more time studying on their own as they mature. A few students, particularly slow learners, may never be able to study mathematics on their own; insisting that they do can cause them to develop a dislike for the subject.

What is the role of the teacher in independent study? This depends upon the ability and maturity of students. Young students will usually need considerable supervision to keep them engaged in learning because of their tendency to be distracted easily. Slow learners should have just enough assistance to help them keep up with the other students and gain self-confidence. For older and more mature students the teacher should be available to give advice and assistance but should not supervise too closely. The purpose of independent study should be to help students become increasingly self-reliant and self-directed.

The kinds of spaces in which a student can undertake independent study in mathematics will vary with the way in which he intends to study, using his own choice of instructional aids. He needs quiet spaces where he can concentrate. He needs spaces where he can view films and filmstrips and listen to recordings. Ideally, these types of spaces should be combined in a mathematics resource center. The student also needs to have access to the equipment and materials in the various mathematics laboratories described earlier. To be useful for independent study, the laboratories should be enclosed and in near proximity to the resource center.

Typically, the quiet area of a resource center should contain study carrels in which students can think, read, write, listen to tape recordings (using earphones), view film cartridges, and consult briefly with fellow students. Some of the carrels should be equipped with listening and viewing devices. Carpeting is a must in the quiet area, and it is advisable in other areas. There should be shelving, book trucks and display equipment to accommodate and display books and pamphlets on supplementary and enrichment topics, programed textbooks, independent study packages, and periodicals. There should be materials for students needing reteaching, for accelerated students working on advanced topics and projects, and for students engaged in exploring ideas (browsing). Small collections of materials on special topics should be on separate shelves. The use of the materials described can be encouraged by displays of new purchases, bulletin board notices, and exhibits of special collections. If books and other

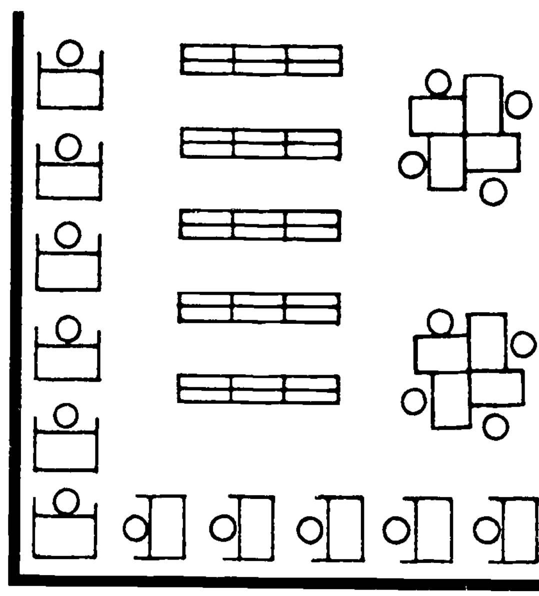


materials are in sight and available, students will be inclined to use them or at least peruse them.

Besides the quiet area, the mathematics resource center should have an area devoted to audio visual resources. There should be projectors for films, filmstrips, and slides, videotape players, and disc players. Students using sound equipment should use earphones so as not to disturb students engaged in other pursuits. This section of the resource center is not an area in which library silence should be expected. However, if functions normal to this area are to be carried on, talking and other sound must be subdued.

Specialized mathematics resource centers of the kind described are more common in secondary schools than elementary schools; however, there are elementary schools that do have spaces of this kind. In elementary schools the mathematics resource center is usually part of one large resource center (sometimes called a learning center) that serves two or more or even all subject areas.

Quiet spaces for independent study can also be improvised outside the mathematics resource center. These should be individual niches away from traffic lanes, such as a small alcove, a



*Adapted from drawings in* The School Library; *printed by permission of the publisher, Educational Facilities Laboratories*

FIGURE 2.13. *Three arrangements of study carrels and bookcases in independent study areas*

screened-off corner of a standard-sized classroom or adjunct space, or even an empty broom closet. It deserves mention that today's concept of independent study spaces originated at Melbourne High School (Florida) in 1957, where the first independent study space was a custodian's broom closet (8, p. 53). This space accommodated six students per day, one student during each of six periods. Where such spaces do not exist and cannot be improvised, study carrels should be provided to ensure privacy. These can be placed in corridors, in lobbies, in the cafeteria, or in special rooms set aside for them (Figure 2.14). Students using these quiet spaces should have access to as many resource materials as possible, either by proximity to the resource center or through dialing access to materials located elsewhere in the school building or even outside the building. Resource materials can also be stored and displayed on book trucks or rolling resource centers.

## *Tutoring*

Tutoring focuses on the individual. As tutor, the teacher engages in direct face-to-face dialogue with a single student. Tutoring may involve a lengthy one-to-one instruction session, or it can be a student's questioning look answered by a teacher's confirming nod. Usually it is something in between.

Tutoring may grow out of the teacher's supervisory role in independent study. Thus the teacher can diagnose difficulties in the setting and at the point at which they occur, lead the student out of misconceptions, redirect him, and finally spur him on to try again to work by himself.

Another tutorial function is to test a student individually, and perhaps orally. Oral testing is especially helpful if the student has written a "bad" test, for the teacher can adjust the questions so that he can tell what the student actually can do. Doing this can also help to reestablish the student's self-confidence; he may fear that since most of his test answers were wrong, he knows nothing.

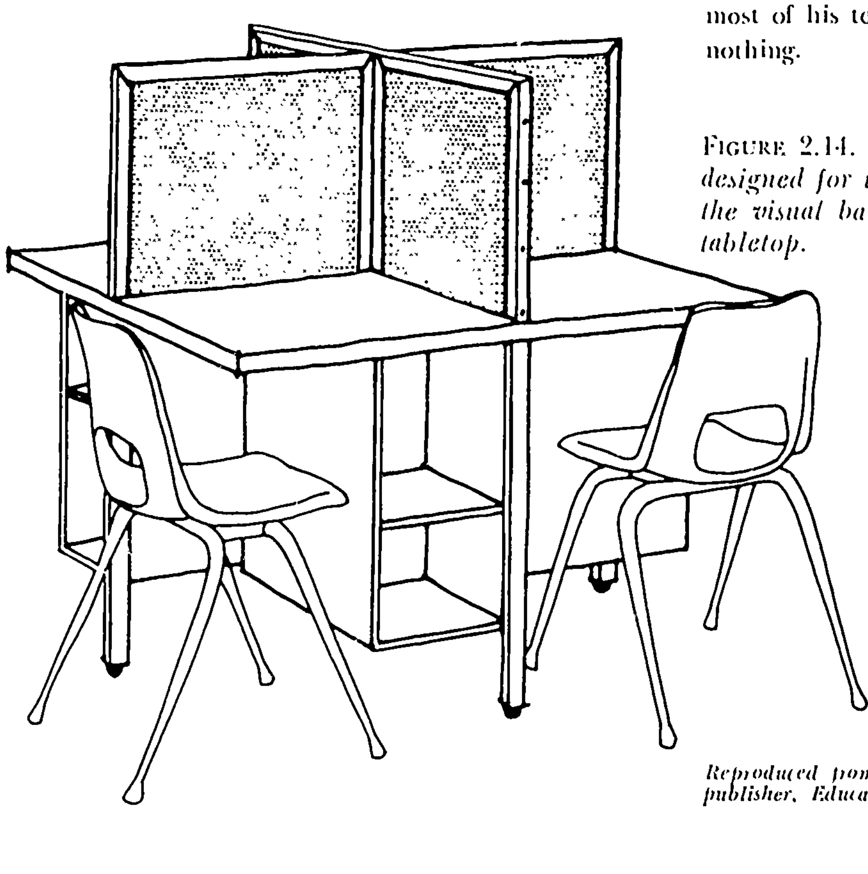


FIGURE 2.14. *A cluster of four study carrels designed for use in a cafeteria. At lunchtime the visual barriers can be lowered into the tabletop.*

*Reproduced from* Middle Schools *by permission of the publisher, Educational Facilities Laboratories*

The teacher may find it necessary to reteach the student concepts and skills that he has learned earlier and forgotten or that he should have learned earlier but failed to comprehend. Much tutoring does involve such reteaching, but tutoring should never be allowed to degenerate into monotonous drill. Sometimes the teacher will need to help a student learn or relearn the steps of an algorithm by parsimonious prompting, or help him analyze a statement to be proved by asking probing questions or giving hints to stimulate the production of hunches. An ingenious tutor will spontaneously generate the dialogue needed to bring the tutee to the "I see!" stage. Following are some techniques that have been used by successful tutors:

1. Varying instruction so that the student does not get restless; breaking up the action
2. Explaining things several times instead of just once, and explaining in several different ways; providing just the right amount of repetition
3. Playing games with the student (suggested games: card, board, and spinner games; battleship; dominoes; simulating a shopping tour)
4. Making up games with problem cards
5. Playing "correct the tutor"
   The tutor should make the same mistake several times in a row, so that the student gets the idea.
6. Asking the student questions about himself that involve numbers
7. Having the student try to guess the answers to problems before solving
8. Working problems with the student; taking turns doing successive steps
9. Making up problems, using a menu or sales advertisement
10. Writing answers to the problems that the student makes up
11. Reading problems to the student and having the student read problems to the tutor
12. Having the student read directions and then checking to see that he can follow them
13. Helping the student to cut down the size of a problem, to get started on it rationally rather than being overwhelmed by it.

The student will learn more if the tutoring sessions are pleasant, meaningful, and efficient. There are several ways the teacher can arrange this. If a series of sessions is necessary, he may wish to prepare a folder containing a record of the student's progress and also some notes pertaining to his interests. He should begin by working on the student's most important problems, without wearing him out by trying to find every weakness. He should be attentive to what the student says and comment on his successful efforts. Finally, to help the student leave the session with a good feeling, the last task should be something that is easy for him.

Tutoring is not just for the student who is having trouble with basic content. It can also benefit the advanced student who is studying independently. Such a tutoring session can develop into a joint attempt by teacher and student to produce a proof with both contributing and weighing one another's arguments. More often, the tutoring session for the advanced student is a time during which he can obtain advice. He may need suggestions for sources of information or possible ways of resolving difficulties he has encountered. He may have become so engrossed in the narrow aspects of a topic that he has lost sight of important relationships that he must consider in order to progress; the teacher can help him broaden his approach. He may need, finally, nothing more than encouragement from his tutor. Tutoring can provide the opportunity for guidance necessary to keep independent study from becoming aimless or bogged down. The dialogue can be a source of inspiration to the student as well as an aid in teaching him to evaluate his own work.

The role of tutor is not limited to certified teachers. Others, such as college students, parents, and community adults can serve as tutors.

During each period of the day A and B+ students can be available in the independent study area to provide help. Students studying the same content can also tutor each other. There are studies to indicate that tutoring of one student by another student often improves the achievement of both.

The amount of tutoring a student needs will vary with the individual, but it should never be allowed to become the steady instructional diet of any student. The teacher should aim at making the student confident and self-reliant, capable of studying by himself and solving problems on his own.

The kinds of spaces needed for tutoring are as varied as students' needs for tutoring. Tutoring can take place at either the teacher's or student's desk, at the student's work space in a separate mathematics laboratory, or at a computer terminal. Occasionally a corner of a space designed for small group meetings will do. For example, a teacher may find that one or two students of the group need individual help, so he may spend some time with them in a corner while the other members of the group continue nearby, working singly or cooperatively. Usually some space more private than these works better, perhaps the teacher's office or a small conference room. Since many of a student's needs for tutoring occur during independent study, a special place close to a cluster of study carrels where the student and teacher can confer is desirable. B. Frank Brown describes a "stretched" carrel, which he has found useful:

> The best description of the stretched carrel is a student carrel which has been lengthened, widened, deepened, and heightened. The desk top dimensions measure six feet in length and two feet in width. This provides ample work space for the activities in which the independent study preceptor engages. Located above the desk top space is a considerable quantity of shelf space that provides storage for teaching aids and books. Half sized filing cabinets (two drawers) fit neatly under each end of the enlarged desk top, affording convenient filing space. A swivel chair completes the arrangement. [8, pp. 121–122]

These blown-up carrels were originally designed for staff members at Melbourne High School, a school that pioneered in the idea of independent study.

## TEAM TEACHING

The technique of team teaching is useful in carrying out a strategy of individualizing instruction by customizing a combination of arrangements for each student. In its simplest form team teaching is a technique whereby two or more teachers use their respective competencies to plan instruction and to teach and evaluate the equivalent of two or more medium-sized groups.

The rationale underlying team teaching is that making each teacher's special competencies available to as many students as possible improves the overall quality of instruction and increases individualization.

Thus, each teacher on a team is encouraged to do "his thing"—teach subject matter in which he has his greatest competence and to project himself in the role (large group presenter, independent study supervisor, laboratory consultant, small group discussion catalyst, or tutor) for which he has his greatest aptitude.

It is argued that team teaching results in improved instruction because it increases input in lesson development; eliminates confusion and boredom that stem from repetition of presentations to several medium-sized groups ("Have I told you this?"); brings about spontaneous combining of the team members' accumulations of supplementary textbooks, programed units, models, manipulative devices, and so on; and provides a base for focusing attention on critical instructional problems.

Individualization is increased because teachers on a team can make up combinations of large group, small group, independent study, laboratory, and tutorial arrangements; use to advantage the specialized tutorial skills of the team; increase availability of team members for giving individual assistance; and use total evaluation techniques.

The single most important factor in the success of any team teaching arrangement is the recognition of each teacher's special talents by all teachers on the team. From time to time teachers assume different roles.

The success of a team teaching arrangement is enhanced by the availability of assistance from professional and para-professional personnel. Needed from time to time is the cooperation of librarians, counselors, and audiovisual and television supervisors. Resource personnel such as college professors, representatives from the Internal Revenue Service, architects, actuaries, and consultants from the computer industry can make contributions through occasional lectures.

Teacher interns, student teachers, student assistants, and teacher aides can assist with small group problem solving sessions and with monitoring independent study. Clerks can be assigned to such tasks as typing and duplicating tests and bibliographies, assuming custody of equipment and other materials, collecting assignments, correcting tests, and recording grades. A graphic arts technician to prepare visuals, charts, and other illustrative materials and a technician to operate machines will help round out a team teaching effort.

Composition of teaching teams varies widely among schools. The bases of cooperation usually involve decisions pertaining to the total number of students and teachers to be involved, the division of responsibility, the amount of student-contact time needed, and the nature of the curriculum (i.e., main track, accelerated, or general).

Various patterns exist. In the high school, teaching teams can be organized for all students registered for a given subject, all students in one grade studying the same subject, all students within a certain ability or achievement range, or all students interested in a special topic. The examples that follow are illustrative.

1. SINGLE SUBJECT TEAM—NINTH GRADE ALGEBRA

   Four teachers team together to teach elementary algebra to 120 students in a fifty-five minute period each day. Students and teachers divide their time among large group instruction, small group instruction, mathematics laboratory work, and independent study. For example, on one day all students begin the period by viewing an eighteen-minute film. Thereafter, 70 students receive a lecture for the remainder of the period. One group of 12 students meets with one teacher in a small group session; 23 students go to a mathematics laboratory space to work on projects; and the remaining 15 students go to an independent study space.

2. THREE PERFORMANCE LEVELS—EIGHTH-GRADE MATHEMATICS

   Three teachers begin the school year with three medium-sized eighth-grade mathematics groups that meet during the same block of time. After four weeks the students are reassigned to three different performance level groups, and thereafter they are regrouped periodically as indicated by individual progress.

3. AD HOC TEAM—TENTH-GRADE GEOMETRY

   Three teachers of tenth-grade geometry team together to show films and give demonstrations involving special equipment such as a logic trainer and large-volume fillable models. Study guides and postviewing tests are produced as a cooperative effort by team members.

4. INTERDISCIPLINARY TEAM—MATHEMATICS-PHYSICS

   One mathematics teacher and one physics teacher team together to teach advanced algebra and physics to two medium-sized groups. Both groups meet during the same block of time. The purpose of the arrangement is to make possible the use of time-shared computer terminals and to work together in small group problem solving sessions.

5. CONTENT SPECIALIST TEAM—SECONDARY SCHOOL PRACTICAL ARTS

   One mathematics teacher, one industrial

arts teacher, and one home economics teacher team together to teach a semester subject in practical arts to three medium-sized groups. The mathematics teacher teaches the basic content on measurement to all students in large group sessions and special interest content to each medium-sized group on request.

6. CONTENT SPECIALIST TEAMS—ELEMENTARY SCHOOL.

This is a descripton of a team teaching strategy structured by Joan Kirkpatrick, Westbrook Elementary School, Edmonton, Alberta.

Eight teachers were given complete responsibility for planning, teaching, and evaluating the total instructional program of eight medium-sized graded elementary school groups. There were three fourth-, three fifth-, and two sixth-grade heterogeneous groups. Each of these groups was identified with one home base teacher.

Each teacher provided large group instruction for a subset of the students from the eight medium-sized groups in physical education, music, art, or science for two hours each week. In addition the teachers for each grade served on a grade-level team for the core subjects, reading and language arts, social studies, and mathematics.

Each team employed a scheme of specialization in which one teacher served as the mathematics specialist, one as the reading and language arts specialist, and one as the social studies specialist. That is, each teacher was the specialist for at least one core subject. The specialist took the lead in planning units and lessons, conducting team meetings, and being the whip for his subject or subjects. However, each teacher on the team taught mathematics as well as all other core subjects.

Mathematics for all three grades was taught during the same block of time in the school's big room (similar to the pods discussed in connection with open space schools later in this chapter).

For mathematics the heterogeneous medium-sized home base groups for each grade were initially regrouped homogeneously into four ability subgroups, but in practice students shifted from one subgroup to another for specific lessons and then rejoined their own mathematics subgroup.

The superior students for each grade level worked by themselves much of the time, either individually or together. After each unit the subgroups in each grade were tested and re-grouped. Groups fluctuated, with a small hard core at each end of the performance range. It was decided to give the top group enrichment instead of permitting free acceleration so as to keep all groups on the same unit at all times. This decision was made in order to make it possible for students to move in or out of any performance group at any time.

The mathematics program at each grade level was planned and taught by units. Doing this facilitated movement of students from one performance group to another. Each unit had a pretest so that students could be grouped in accordance with their performance. The introduction to a unit was often presented to a total grade level group, that is, a large group consisting of either two or three medium-sized groups.

The description that follows tells how a fifth-grade unit on fractions was handled.

Teachers on the fifth-grade team planned the units, pretested the students, and put them into three performance groups. The sequence of lessons in the unit was then timetabled. Next the plan was discussed with the sixth-grade team, and any sixth-grade student needing work on fractions was sent to work with the fifth-grade students whenever the necessary skills or topics were being presented.

## SCHEDULING

To ensure that each student will have the advantage of receiving instruction in the various instructional arrangements prescribed for him, his program must be systematically planned. This involves scheduling.

Scheduling is a scheme of bringing together the various components of instruction: students, teachers, time, spaces, textbooks, models, computers, and so on. Ideally, it should be focused on the educational needs of each student, on placing him with teachers who are right for him in the right spaces for the right periods of time and with the right frequency. In practice, however, there are many compromises because of spatial constraints, scarcities of teacher time, and a lack of adequate materials. Thus scheduling often focuses on subjects, in the belief that the needs of each student can be met by allowing him his own selection of subjects from among those in the curriculum.

Three types of scheduling patterns have evolved: scheduling based on daily, uniform length of periods (referred to as conventional scheduling), modifications of the conventional scheduling pattern, and modular flexible scheduling.

### *Conventional Scheduling*

The conventional schedule has been in common use for some time. In the high school it usually consists of a daily routine of six fifty-five minute periods, with the same sequence of classes repeated every day. Students enrolled for a particular subject meet in the same group, usually a medium-sized group of twenty-five to thirty-five, during the same period every day. Many elementary schools follow a similar daily uniform period-length pattern, except that periods are shorter because the elementary school must divide time among nine subjects. Also, there may be variations in period length from grade to grade in such subjects as reading and mathematics.

### *Modifications of the Conventional Scheduling Pattern*

Although the conventional scheduling pattern is extremely rigid, a slight change will give schools a more adaptable schedule without replacing the basic pattern. For example, if three or more medium-sized groups are scheduled for the same or related subjects during the same period, teachers can then regroup the students into three performance levels. This arrangement works well, provided that the performance level groups are continually re-formed as students progress.

A second alteration in the conventional scheduling pattern will give each medium-sized group one "long" period each week (14, p. 91). This can be accomplished by making an even exchange of periods on Mondays and Tuesdays or any other pair of days. For example, a class that normally meets during first period could be scheduled to meet during both first and second periods on Mondays and the class that meets during second period could meet during both first and second periods on Tuesdays. The same even exchange pattern would be followed by third and fourth period classes, and so on. During the remainder of the week all students would meet during their regular periods.

Sequence rotation is another variant of the conventional schedule (14, p. 94). This involves having a class that regularly meets during the first period meet during the first period as usual on Mondays, during the second period on Tuesdays, during the third period on Wednesdays, and so on. All by itself this scheme seems to have little purpose except to equalize distractions and interruptions associated with different periods and different days of the week.

A more fruitful modification of the conventional schedule combines variation of period length with ordinary sequence rotation. This scheme permits class groups to be scheduled in short, medium, and long periods. While this pattern is somewhat rigid because of its uniformity it does make possible scheduling a variety of instructional arrangements.

| Minutes per Period | M | Tu | W | Th | F |
|---|---|---|---|---|---|
| 40 | 1 | 6 | 5 | 4 | 3 |
| 40 | 2 | 1 | 6 | 5 | 4 |
| 60 | 3 | 2 | 1 | 6 | 5 |
| 60 | 4 | 3 | 2 | 1 | 6 |
| 80 | 5 | 4 | 3 | 2 | 1 |
| 80 | 6 | 5 | 4 | 3 | 2 |

FIGURE 2.15
*A schedule that combines varied length of periods with sequence rotation. Each day has six periods: two forty-minute periods, two sixty-minute periods, and two eighty-minute periods. On Monday the sequence of six periods is the same as in a conventional schedule. On Tuesday, a class that met during the sixth period on Monday meets during first period; a class that met during the first period on Monday meets during the second period, and so on.*

### *Modular Flexible Scheduling*

To realize the full advantages of the team teaching technique and the separation, accentuation, and expansion of instructional functions underlying the organization of different instructional arrangements, greater flexibility in scheduling is needed than can be attained using the conventional schedule or modifications of it. Some schools achieve such flexibility through modular flexible scheduling.

The basic concept of modular flexible scheduling is that group size and session length can be varied, subject by subject, to suit instructional objectives. Under a modular flexible schedule students studying a particular subject will meet at different times in large groups, medium-sized groups, small groups, in the laboratory, for independent study, or with a tutor. It is a scheme that allows groups to meet with varying frequency and for varying lengths of time. Flexible scheduling is an innovation that generally affects the whole school. Changes in scheduling one subject must be compensated for by changes in scheduling another subject.

To build a modular flexible schedule a time group-size scheduling unit must be defined. The time module of this unit is usually chosen to be the number of minutes of the shortest instructional session in the scheduling time cycle. For convenience the group-size module can be defined as the average number of students in the smallest group assigned to any instructional session during this time cycle. A practical choice of a scheduling unit is one with a period length of twenty minutes and a group size of twelve students.

A critical part of scheduling a subject is determining a scheduling structure for the subject—that is, determining under which instructional arrangements the subject should be offered, the length of sessions under each arrangement (stated as a multiple of the time module), frequency of sessions under each arrangement, optimal group size under each arrangement (stated as a multiple of the group-size module), and the number of teachers and teacher aides needed for each session. Time to be spent in independent study and tutoring is not considered part of the scheduling structure for a subject. The person making these decisions should examine the objectives of the subject to be scheduled in the light of what he knows about the instructional functions that can be performed in the different arrangements.

Suppose, for example, that 192 students are to be scheduled for geometry and that eleven twenty-minute time modules are to be available in a weekly time cycle. One possible scheduling structure for this subject is displayed in Figure 2.16.

Each segment on the horizontal scale represents one group-size module of twelve students, and each segment on the vertical scale represents one time module of twenty minutes. Each square represents one time-group-size scheduling unit. Under the specifications chosen, each student can be scheduled in geometry for eleven time modules in large group, small group, and laboratory arrangements. Time to be spent in independent study and tutoring is determined by either the teacher or the student as the need arises.[7]

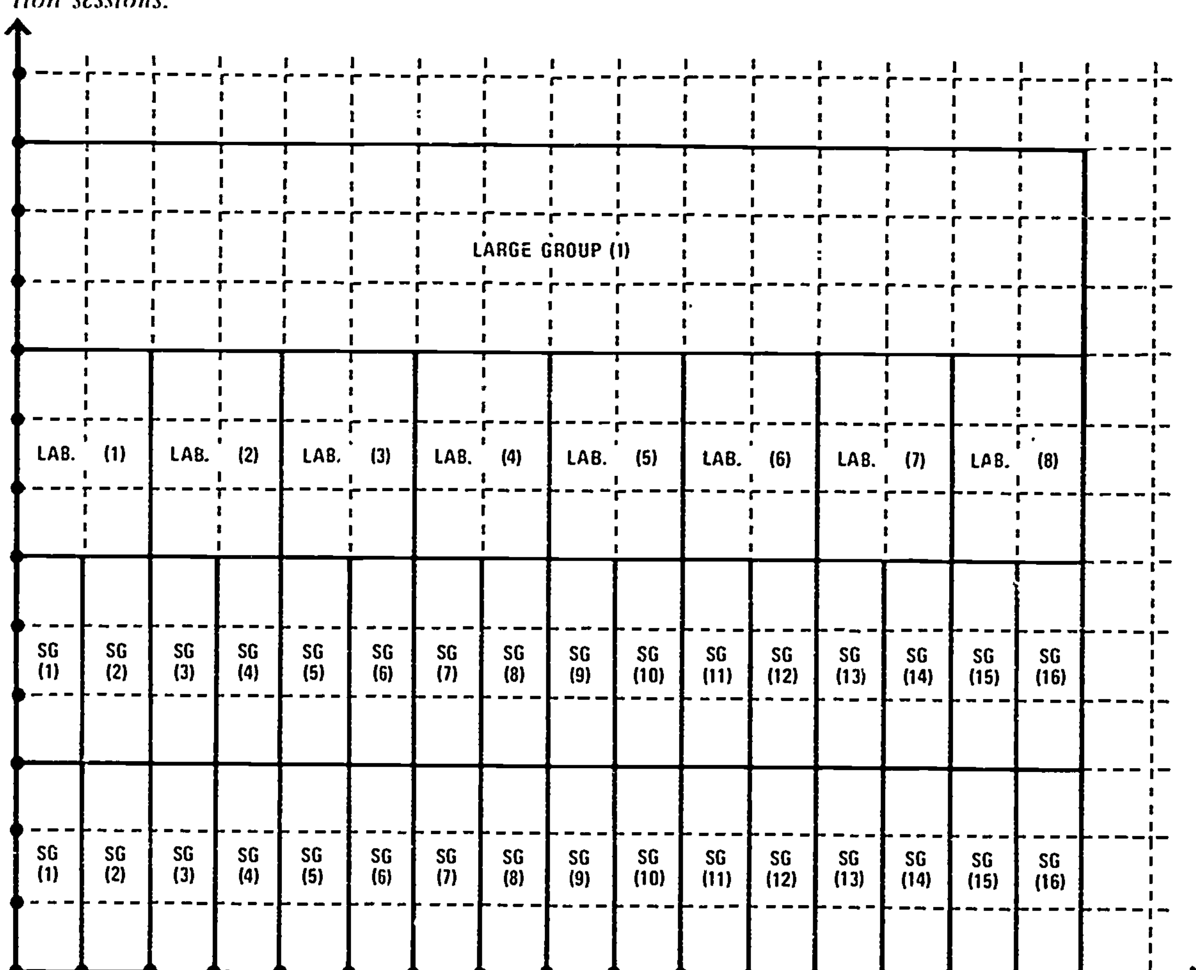
When the scheduling structures for all subjects have been defined, a school master schedule cir-

cumscribing all subject offerings is synthesized. The master schedule can be mapped on a large sheet of squared cross-section paper and each school subject plotted (scheduled) as part of the total program.

Needless to say, this is a very complicated procedure if done by hand. Making use of a computer considerably reduces the tedium and in addition enhances the prospect of designing and implementing a truly flexible schedule.

Proponents of flexible scheduling believe that it increases the effectiveness of team teaching and thereby enhances individualization of instruction. They claim that flexible scheduling promotes greater interest in independent study, gives teachers more time for tutoring, and ultimately leads to reorganization of content into sequences of performance levels.

FIGURE 2.16
*Scheduling structure for 192 students in geometry. The structure consists of one large group section of 192 students meeting forty minutes per week in one session; eight laboratory sections of 24 students with each section meeting for one sixty-minute session per week; and sixteen small groups of 12 students each meeting for sixty minute periods twice weekly. A student can be scheduled in geometry for eleven twenty-minute modules per week: forty minutes in large group instruction, sixty minutes for laboratory work, and two sixty-minute small group instruction sessions.*



| Time | MOD | Monday | MOD | Tuesday | MOD | Wednesday | MOD | Thursday | MOD | Friday |
|---|---|---|---|---|---|---|---|---|---|---|
| 8:30-8:40 | H | Home Room | H | Home Room | H | Home Room | H | Home Room | H | Home Room |
| 8:40-9:00 | 1 | Music (MG) | 1 | Work on my Science Experiment (IS) | 1 | Music (MG) | 1 | Help First Grade with Reading | 1 | Meet with Advisor |
| 9:00-9:20 | 2 | | 2 | | 2 | | 2 | | 2 | Listen to Resource Person (LG) |
| 9:20-9:40 | 3 | Mathematics (LG) | 3 | | 3 | Listen to Spelling Tapes (IS) | 3 | Language (SG) | 3 | |
| 9:40-10:00 | 4 | | 4 | Physical Education (Free Play) | 4 | | 4 | | 4 | View Filmstrips (IS) |
| 10:00-10:20 | 5 | Language (SG) | 5 | | 5 | Work on my Puppet Story for Reading (IS) | 5 | Guidance and Counseling (SG) | 5 | Mathematics (SG) |
| 10:20-10:40 | 6 | | 6 | Work on my Social Studies (IS) | 6 | | 6 | | 6 | |
| 10:40-11:00 | 7 | Physical Education (LG) | 7 | | 7 | Physical Education (LG) | 7 | Work on my Social Studies (IS) | 7 | Do my Mathematics (IS) |
| 11:00-11:20 | 8 | | 8 | Music Appreciation (LG) | 8 | | 8 | | 8 | |
| 11:20-11:40 | 9 | Science (SG) | 9 | Science (SG) | 9 | Meet with my Social Studies Project Committee (SG) | 9 | Social Studies (SG) | 9 | Social Studies (SG) |
| 11:40-12:00 | 10 | | 10 | | 10 | | 10 | | 10 | |
| 12:00-12:20 | 11 | Lunch | 11 | Lunch | 11 | Lunch | 11 | Lunch | 11 | Lunch |
| 12:20-12:40 | 12 | | 12 | | 12 | | 12 | | 12 | |
| 12:40-1:00 | 13 | Library Study (IS) | 13 | Reading (SG) | 13 | Mathematics (LG) | 13 | Reading (SG) | 13 | Language (SG) |
| 1:00-1:20 | 14 | | 14 | | 14 | | 14 | | 14 | |
| 1:20-1:40 | 15 | Reading (SG) | 15 | Mathematics (SG) | 15 | Language (SG) | 15 | Do my Reading (IS) | 15 | Browse in the Resource Center (IS) |
| 1:40-2:00 | 16 | | 16 | | 16 | | 16 | | 16 | |
| 2:00-2:20 | 17 | Do my Reading (IS) | 17 | Do my Mathematics (IS) | 17 | Listen to Music Records (IS) | 17 | Science (SG) | 17 | Reading (SG) |
| 2:20-2:40 | 18 | | 18 | | 18 | Art (MG) | 18 | | 18 | |
| 2:40-3:00 | 19 | Social Studies (SG) | 19 | Work on my Mathematics with Teacher Help (IS) | 19 | | 19 | Listen to Spelling Tapes (IS) | 19 | Art (Art Lab.) |
| 3:00-3:20 | 20 | | 20 | | 20 | | 20 | Meet with Advisor | 20 | |

FIGURE 2.17

*An example of a sixth-grade student's schedule. The white areas represent structured time and the shaded areas represent independent study time that the student has scheduled for himself.*

| Time | MOD | Monday | MOD | Tuesday | MOD | Wednesday | MOD | Thursday | MOD | Friday |
|---|---|---|---|---|---|---|---|---|---|---|
| 8:00-8:20 | 1 | English (SG) | 1 | English (SG) | 1 | Biology (Biology Lab.) | 1 | English (SG) | 1 | Geometry (Math Lab.) |
| 8:20-8:40 | 2 | | 2 | | 2 | | 2 | | 2 | |
| 8:40-9:00 | 3 | Spanish (Language Lab.) | 3 | Geometry (LG) | 3 | Driver's Education (LG) | 3 | Geometry (LG) | 3 | |
| 9:00-9:20 | 4 | Physical Education (MG) | 4 | | 4 | | 4 | | 4 | Study Driver's Manual (IS) |
| 9:20-9:40 | 5 | | 5 | Biology (LG) | 5 | Driver's Education (SG) | 5 | Biology (LG) | 5 | Driver's Education (SG) |
| 9:40-10:00 | 6 | | 6 | | 6 | | 6 | | 6 | |
| 10:00-10:20 | 7 | Biology (SG) | 7 | Physical Education (MG) | 7 | Biology (SG) | 7 | Spanish (LG) | 7 | Biology (SG) |
| 10:20-10:40 | 8 | | 8 | | 8 | | 8 | | 8 | |
| 10:40-11:00 | 9 | Spanish (Language Lab.) | 9 | | 9 | Spanish (Language Lab.) | 9 | Sophomore Seminar | 9 | Spanish (Language Lab.) |
| 11:00-11:20 | 10 | Spanish (SG) | 10 | Spanish (SG) | 10 | Spanish (SG) | 10 | | 10 | Spanish (SG) |
| 11:20-11:40 | 11 | | 11 | | 11 | | 11 | Work Out in Weight Room (IS) | 11 | |
| 11:40-12:00 | 12 | Lunch | 12 | Lunch | 12 | Physical Education (MG) | 12 | | 12 | Lunch |
| 12:00-12:20 | 13 | Study Geometry (IS) | 13 | Study Driver's Manual (IS) | 13 | | 13 | Lunch | 13 | Work on Biology Experiments (Biology Lab.) |
| 12:20-12:40 | 14 | World Studies (LG) | 14 | | 14 | | 14 | | 14 | |
| 12:40-1:00 | 15 | | 15 | Plan Wrestling Strategy (IS) | 15 | Lunch | 15 | Physical Education (MG) | 15 | |
| 1:00-1:20 | 16 | | 16 | | 16 | Study Geometry (IS) | 16 | | 16 | Study my English (IS) |
| 1:20-1:40 | 17 | | 17 | World Studies Project (IS) | 17 | | 17 | | 17 | |
| 1:40-2:00 | 18 | Geometry (SG) | 18 | | 18 | Geometry (SG) | 18 | Work on World Studies (IS) | 18 | English (LG) |
| 2:00-2:20 | 19 | | 19 | World Studies (SG) | 19 | | 19 | World Studies (SG) | 19 | |
| 2:20-2:40 | 20 | | 20 | | 20 | | 20 | | 20 | |
| 2:40-3:00 | 21 | English (IS) | 21 | | 21 | English (IS) | 21 | | 21 | |

FIGURE 2.18

*An example of a high school student's schedule. The white areas represent structured time. The shaded areas represent independent study time that the student has scheduled for himself.*

## COORDINATING FORM AND FUNCTION

This final section brings together three exemplary ideas for individualizing instruction, using different kinds of plant facilities. The first example is concerned with the possibility of modifying a conventional building to make it more functional for housing individualized programs. The second example contains an actual account of how an individualized mathematics program is carried on in a varied-sized space complex incorporated into the structural design of an elementary school. The third example is really a short discourse on the latest innovation in schoolhousing design, the open space school. Emphasis is given to the special capability of this design concept for accommodating an individualized program in mathematics.

### *Modifying Conventional Buildings*

Individualized programs are most easily organized in schools that have specialized spaces for large group instruction, small group instruction, independent study, laboratory work, and tutoring. However, such programs can also be organized in conventional buildings that have only standard-sized classrooms. Let us see how a conventional building made up of standard-sized classrooms can be recommissioned to accommodate an individualized program.

One standard-sized classroom can be separated into two or three small group spaces, each capable of holding from seven to twelve students. Room dividers can be installed to reduce noise and distraction, and individual deskchairs or two-chair tables arranged informally for the separate groups. A second room can be made usable for independent study by arranging deskchairs so that they face away from each other or toward outer walls. A third room can be outfitted as a mathematics laboratory as shown in Figure 2.19.

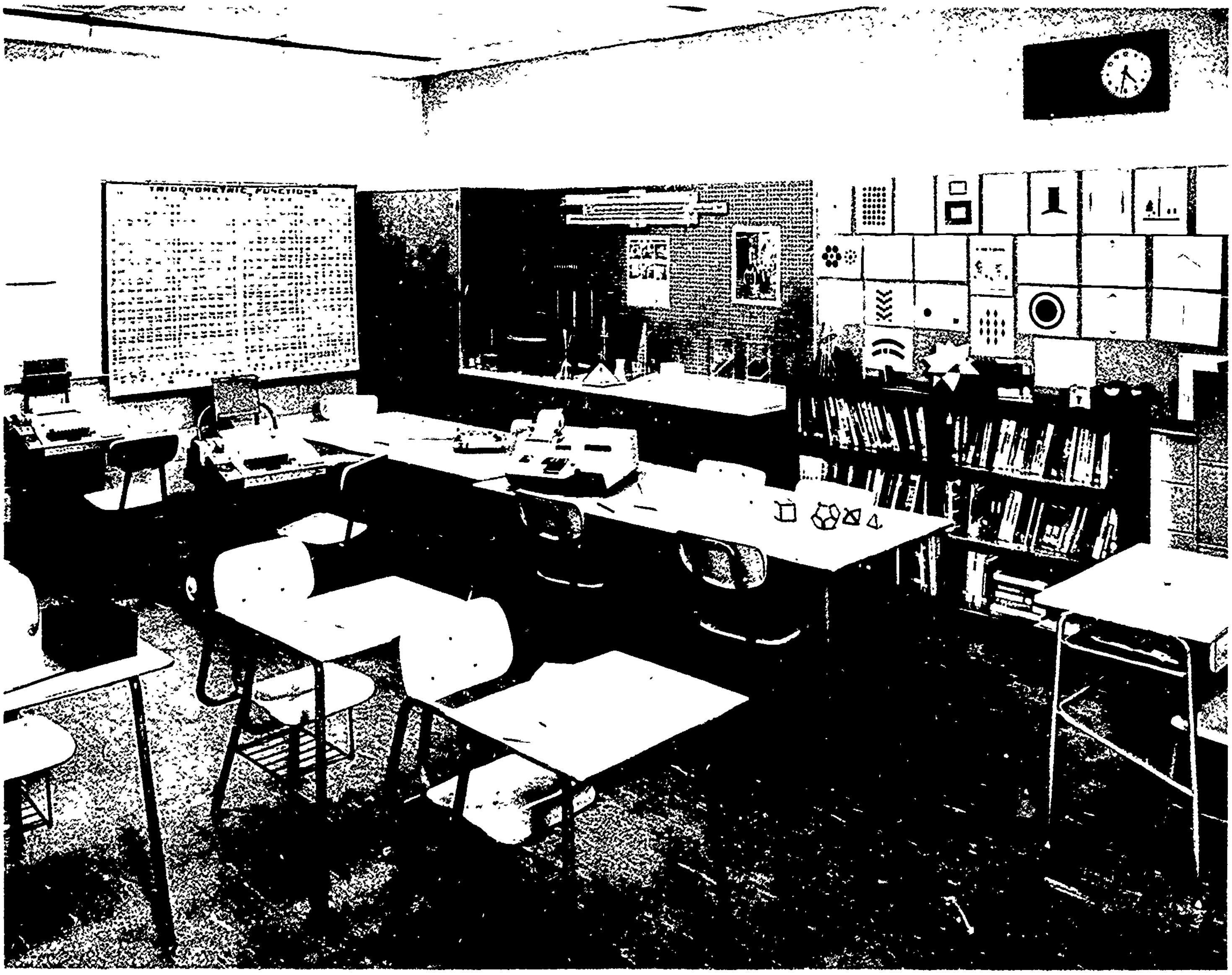
Providing space for large group instruction presents more of a problem. A standard-sized classroom can accommodate a reasonably large group if students' deskchairs are replaced by folding chairs; of course, the kinds of instructional functions that are possible under such an arrangement will be limited. There is also the possibility of knocking out a nonbearing partition wall to make one large group space out of two standard-sized classrooms. Removing corridor walls near the end of a wing or using the cafeteria or school auditorium are other possibilities. Acoustical problems can be reduced by carpeting the floors.

In an individualized program, textbooks and other instructional aids should be collected in one place in order to be readily accessible. This means that at least one standard-sized classroom needs to be used as a mathematics resource center (or perhaps two or three such rooms with adjacent interior walls removed). There should be areas in the resource center for housing and for using pertinent mathematics textbooks, films, filmstrips, records of facts, drill tapes, manipulative materials, models, and construction materials for projects and exhibits.

### *An Individualized Elementary School Mathematics Program Conducted in a Varied-Sized Space Complex*

An individualized elementary school mathematics program developed under the direction of M. Vere DeVault is planned to enable each student to select the mathematics he should study and to establish his own pace and sequence for learning the content he has selected. The major concepts of this mathematics program are organized into eight strands, with each strand organized into units. Instructional materials include booklets listing possible choices of textbooks for each unit, a library containing multiple copies of seven selected textbook series, tests with parallel forms for each unit (available at two reading levels), and answer sheets and keys for students. Record forms include student folders, test performance records, assignment certification records, and student-teacher conference guides.

FIGURE 2.19
*A standard-sized classroom furnished and equipped as a mathematics laboratory*



The school in which this program was implemented is the Giese School in Racine, Wisconsin. The structural design of the school includes two instructional space configurations, each of which is an example of a varied-sized space complex (Figure 2.20). Each of these complexes consists of a hexagonal shaped learning center surrounded by ten doorless pentagonal shaped standard-sized classrooms designed for use by small and medium-sized groups. Partition walls between three pairs of adjacent classrooms can be folded back to make large group spaces. The learning center, shown enlarged in Figure 2.21, may be used by large, medium-sized, and small groups, and has carrels available for independent study. Some of the areas behind the work islands are arranged for laboratory work. The following quotation from "Description of the Individualized Mathematics Curriculum Project" by DeVault, Buchanan, and Nelson (22) depicts what students and teachers do when they are at work in the program.

> About 40 students and a teacher are seated in an informal arrangement in the large hexagonal learning center surrounded by individual classrooms. Some students are reading independently in study carrels or at desks, while others are in small groups of two or more at tables. The auxiliary aide is at work near the files, records, and materials of the program. A few students are gathered about the movable library carts, looking up suggested unit assignments in various text series and selecting text assignments in

FIGURE 2.20. *Floor plan of the Giese Elementary School, Racine, Wisconsin. The distinctive design concept of this school is embodied in two varied-sized space complexes.*

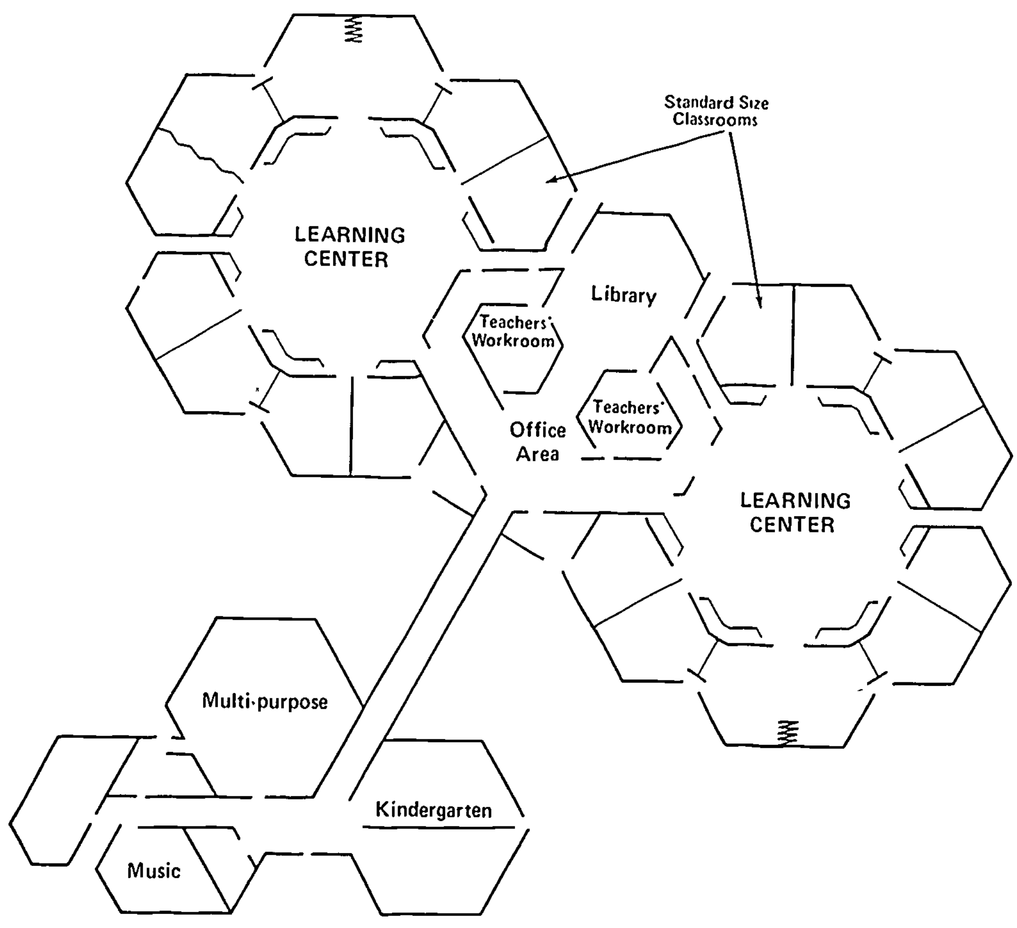
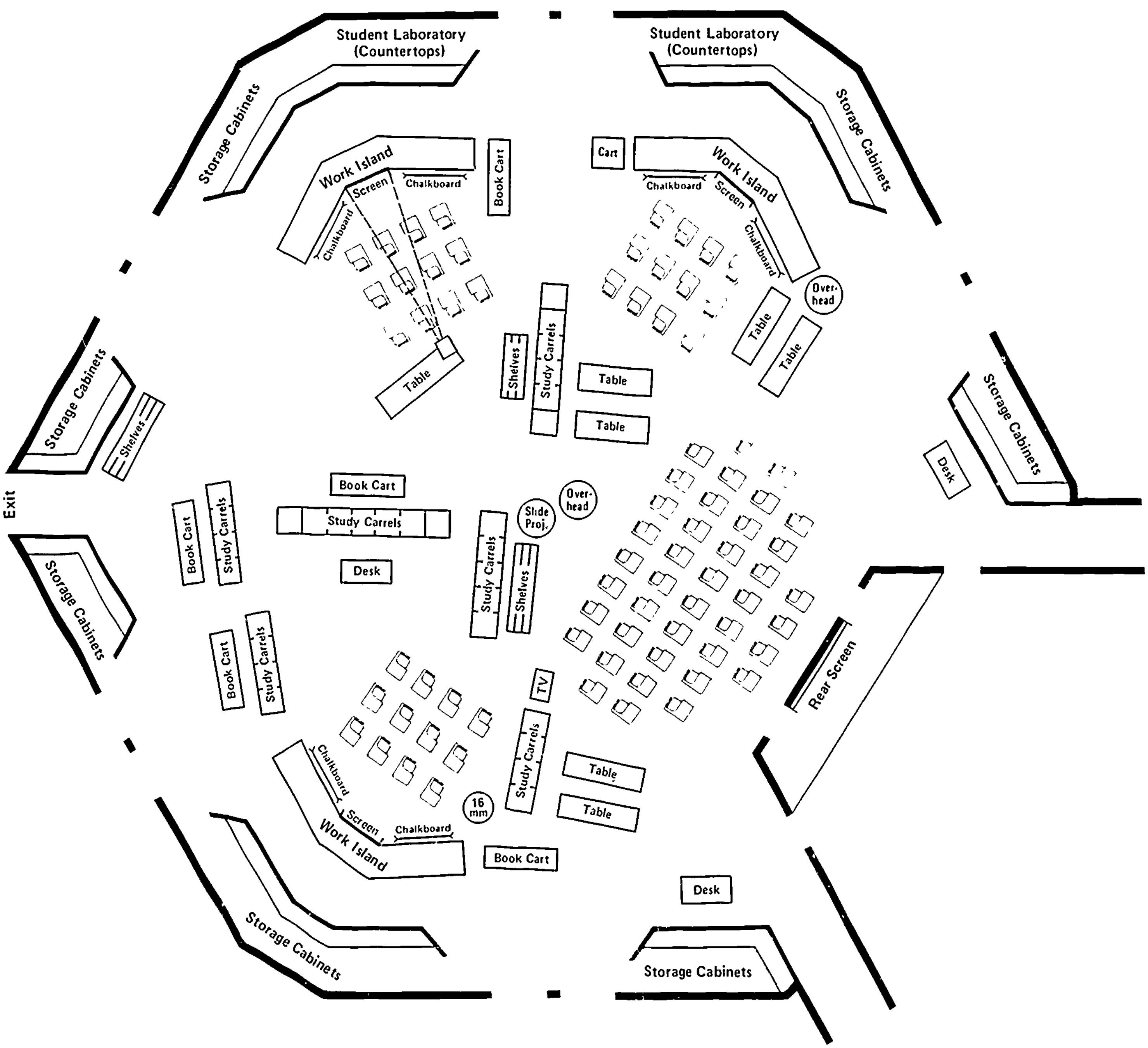


FIGURE 2.21. *A Learning Center in the Geise Elementary School, Racine, Wisconsin*



which the approach to the topic and the reading level are appropriate for their individual study.

One student has sought the teacher's assistance with some material on addition. After questioning the student briefly and reviewing his record folder, the teacher suggests that he lay aside his present work temporarily and choose further assignments on regrouping tens and ones. Another student is working a page of problems which emphasizes an understanding of the distributive property. As he finishes that page, he finds the teacher's manual, checks his answer, records the assignment in his folder, looks at his strand booklet marked Multiplication-Division, selects another textbook from the cart and returns to his desk. He opens the book and is once again confronted with some material on distributivity, but this time a different approach is taken to explain the concept.

The aide has just provided a progress test and answer sheet for one student, and is presently helping another locate a page of answers for an assignment in a teachers' manual, while yet another student waits momentarily to have a completed assignment certified. Seated at a listening center, a student who lacks certain reading skills is listening to a unit test previously tape-recorded by the aide.

Another student is puzzling to himself over a decision to move to the next concept in a unit. He checks his record folder and sees that he has done only two assignments related to this concept; he also notes that when he checked his work he had several errors on each assignment. He makes an independent decision to work further on the same concept but to use a text at an easier reading level.

Still another student is about to complete a unit test. After the items are checked by the aide, who will indicate from the annotated answer sheet the areas in which he needs more work, he will discuss the results with his teacher, who may then recommend additional or background assignments; or, the teacher may suggest that the student move on to another unit but that he attend the next small group seminar on the critical concept.

In one of the rooms adjoining the learning center a group of 20 children is at work with a teacher on one of the computational skills—this group is using a regular text series for reference and as a source of assignments. In another room seven children and a teacher are in the midst of a small group seminar on numeration systems. The students have all recently finished a unit on uncommon numeration systems and are working together on devising their own new numeration systems. On another day one might see a problem solving seminar comprised of a teacher and five students who have individually been working on the concepts in the measurement unit called "How Is Space Measured?" and who now have been confronted with a unique problem for group solution.

## *Open Space Schools*

Efforts to obtain flexible space arrangements needed to individualize instruction led to the open space concept in schoolhousing design. So far this concept has been used mainly in the construction of elementary schools; however, it has also been used in the construction of a number of junior high schools and even a few senior high schools.

In most schools built according to this concept, the teaching and learning areas are separated into large open spaces called pods. Usually, the floor area of a pod is equivalent to that of from three to ten standard-sized classrooms. Various shapes have been tried for pods: rectangular, hexagonal, circular, caracole (spiral), square, or a combination of these, such as the tri-pod, which consists of three rectangular spaces extending out from a common center area. There may be one pod within a school—or several, as in the Diamond Path Elementary School in Rosemount, Minnesota, which has three pods. The largest of these covers approximately 21,000 square feet of floor space and is capable of accommodating 225 students. Theoretically the size of a pod should be determined by the number of teachers who are cooperating as a team in teaching a particular grade, performance level, interest area, or subject. In practice, however, its size is usually determined by the anticipated enrollment for one or two grades or subjects.

Some open space schools are designed to be

completely open, even the larger ones. For example, the second floor of the Matzke Elementary School in Cypress, Texas, covers 56,000 square feet, is capable of accommodating 600 students, and has no interior walls.

There is, however, occasionally some hedging on the open space concept. It is not uncommon to see accordion or folding partitions, and in some schools movable coat racks and bookcases are used as sight dividers. Some designs provide for adding partitions in the future if they should be desired.

Such spaces as administrative offices, teachers' offices, teachers' commons, team planning centers, materials preparation spaces, and storage spaces for expensive equipment are enclosed by permanent partition walls.

An open space pod can be organized and furnished in various ways. Usually, it includes an open, easily accessible resource center where students and teachers can obtain textbooks and other instructional aids as needed. Around its perimeter and in its interior are located teaching stations, where small and medium-sized groups can gather. These stations are sometimes furnished and carpeted in different hues to enable students to locate the different stations without difficulty. The furniture in open space schools is usually movable and is arranged in islands to avoid corridor effects. Trapezoidal and semicircular tables used by students are arranged to form various configurations. Chalkboards are usually attached to the walls and movable chalkboard panels supplement these. Remaining wall areas may be covered with pegboard and some surface on which tacks and staples can be used.

Open space schools have a number of advantages over egg-crate schools and even over schools with varied-sized group spaces. First of all, they are more inviting to newer instructional strategies. Almost any desired combination of large group, small group, laboratory, independent study, and tutorial arrangements can be accommodated without concern for constraints imposed by existing walls.

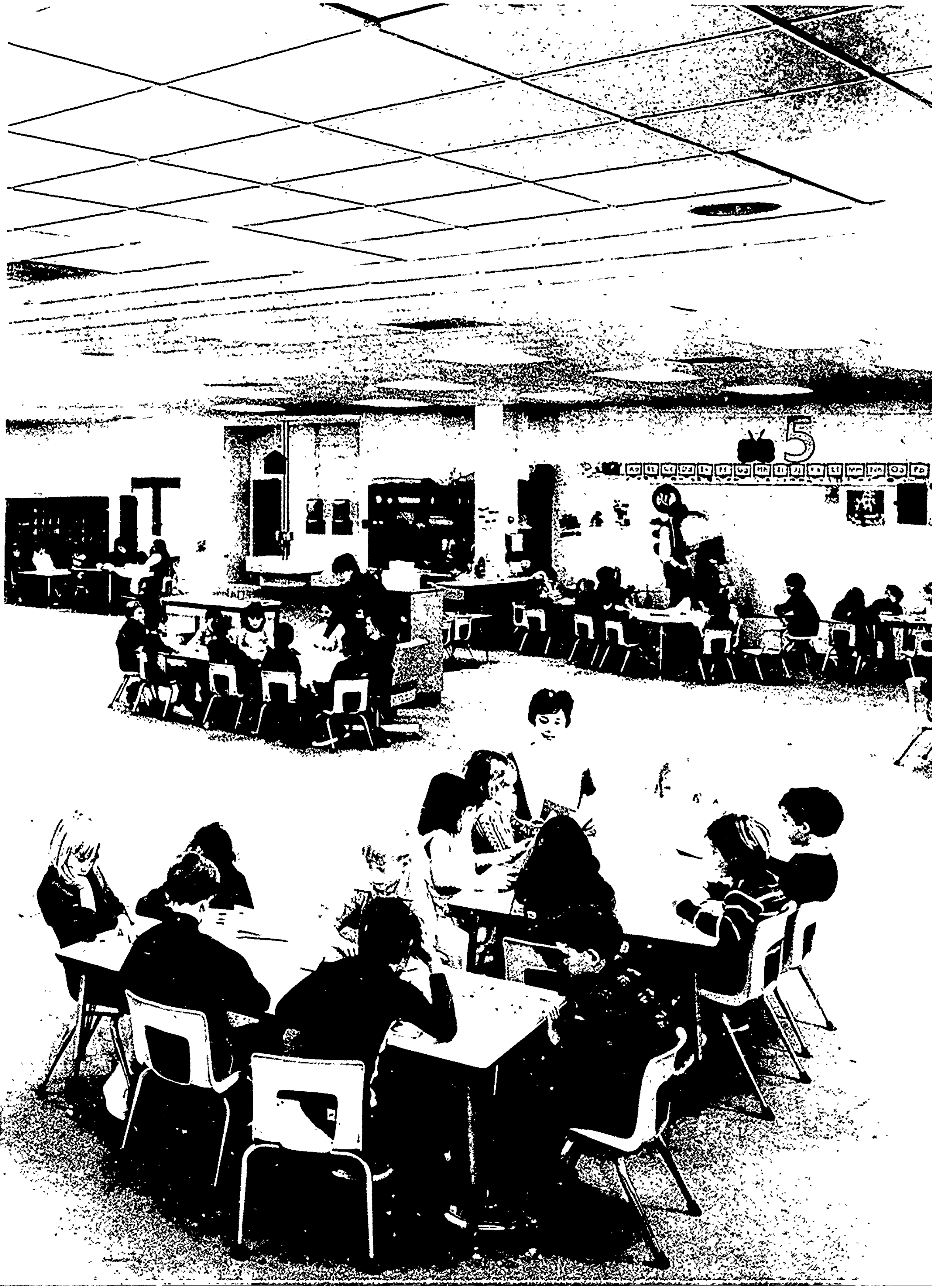
A second advantage of open space schools is that they make possible a bigger, better, and more varied inventory of instructional aids than schools of other design, and at no substantial increase in cost. With resource materials located at the center of a pod, more of the expensive equipment is readily available for use by large or small groups, thus lessening the need for duplication. This leaves money available to provide a more complete and elaborate resource center. In addition, teachers and students can make frequent and immediate use of audiovisual equipment as needs arise, for the machines are always available and do not have to be obtained in advance from a separate department. The easy access to whatever audiovisual materials a teacher or student wants should be helpful in customizing instruction to meet individual needs.

The principal advantage offered by open space schools is that they make performance level grouping more feasible than in schools of other design. Open space encourages flexible grouping patterns on an unconstrained and continuing basis, so that the uneven growth of a student in various subjects can be readily accommodated at any time. For instance, a student who is reading at the fifth-grade level but doing mathematics at the third-grade level can easily join an appropriate group for each subject (or start one of his own). If he should suddenly begin making progress in mathematics more rapidly than the members of his present group, then he can move immediately to a group more suited to his rate and level of learning.

An open space school maximizes the benefits attainable in a small group problem solving session because students working on the same problems are able to recognize each other's whereabouts and get together. Such clustering can occur with a minimum of commotion, and teachers are able to observe and direct students' movements without giving all their attention to them.

A further benefit of the open space school is that it encourages greater interaction among teachers, among students, and between students and teachers. A teacher is impelled to walk over

Figure 2.22
*An open space pod in the Diamond Path Elementary School, Rosemount, Minnesota*



and talk to another teacher if the two are near each other. The easy exchange of ideas and instructional aids, as well as the benefits of observing the teaching of nearby colleagues, offers a spontaneous brand of in-service training for teachers. And the ease of coordination among teachers in open space naturally encourages team teaching.

The open space school brings with it certain problems as well as benefits, the chief of these being noise and distraction. Since overpopulation can increase noise beyond a permissible level, an open space school cannot be used to hold more students than an equivalent combination of spaces in a conventional school. It is inevitable that the level of background noise is slightly higher than in the self-contained classroom because movement, which is not usually allowed in conventional buildings, is accepted as a normal part of the instructional program. Teachers and students seem to expect this additional noise and adjust to it rather easily. It is necessary, however, for teachers to adjust the volumes of their voices so they do not carry over into adjacent teaching stations, and high pitched voices can cause problems. Unwanted sounds are partially absorbed by installing carpeting over the entire open space. Carpeting eliminates the noise that normally results from books being dropped, chairs being pushed or knocked over, the scraping of feet, and the moving of furniture. It also encourages students to sit on the floor and assemble in small groups. Such congregating enables the teacher to lower his voice, thus reducing vocal interference with other groups.

A related problem in open space schools is that of scheduling. Care must be taken to avoid scheduling in proximity, at the same time, groups of students studying subjects that require concentration, such as mathematics or creative writing, and groups of students engaged in singing or viewing motion pictures with amplified sound. And scheduling brings with it not only the problem of avoiding distraction but also a certain lack of flexibility. Once a schedule has been agreed on by the members of a teaching team, it may not be possible to change it without the consent of all members of the team.

These problems are sometimes met by incorporating adjunct spaces into the open space design or using operable walls to enclose spaces for special work. Actually these increase the flexibility of the open space school, making scheduling easier. Adjunct spaces include student conference alcoves, specialized spaces for noisy laboratory equipment (for example, calculators, computer teletype terminals, typewriters, and power tools for making models and displays), space for unfinished or extended projects, a closed-off room for film viewing, and separate spaces for music and gymnasium. Some of the most recently constructed open space schools have eliminated the need for a separate closed-off space for film viewing by making it possible to dim the lighting in each teaching station separately and equipping students with plug-in headsets. Another innovation that permits broadcast of lessons, quizzes, musical programs, or tapes to various teaching stations, without distracting those in adjacent areas, is a wireless listening system. This can be achieved by encircling each teaching station with a wire loop strung near the ceiling or buried in the floor. Using a wireless headset, anyone within the loop can pick up the broadcast transmitted to that station.

In Matzke Elementary School, the open space school mentioned earlier, curricula and instruction are nongraded.[2] A team of teachers is assigned a set of students for a nine-month period, and the teachers are jointly responsible for planning and implementing each student's instructional program. The entire design of both building and program is planned to provide an individualized program for each student.

During the school day the students within a team are redeployed to form small homogeneous performance groups in each of the subjects, mathematics, reading, and language arts, with

2. This description of the mathematics program in Matzke Elementary School is based on information received from Kay Killough, the principal.

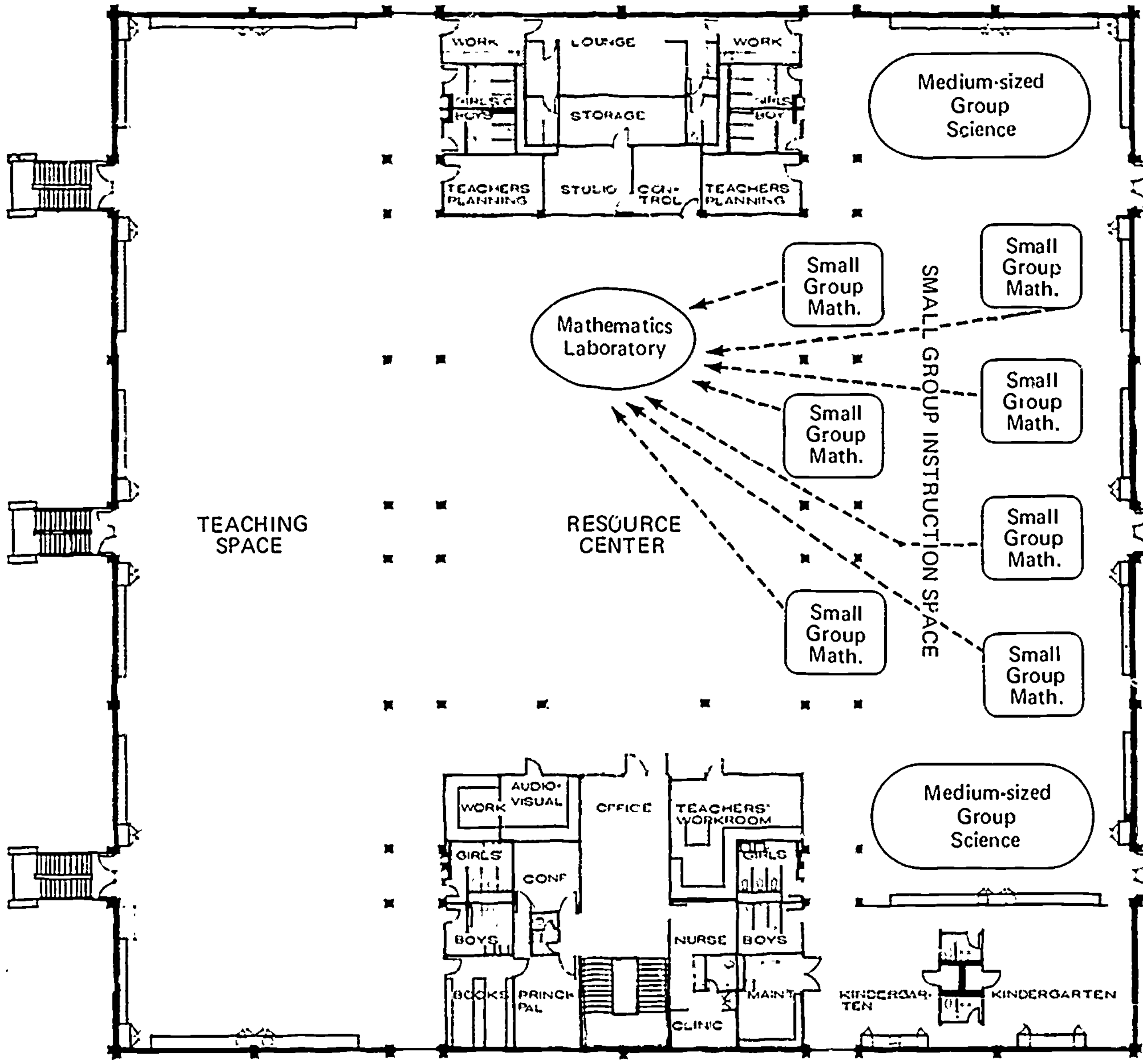



FIGURE 2.23. *The second floor of the Matzke Elementary School in Cypress, Texas. The area of this space is 56,000 square feet and will accommodate two teaching teams, each composed of at least nine teachers and 240 students. The diagram displays the deployment of one such team for science instruction in two medium sized groups and for mathematics instruction in seven small groups. A large group of 60 students is sent to the lower floor or outdoors for physical education. A team of the same size with the same number of students can be deployed on the left side of the open space.*

large group instruction in art, music, and physical education. The mathematics program is built around a continuum of skills defined in terms of behavioral objectives with a pretest and posttest for individual diagnostic information. Students are allowed to progress at their own rates through the sequential program. They are afforded instructional time with a diagnostic teacher and reinforcement time in individually prescribed independent study in the open learning center. The progress of each student is reported to parents, with ratings on individual mathematics skills rather than letter grades.

## EPILOGUE

The quest for ways of coordinating form and function continues . . .

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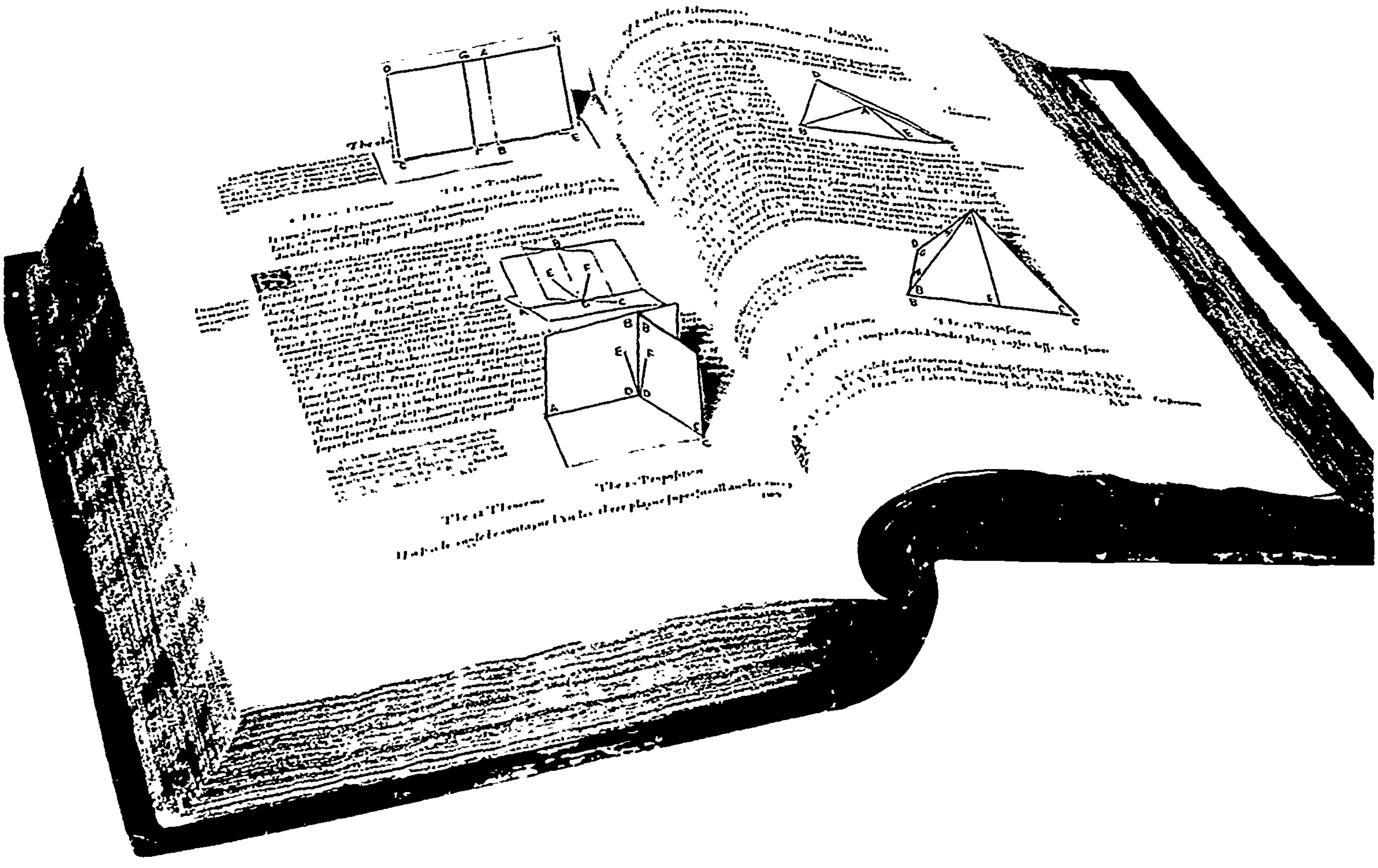
# 3. THE TEXTBOOK AS AN INSTRUCTIONAL AID

THE TEXTBOOK achieves, on rare occasions, both function and beauty. The first English translation of Euclid includes such a volume. Volume 2 of this translation contains magnificent woodcuts depicting solid geometric figures. This volume also contains a collection of pop-ups that rivals the efforts of many contemporary publishers. Certain figures are made of paper and so pasted in the book that they may be opened up to make actual three-dimensional models as shown in the illustration. Few textbooks combine beauty, novelty, and information as well as this volume of the first English translation of Euclid. It is a model to be emulated.

VOLUME 2 of *The Elements of Geometrie of the most auncient philosopher Euclide of Megara faithfully (now first) translated into the English toung*, by H. Billingsly, London, 1570. One copy is currently in the American University Library, Washington, D.C.



CHAPTER 3

# THE TEXTBOOK AS AN INSTRUCTIONAL AID

*by*
HENRY H. WALBESSER
University of Maryland, College Park, Maryland

*Not the violent conflict between parts of the truth, but the quiet suppression of half of it, is the formidable evil; there is always hope when people are forced to listen to both sides. It is when they attend only one that errors harden into prejudices, and truth itself ceases to have the effect of truth by being exaggerated into falsehood.*

JOHN STUART MILL

The textbook has the capability to present or suppress; to help develop objective or bigoted individuals; to maintain a free society or to hasten its demise. This chapter addresses itself to the problem of textbook selection.

# 3. THE TEXTBOOK AS AN INSTRUCTIONAL AID

## PROLOGUE

"Everyschool" is considered part of a contemporary, forward-looking (but not taken to extremes) school system located in a prosperous suburb of a not-too-large metropolitan region. The exterior of each school plant displays that usual blend of school architecture—large glass windows periodically punctuated by clean, new brick walls enclosing a most unimaginative set of rectangular shaped classrooms.

As one walks, pauses, listens, and looks within this school—what of the mathematics instruction he observes? The exhilarating sounds of students learning mingle with exhibitions of teachers telling. The most singularly striking element of commonness, however, is the deployment of textbooks in each of the classes and the obvious instructional dependence upon the textbook. In short, this is a school not so very much unlike every school.

## INTRODUCTION

Why does the textbook hold the instructional system in such a powerful grasp? Must the textbook be such a dominant force in shaping the mathematics curriculum? Are there alternative models for the instructional system in which the function of a textbook is altered?

Just what are textbooks? What a ridiculous question! Everyone knows what textbooks are. Or do we? If a learner reads a book in a classroom, is the book a textbook? If a book is chosen from a library shelf with the purpose of browsing through it, is such a book a textbook? If study material is selected from a newspaper, is the newspaper a textbook? If a learner uses a self-instructional program from a computer-assisted instructional system, is the computer program a textbook? Yet, with this acknowledged ambiguity, most individuals have some idea of what textbooks are and how they are used. That is, the meaning is clear until one is called upon for an explicit description.

## A HISTORICAL PERSPECTIVE

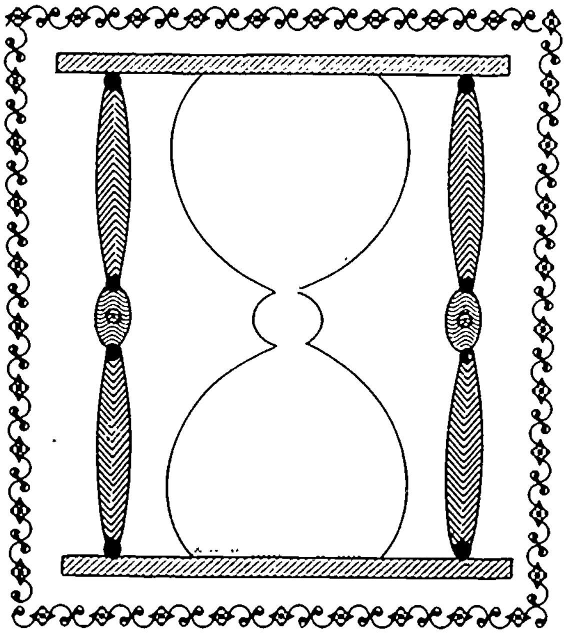
What are the potential sources of evidence that might yield information helpful to the interpretation of the textbook as an instructional aid? One means of acquiring information about the textbook is to examine its historical development.

### *Early Forms and First Purposes*

One of the first textbooks, if not the first textbook in a modern sense, was the elementary instructional program specified by the Council of Mainz in A.D. 813 (5). The Council's decree required that children be taught the *Fidem Catholican et Orationem Dominican*. This was the first required program of elementary instruction in recorded Western European history. It is interesting that this early "textbook" described the extent of the training for the learner as well as the content of instruction. Viewed in this training context, this first textbook came remarkably close to constituting a behavioral description of the tasks the learner was to accomplish—a most modern view.

Through the next few centuries, the textbooks consisted mostly of books devoted to basic literacy training. For this reason these texts are often referred to as the ABC books. In this developmental period, the expressed purposes and actual uses of the ABC books continued to retain a direct relationship to learner training.

The fundamental literacy textbooks were soon sequenced with readings of a more religious character, such as the Credo, the Pater Noster, and somewhat later the Ave Maria. By the fourteenth century the Benedictus and the Gratias had been added to this literacy-religious training sequence (6).



G

As runs the *Glass*
Man's life doth pass.

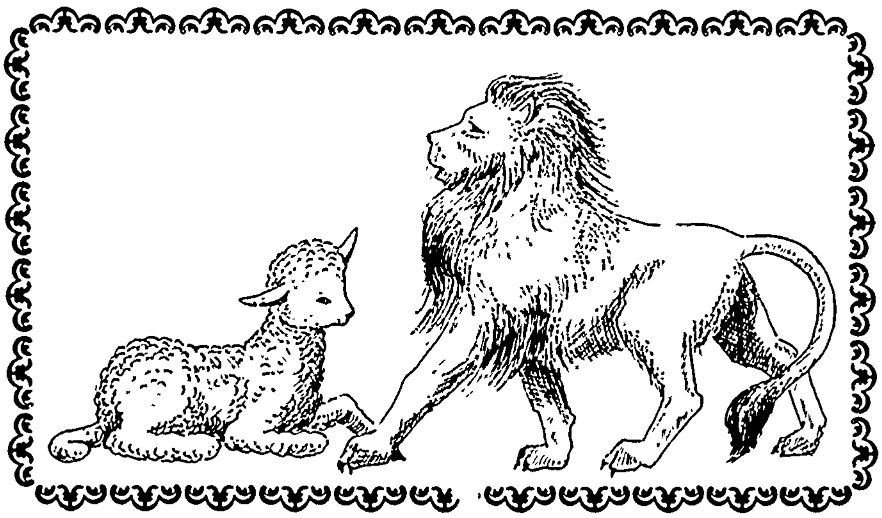
FIGURE 3.1

Among the first type-printed books was an elementary textbook of the Roman Catholic Church (4). This volume contained the alphabet, the Credo, the Pater Noster, the Ave Maria, and a few other prayers.

The early Reformation period found the elementary textbooks mostly manuals of church services. There was, however, an occasional insert of what might be termed secular material—principally in the form of classical sayings. Hence, the textbooks had slowly undergone a transformation over these five centuries until they had now become descriptions of information to be mastered.

### *Appearance Becomes a Factor*

The early sixteenth century brought a significant contextual shift in the textbook. Marens Schulte published a basic literacy text in 1532 which presented two radical changes in format (20). First, the ABCs were organized in vertical arrangement on the page. Second, there appeared a print and a line of reading adjacent to each letter of the alphabet. The prints illustrated an object for which the letter could stand. The lin of reading was also related to the particular letter of the alphabet. Several



L

The *Lion* bold
The *Lamb* doth hold

variations of this basic format were soon forthcoming. Perhaps the most significant was the presentation of rhymed couplets for each letter, as shown in Figure 3.1, rather than simple lines of reading. The advent of this change signals an awareness of the relationship between the textbook format and the learner's use of it.

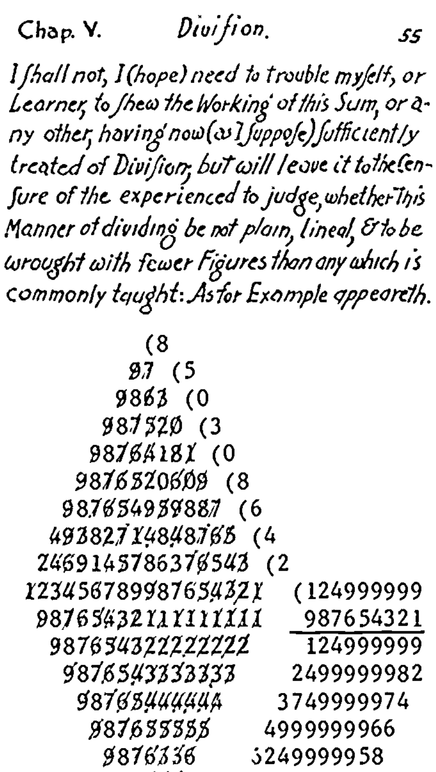
The first popular arithmetic in English appeared about ten years after Schulte's text—Recorde's arithmetic, *The Grounde of Artes*—but it is a collection of rules to be mastered. The mathematics texts, unlike the literacy volumes, did not reflect an awareness of the relationship between the format of the textbook and the learner's use of it. The organization of the arithmetic text was a "rules to be mastered" conceptualization.

One of the earliest editions of elementary mathematics actually published in colonial America was *Arithmetic, or That Necessary Art Made Most Easy* by James Hodder (18). This volume first appeared in the New World in 1719, where it was printed by J. Franklin, brother of Benjamin Franklin. Hodder's textbook contained an interesting and often challenging collection of arithmetic algorithms. One example of such an algorithm is the "plain, lineal, and wrought with few figures" division algorithm (Figure 3.2).

The arithmetic written by Thomas Dilworth enjoyed a great popularity in the colonies among all of the early foreign imports. The first American edition of the *Schoolmaster's Assistant* (12) was published in 1773. The contents of this volume are quite representative of the material treated by most of the contemporary works. This *Compendium of Arithmetic, Both Practical and Theoretical* included five parts, which (in Dilworth's words) discussed:

I. Arithmetic in Whole Numbers, wherein all the common Rules, having each of them a sufficient Number of Questions, with their Answers, are methodically and briefly handled.
II. Vulgar Fractions, wherein several Things, not commonly met with, are there distinctly treated of, and laid down in the most plain and easy Manner.
III. Decimals, in which, among other Things, are considered the Extraction of Roots; Interest, both Simple and Compound; Annuities; Rebate, and Equation of Payments.
IV. A large Collection of Questions with their Answers, serving to exercise the foregoing Rules; together with a few others, both pleasant and diverting.
V. Duodecimals, commonly called Cross Multiplication; wherein that Sort of Arithmetic is thoroughly considered, and rendered very plain and easy; together with the Method of proving all the foregoing Operations at once by Division of several Denominations, without reducing them to the lowest Term mentioned.

FIGURE 3.2



Chap. V. *Division.* 55

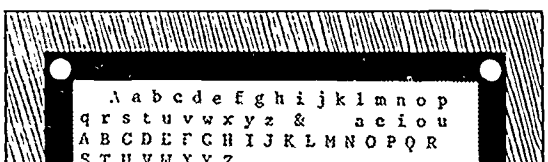
*I shall not, I (hope) need to trouble myself, or Learner, to shew the Working of this Sum, or any other, having now (as I suppose) sufficiently treated of Division, but will leave it to the Censure of the experienced to judge, whether This Manner of dividing be not plain, lineal, & to be wrought with fewer Figures than any which is commonly taught: As for Example appeareth.*

```
            (8
           97 (5
          9863 (0
         987520 (3
        98764181 (0
       9876520609 (8
      987654959887 (6
     49382714848765 (4
    2469145786376543 (2
   1234567899876543 21   (124999999
   9876543211111111       987654321
    987654322222222      124999999
     98765433333333     2499999982
      9876544444444    3749999974
       987655555     4999999966
        9876336     5249999958
         98777     7499999940
          988     9749999933
           9     9999999920
                11249999915
                          8
Proof         123456789987654321
E 2                          CHAP.
```

John Ward's *Young Mathematician's Guide* (41) was the volume in common use at Yale and Harvard as a mathematics text used at these universities for much of the eighteenth century. The edition commonly used at the university level included sections on arithmetic, algebra, the elements of Euclidean geometry, conic sections, the arithmetic of infinities (repeating decimals), and logarithms.

Literacy volumes following the letter-rhyme format (letter accompanied by illustration and rhymed couplet) were brought to the New World by the Puritans. The appearance of illustrations and a more appealing visual format is a significant development beyond the mere physical change. Their appearance assumes significance also because the image of total mastery of textbook items was now tempered by the introduction of aids to the learner: to be sure, the shift from instruction by mimicry was small. Nevertheless, their appearance did forecast the changes that were about to occur in the textbook.

FIGURE 3.3




Among the most popular texts published in the New World and used by the eastern seaboard colonies was the *New England Primer* (21). The first edition of the *Primer* appeared toward the end of the seventeenth century, in 1673. The hornbook and the sampler were considered by some educational historians not to be "real" textbooks because of their physical format. Nevertheless, the hornbook and the sampler did enjoy widespread popularity at this time and were competition for the *New England Primer*. The hornbook was shaped like a paddle. The example depicted in Figure 3.3 is typical of the hornbook's appearance.

The book's construction was usually of wood or heavy cardboard. On the paddle was pasted a piece of paper containing the text to be mastered. The text was then covered by a thin, transparent slice of horn attached to the wooden base. The student would practice his imitations of this lesson until he had mastered it. Some hornbooks included a simple counting frame, usually attached to the back of a paddle as in Figure 3.4.

Young girls studying at home or in a dame school—a type of finishing school enrolling young girls at this time—would construct samplers on which they embroidered texts. Among the materials embroidered on the samplers were rhymed couplets, verse of one form or another, the alphabet, and usually a few names for natural numbers.

The learning emphasis one can infer from both the hornbook and the sampler is one of learning viewed as mastery of materials—and mastery accomplished by repeated duplication of a given pattern. All in all, it was not a particularly exciting environment for the learner.

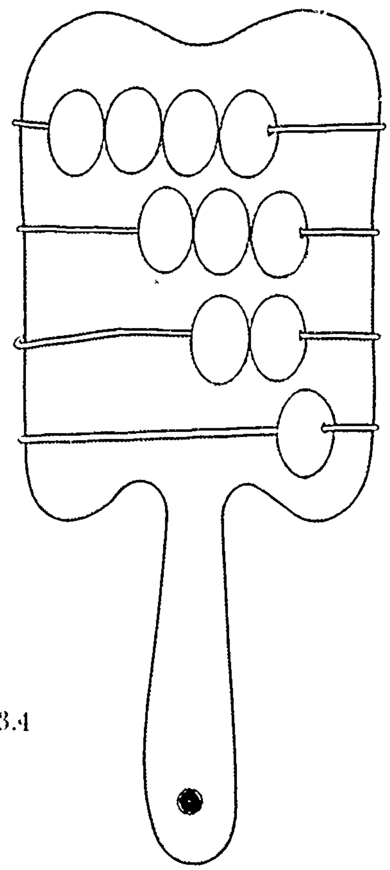


FIGURE 3.4

## *Early Mathematics Textbooks in the United States*

The description of the mathematics instructional materials of the colonial and early national periods is little different from that of the imports to the eastern seaboard colonies, the largest proportion of these being British. Among the most popular of these arithmetic works were the editions of James Hodder, Edward Cocker, John Ward, George Fisher, and Thomas Dilworth.

Ward's introductory statements in his volume (41) are of some interest to contemporary mathematics education, considering his avowed purposes of mathematics instruction:

> As to the usefulness of arithmetic, it is well known that no business, commerce, trade, or employment whatsoever, even from the merchant to the shopkeeper and so on can be managed and carried on without the assistance of numbers.
>
> As to the usefulness of geometry, it is as certain, that no curious art, or mechanical work, can either be invented, improved, or performed without its assisting principles; tho' perhaps the artist, or workman, has but little (nay, scarce any) knowledge of geometry.
>
> Then as to the advantages that arise from both these noble sciences, when duly joined together, to assist each other, and then applied to practice, (according as occasion requires) they will readily be granted by all who consider the vast advantages that accrue to mankind from the business of navigation only. As also from that of surveying in dividing lands betwixt parties and party. Besides the great pleasure and use there is from time-keepers, as dials, watches, clocks, and so on. All of these and a great many more useful arts, (too many to be enumerated here) wholly depend upon the aforesaid sciences.

The first arithmetic textbook written and published by a North American was *Arithmetick, Vulgar and Decimal*, by Isaac Greenwood (17). This volume, which appeared in 1729, provided an interesting organizational format which helped to distinguish it from the popular foreign texts such as Dilworth's or Fisher's. Greenwood's method was to give the rule, present examples illustrating the rule, provide examples for the student, and finally suggest a formal proof. Each student example had a blank space beneath it for the student's practice of the rule. In some ways this text foreshadows the self-instructional format of this century. The technique of rule-example-practice is still with us.

The content of Greenwood's volume was similar to that covered by most of the arithmetic textbooks of this period. The material included work on base-ten numeration of natural numbers, as well as the naming of positive rational numbers by fractional numerals and by decimal numerals; and it presented algorithms dealing with the operations of addition, subtraction, multiplication, and division first performed on elements from the system of natural numbers and later on elements from the system of positive rational numbers. The monetary system of pounds, shillings, and pence also consumed a fair measure of Greenwood's presentation, with many of the problems set in the contexts of trade, partnerships, or the transfer of merchandise.

A sequencing of arithmetic followed by algebra was established quite early in the colonies. In fact, the first treatment of algebra to appear in the eastern seaboard colonies was combined with an arithmetic—*Arithmetica of Cyffer-Konst* by Pieter Venema (38). The New World edition of this volume was first presented in 1730 and was printed in Dutch. Venema firmly established a link between arithmetic and algebra by commenting that he had undertaken to make a "clear and distinct cyphering book" and to this treatment had "added the elements of algebra." Algebra was now instituted as a part of the mathematics curriculum. The purpose of algebra was to be the key to unlocking all the obscure propositions of arithmetic.

The first separate algebra textbook printed in English in the United States came some eighty-four years later—it was not until 1814, in the publication of *An Introduction to Algebra* by Jeremiah Day (10), that there was a New World publication of algebra in English, written by an American.

## *New World Editions*

Toward the end of the eighteenth century elementary mathematics volumes by New World inhabitants began to replace the imported mathematics textbooks. Among these early editions which gained widespread acceptance was *A New and Complete System of Arithmetic* by Nicholas Pike (26). The first edition of this material was published in 1788.

"Old Pike," as it affectionately came to be known, is significant in the development of mathematics textbooks in the United States for several reasons. First, the breadth of material gathered together into a single program and described in Pike's arithmetic served as a model to be emulated by most of the textbooks for the next one hundred years in the United States. Second, Pike's effort served to affect mathematics instruction for an even more extended period of time in our national history than did the model of topics to be included. The textual material presented mathematics as collections of rules to be mastered. One rule per page was about the average for Pike's volume, and this scheme appears to have been quite thoroughly mastered by the succeeding authors of mathematics textbooks during the nineteenth century and most of the early twentieth century. More unfortunate than this organization of mathematics "by the rules" was the adherence to the mastery of these rules as the characteristic goal of mathematics instruction. (This form of instructional torture is, of course, no longer practiced.) The description of the rules was often so ambiguous or so involved as to be beyond any reasonable assumption concerning the capabilities of the learners involved. One supposes these complicated formats and ambiguous descriptions were not intentionally confusing, although their lack of clarity and the length of some of the rules strains the credibility of such a supposition.

Most learners who were exposed to "Old Pike" did not acquire computational facility beyond that of being able to perform the operations of addition, subtraction, multiplication, and division on natural numbers. Some competency was usually acquired with the identification and naming of a few positive rational numbers expressed in the form of fractional numerals (commonly called vulgar fractions in Pike's time). A student was considered remarkable if he managed to interpret and demonstrate the ability to use the "Rule of Three," which in Pike's version went something like this—"The Rule of Three teaches, by having three numbers given, to find a fourth, that shall have the same proportion to the third, as the second to the first."

## *Mastery: the Diet of the Day*

As best they can be reconstructed, the descriptions of activities, the instructions to the students, and the exercises to be practiced by the students contained within the volumes of Pike and his contemporaries strongly suggest that the instructional emphasis was placed upon

the mastery of rules. The actual instructional procedure often assumed the form of presenting the learner with a statement of the rule to be mastered, followed by one or more illustrative examples, and finally relentless repetition of the procedure by the learner until he could successfully demonstrate a mimicry of the rule.

Long computational problems were also characteristic of the early and mid-nineteenth-century arithmetics of native origin. Often a long series of computations would be carried out by the entire class of students with frequent checks on one another's work. This mass drill did offer one alternative to the individual drill practice and for that reason could be considered of some instructional merit. All in all, however, the instructional procedures were devoted, with almost monotonous exclusiveness, to the mastery of given rules.

Congress distinguished itself in 1786 by establishing a federal monetary system based on the decimal system of numeration. This decision had little immediate effect on the presentation or discussion of the monetary system in the elementary mathematics textbooks. However, this decision proved to have a significant impact on the content of the nineteenth-century elementary mathematics textbooks. Among the first evidences that this congressional decision had an impact on the presentation and discussion of the monetary system in the elementary mathematics textbooks was the appearance of Errastus Root's volume *An Introduction to Arithmetic* (34) in 1796.

*An Introduction to Arithmetic* was among the first elementary volumes to give a prominent position to the decimal monetary system together with a strong nationalistic plea for its adoption and use. This particular plea can be assessed in remarks such as the following, taken from the preface.

> It is expected that before many years shall elapse, this method of reckoning will become general throughout the United States. Let us, I beg of you, fellow citizens, no longer meanly follow the British intricate mode of reckoning. Let them have their own way and us, ours. Their mode is suited to the genius of the government for it seems to be the policy of tyrants to keep their accounts in as intricate and perplexing a method as possible; that the smaller number of their subjects may be able to estimate their enormous impositions and exactions. But Republican money ought to be simple and adapted to the meanest capacity.

Root's emphasis on the decimal monetary system remained the exception to the monetary presentations developed in most elementary mathematics volumes for a number of years after the beginning of the nineteenth century. A traditional presentation persisted in most volumes, with the emphasis being given to the treatment of unbelievably complicated business transactions involving English coins, Irish coins, pistoles, doubloons, and moidores, as well as all manner of possible (if often improbable) combinations. Figure 3.5 shows a problem of little currency complication but suggestive of the business transaction problem.

*Daboll's Schoolmaster's Assistant* (9), by Nathan Daboll, followed in the tradition of Pike's and Root's volumes and proved to be among the most popular elementary mathematics volumes of the first half of the nineteenth century. The extensive use of the *Schoolmaster's Assistant* can be attributed to a number of separate but significant factors. The limitation of material to a collection of items which was more realistically and practically suited to the actual school circumstances was one. The smaller size of the volume needed to present such material undoubtedly was among the more prominent of these factors. Another factor of some importance was the volume's emphasis on the new decimal currency. The value of this particular factor was reduced somewhat by the occasional intrusions of problems that reverted to mentioning pounds and shillings.

It is worth noting that most mathematics volumes of this period included problems related to gambling situations as well as some that had something to do with the manufacture and distribution of alcoholic beverages.

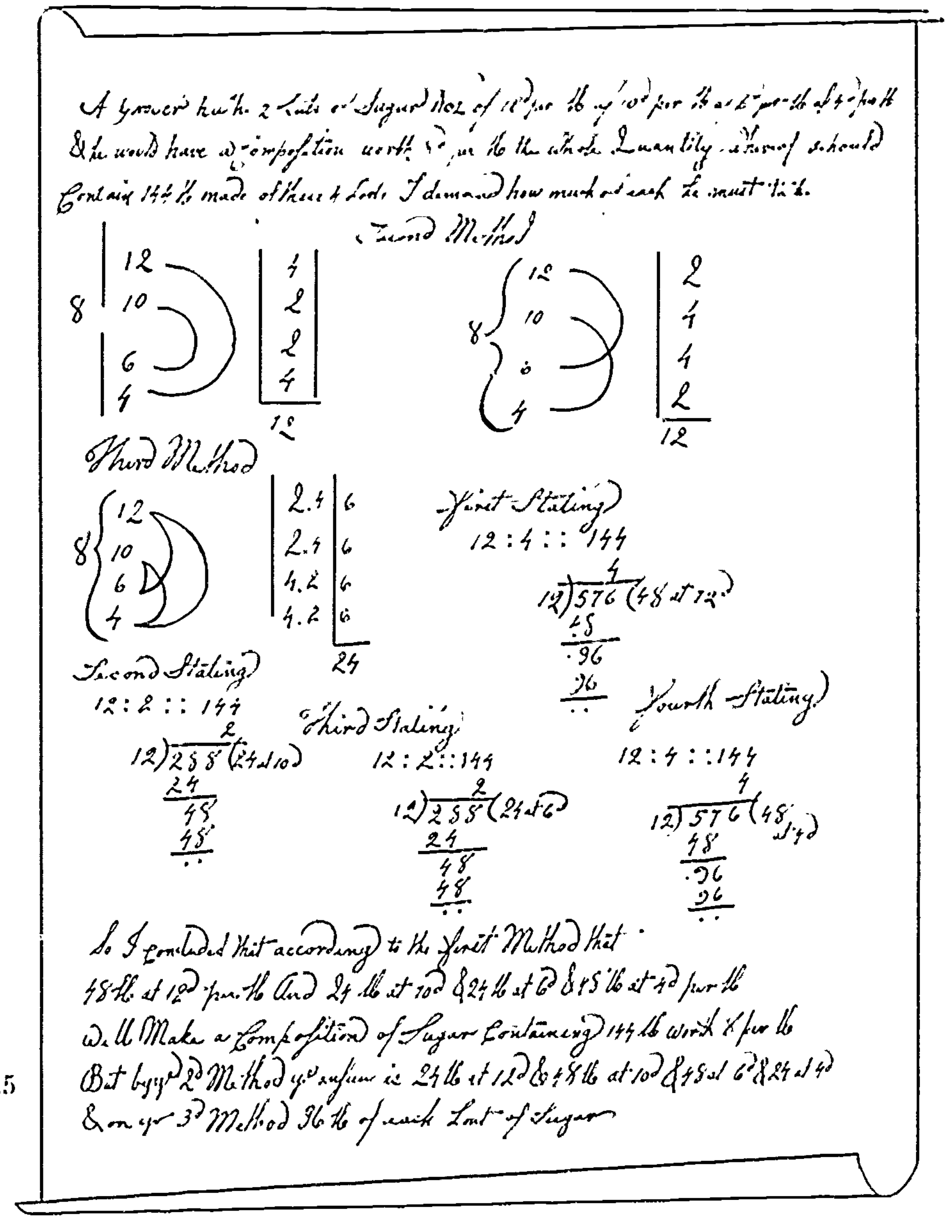



FIGURE 3.5

## *Deviation From the Rules*

The turn of the century brought with it several innovations of note. The first of these innovations was Daniel Adams's text *The Scholar's Arithmetic* (1). Adams attempted to produce a collection of clear and simple presentations within practical orientations. In addition to the simplification and clarification attempted by Adams, he included a workbook-like feature within his presentation by leaving blank spaces between problems for the learner to insert solutions.

One of the most significant breakthroughs in the development of nineteenth-century versions of elementary mathematics textbooks in the United States occurred in 1821 with the publication of *First Lessons in Arithmetic* (8) by Warren Colburn and his *Sequel to the First Lessons* (7), which appeared a year later. Unlike the preceding elementary mathematics volumes, which were in the tradition of rule presentation,

group and individual practice by a sing-song repetition, and demonstration of the mastery of a memorized rule, Colburn attempted to provide physical experiences whose purpose was to expose a learner to tasks he was to perform. These tasks were related to the possible origin of a particular arithmetic rule and attempted to place the learner in situations highly suggestive of a need for it.

The organization of Colburn's work reflects the influence of the pedological position espoused by Pestalozzi. Colburn's observations concerning the importance of the number 1 in the development of counting, as well as the development of an intuition concerning order relations, appear quite contemporary. For example, Colburn observes in the preface of his first volume that "as soon as a child begins to use his senses, nature continually presents to his eye a variety of objects; one of the first properties which he discovers is the relation of numbers. He intuitively fixes upon unity as a measure, and from this forms the idea of more and less."

Direct physical observation and tactile experience are the keynotes of Colburn's presentations. Colburn introduced natural numbers, not by statements of rules to be remembered, but by a series of questions related to the child's physical environment. He accomplished this by asking questions such as "How many thumbs have you on your right hand? How many on your left? How many on both together? How many hands have you?"

Colburn included a large number of simple computational problems to be done mentally. He was among the first to introduce mental arithmetic to the elementary mathematics curriculum of the United States. The staying power of this idea of mental arithmetic is recognizable to anyone familiar with the elementary mathematics curriculum.

Colburn's genius with respect to instructional innovation and textbook organization shifted the focus of attention from rules to the learner. In many respects Colburn's volumes anticipate the instructional segments of the contemporary mathematics revolution by more than a hundred years. The *First Lessons* and the *Sequel* presented the elementary school teacher with a new perspective on mathematics instruction. These volumes were a departure from the tradition of mathematics-by-the-rules which had held sway for many centuries.

In assessing the actual impact of Colburn's volumes it must be said that the changes in instructional practice were slow in coming, and the shift from rule memorization to learner-oriented instruction was by no means universally adopted. However, the appearance of these two volumes does represent a major attack upon this staid instructional emphasis. The advent of these materials signaled the beginning in a lengthy sequence of reappraisals of the relationship between the discipline, the learner, and the mode of instruction which is continuing up to the present day.

## *Visual Appeal of Textbooks*

The general attractiveness of eighteenth- and nineteenth-century schoolbooks as well as the visual appeal of the material contained within these volumes was seldom, if ever, a consideration of the authors. Nor could the attractiveness of presentation be treated as a strong point of the publisher's concern for elementary mathematics textbooks in the United States during the eighteenth and early nineteenth centuries. The appearance of Frederick Emerson's *North American Arithmetic, Part 1, Part 2,* and *Part 3* (13), marks a change in concern over the attractiveness of texts.

The first of Emerson's volumes appeared in 1830 and was quickly followed by two additional volumes, so that by 1831 three parts were available. These volumes are unique in that Emerson scattered drawings throughout the first part. The first volume of this three-part series represented a genuine attempt to produce an attractive beginner's book. This is one of the first calculated efforts to capitalize on illustrations as a functional part of the instructional dialogue

in a mathematics text. The preface to *Part I* supports the claim that Emerson purposely compiled the three volumes with such a view to illustrations: "The method employed for illustrating the subject, it will be seen, is original and peculiar." The separation of an arithmetic into separate parts represents a significant innovation introduced by Emerson. This particular innovation proved to be a most lasting organizational format, as the numerous mathematics series of today will clearly attest.

Although Emerson's volumes represent the first to contain a substantial number of illustrations purposely related to instruction, many of the earlier textbooks did contain a few formal diagrams, and some volumes such as Thompson's *The American Tutor's Guide* (36) even made use of an occasional print to point up the situation in which a problem was set. One such example from the Thompson volume is shown in Figure 3.6, with a problem set in verse and accompanied by an illustration.

A second interesting example of a pre-Emerson use of illustrations for the purpose of augmenting instruction is given by Barnard's volume *A Treatise on Arithmetic* (2). The quotation and accompanying illustration shown in Figure 3.7 are typical of Barnard's use of drawings.

Such sequences of questions were terminated by a question more directly associated with an arithmetic skill. In this case, the sequence concluded with the question "3 times 3 is how many?"

The elementary mathematics textbooks of Joseph Ray achieved the distinction of being the most popular of the elementary mathematics textbooks published during the nineteenth century in the United States (using as an index

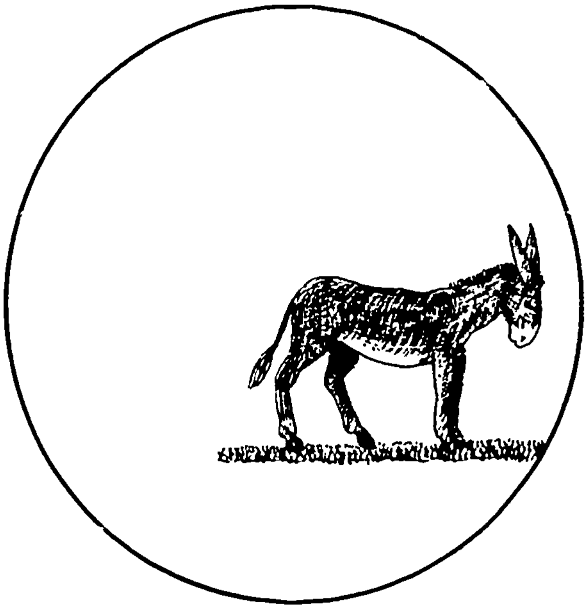


FIGURE 3.6

In the midst of a meadow,
  Well stored with grass,
I've taken just two acres
  To tether my ass;

Then how long must the cord be,
  That feeding all round,
He mayn't graze less or more than
  two acres of ground?

FIGURE 3.7

"Here are three boats, and each boat contains three men. How many men in all?"

the number of copies printed). Ray's first volumes, *The Little Arithmetic* (29) and *Table and Rules in Arithmetic for Children* (32), appeared in 1834 and served as an introduction to his *Eclectic Arithmetic* (28), first published in 1837. These volumes were incorporated into a series that proved to be the most popular of Ray's works. *Arithmetic, Part First* was published in 1842; *Arithmetic, Part Second* appeared a year later, and *Arithmetic, Part Third* completed the series (27).

Ray's texts resembled Colburn's in that they adhered to a simplicity and directness of presentation. Ray did not attempt to construct difficult elementary mathematical situations and, in fact, introduced nothing more puzzling than word problems. The impact of Colburn is easily identifiable in the comments of Ray on intended purposes such as providing a "thorough course of mental arithmetic by induction and analysis" and in the provision for "the inductive and analytic methods of instruction." The use of "mental" and "intellectual" in the titles of the various elementary mathematics volumes also reflects the Colburn influence. The title of *Arithmetic, Part Second*, for example, was changed to *Mental Arithmetic* in the 1853 edition and *Intellectual Arithmetic* in the 1857 edition. The modernity syndrome so characteristic of the mid-twentieth-century volumes also manifested itself in the last editions of Ray's texts with the appearance of *Modern Elementary Arithmetic* (30) and *Modern Practical Arithmetic* (31) in 1903.

One of the significant contributions of Ray's volumes is not a consequence of the content or its mode of presentation, but rather its program of revision. Frequent and well-planned revisions of these elementary mathematics textbooks were conducted. The revision mechanism provided a means of reflecting changes in the emphasis of topics as well as correcting errors discovered in the field tryout. The publishers of Ray's texts did not initiate this technique, but they brought revision of elementary mathematics books to a level of sophistication not previously attempted by publishing houses in the United States.

The popularity of Ray's textbooks as well as the success of the numerous revisions conducted during the latter half of the nineteenth century and at the beginning of the twentieth century is reflected by the estimate of 120 million copies sold over the lifetime of these volumes. In fact, the average annual sale of Ray's volumes between 1900 and 1913 is estimated at 250,000 copies. The apparent popularity of the revised volumes must be balanced with the observation that although these frequent revisions did correct errors, they did not significantly affect the content of the volumes. The title and vocabulary changes were often measures prompted more by sales expediency than anything else. This motivation is characteristic still in evidence today. The field tryout was conducted by those who purchased the volumes.

The appearance of novel forms of organization for the elementary mathematics curriculum during the latter part of the nineteenth century was meager. John French prepared a series of volumes that typify the better textbooks of this period. *The First Lessons in Numbers* (14) was the primary-level volume of this series; it required the child to perform mostly at an oral level.

Much more interesting than the contents of these volumes were the comments of French characterizing the role of rules. "If principles are understood, rules are useless" is one comment that reflects his view, as does this remark in his fourth book: "If the understanding is thoroughly reached, the memory will take care of itself." This represents a remarkable shift from the "rules to be mastered" view of the earlier centuries. Whether such remarks actually affected instruction is another matter.

The "Oswego Object Lesson" movement introduced one of the most interesting innovations of this period. Milton Goff's work is typical of the form taken by this innovation. The idea was to use objects and pictures as the source of instruction.

### *The Historical Development of the Textbook, in Summary*

The pre-twentieth-century textbooks in elementary mathematics were organized to be used in an order-rule statement, example-of-rule practice, demonstration of mastery sequence. A few texts went to the extreme of presenting the rules like a catechism—stated questions followed by answers to be committed to memory, such as:

*Q.* How many kinds of addition are there?
*A.* Two—simple and compound.

Colburn introduced an "inductive" strategy for instruction which appealed to the examination of situations and exercises as a source of information to be used in constructing the rules, or at least to be suggestive of the reasonableness of the rules. Mental or oral arithmetic became a part of the early exposure to mathematics for the child. By the end of the nineteenth century, most texts were a blend of Colburn's induction and the older rule-mastery.

Texts were made more attractive by the introduction of frequent illustrations; and for some volumes, such as those adopting the Oswego Object Lesson strategy, the illustrations were made a functional part of instruction.

The teacher was often given advice as to how best to use a textbook by statements from the author, usually printed in the preface. The majority of texts contained some comments directed at instructional use. Toward the end of the nineteenth century it was becoming more and more common practice for volumes to include special sections at the end of the book such as "Helps to Teachers" or "Suggestions to Teachers." *Elementary Arithmetic* by French, for example, contained a "Manual of Methods and Suggestions" in the back which referred to the treatment or explanation of specific problems in the text.

Because of the inadequate preparation of teachers, the practice of publishing "keys"—volumes including the solutions to the most difficult problems as well as the answers to the problems became widespread. The "Hints to the Teacher" section combined with this expanded version of the "key" is the obvious forerunner of the teacher's edition common to mid-twentieth-century elementary mathematics volumes.

What is the stability of the content of the elementary mathematics volumes up to the twentieth century in the United States? The studies of Smith and Eaton (35), Burns (5), and Futcher (15) summarized in Table 3.1 suggest some dramatic shifts of emphasis. Perhaps the most dramatic of these changes occurred in the study of monetary systems—there were declines in the emphasis on instruction in federal money and in foreign exchange. This change is interpretable in the context of the isolation that evolved during this time in our national history and the growing familiarity with the decimal monetary system. The solidification of the objective of developing competency in the performance of arithmetic operations on the positive integers and the positive rationals as well as demonstrating skill in the manipulation of decimal numerals and cases of percentage is also observable in the proportions reported by the investigations. The emphasis on "practical adult" application of these skills is not observable from these data, other than in the obvious decline in emphasis on the colonial-period applications such as tare, tret, and cloff; partnership; and barter. The impact of the twentieth-century social utility on the mathematics curriculum was still some decades off.

## TWENTIETH-CENTURY CONTRIBUTIONS TO THE EVOLUTION OF THE TEXTBOOK

The twentieth century brought tremendous increases in the school population, with the common school growing in the direction of including fourteen years of formal instruction. This growth has been accompanied by fads such as the attempt to justify a place for mathematics in the common-school curriculum by fitting

TABLE 3.1
*Percentage of Total Pages*
*Devoted to Various Elementary Mathematics Topics*
*in Publications prior to 1917*

| TOPIC | To 1810 (Burns) | 1821–51 (Smith Eaton) | 1851–80 (Smith Eaton) | 1877–1917 (Fulcher) |
|---|---|---|---|---|
| Algebra | . | 0.87 | . | 0.99 |
| Alligation | 1.79 | 1.55 | 0.97 | 0.19 |
| Answers | | | .... | 2.91 |
| Arithmetic: General | 0.29 | 0.30 | 0.65 | ...... |
| Arithmetic Symbols | 0.39 | 0.34 | 0.45 | .... |
| Barter | 0.96 | 0.49 | 0.01 | ..... |
| Bookkeeping | 1.18 | 1.98 | 0.81 | 2.32 |
| Decimals | 6.05 | 4.91 | 5.66 | 5.42 |
| Denominate Numbers | 16.00 | 10.84 | 9.02 | 4.17 |
| Duodecimals | 1.23 | 0.84 | 0.50 | .... |
| Factors (H.C.F. and G.C.M.) | | . | ... | 1.10 |
| Federal Money | 1.93 | 2.03 | 2.41 | 1.06 |
| Foreign Exchange | 5.73 | 1.55 | 1.52 | 0.95 |
| Fractions | 5.59 | 11.75 | 11.94 | 11.30 |
| Fundamental Principles | 9.30 | 17.65 | 16.62 | 20.11 |
| General Reviews | .. | ..... | . | 5.47 |
| Introductions | . | .. | ... | 3.57 |
| Measurement of Geom. Shapes | 1.65 | 3.50 | 5.18 | 7.97 |
| Metric System | .. | | | 1.35 |
| Miscellaneous Items | 4.86 | 1.62 | 4.76 | 1.07 |
| Numeration | 1.50 | 2.50 | 3.12 | 3.67 |
| Partial Payments | 0.71 | 1.27 | 3.76 | 0.39 |
| Partnerships | 2.11 | 1.21 | 0.94 | 0.15 |
| Percentage | 12.30 | 8.59 | 13.41 | 14.36 |
| Perm. and Combinations | 0.41 | 0.33 | 0.08 | . |
| Position | 1.21 | 0.59 | .... | ..... |
| Powers and Roots | 3.38 | 5.83 | 4.12 | 2.53 |
| Practice | 4.13 | 1.59 | 0.46 | ..... |
| Progressions | 2.01 | 3.51 | 1.30 | 0.42 |
| Proportion and Ratio | 7.70 | 6.70 | 4.40 | 2.69 |
| Tare, Tret, and Cloff | 1.47 | 0.58 | 0.09 | ..... |
| Weights and Measures | 3.09 | 2.80 | 4.11 | 4.45 |
| Written Problems | 2.51 | 4.26 | 3.68 | 0.89 |

it to the "cardinal principles of education," the view of mathematics within the purpose of social utility, and the view of mathematics as the promotion of disciplinary formalism and rigor. The textbooks, at the same time, underwent technological refinements and introduced several innovations—the teacher's editions became an expected part of an elementary mathematics series, color was generously used throughout the volumes, print was made larger, consumable volumes for students were marketed, and paperback volumes were made available.

The late nineteenth and early twentieth centuries also brought several important changes in the textbook. Man's view of himself began to undergo a radical change. The work of William James (19), John Dewey (11), G. Stanley Hall, Edward Thorndike (37), and Ivan Pavlov (25) is of special significance. Pressures to alter the textbooks from the experimental psychologists were complemented by the business community's desire to eliminate the study of what was considered "obsolete and unnecessary topics." Fincher identifies these changes when his report is compared with the Burry and Smith Eaton studies. Topics such as partial payments, partnerships, permutations, and combinations were given less space than fractions, the metric system, and bookkeeping. Appendices were enlarged and often acquired material slated to be removed from the next edition of the text. Many times the topics that found their way into the appendices were old familiar content that often influenced the selection of a book. The number of rules to be memorized was reduced.

Lessons attempted to utilize play in the design of instructional activities for younger children. The introduction of actual measuring instruments for field work began to be commonplace. Bookkeeping related to personal accounts as well as business was introduced with the inclusion of topics such as family budgeting, postal money orders, order forms, bills, and receipts.

In 1895 McLellan and Dewey published *The Psychology of Number* (22). In this text, they presented the position that attempted to link problems in elementary mathematics to the physical environment.

The mid-twentieth century saw the introduction of a strikingly new mode for the creation of mathematics textbooks, the well-financed massive materials development projects. Teams of mathematicians and mathematics educators began to collaborate in the production of mathematics volumes first for the secondary school level and subsequently for the elementary school levels. The spirit of "change the content to make it modern" swept the land. School systems throughout the country participated in field tryouts of "experimental" materials.

By the sixth decade of the twentieth century, the common-school educational institution had experienced a fantastic growth. It now was a big business with the choice and implementation of instructional programs a substantial portion of this institution's annual business. The elementary mathematics textbook with an accompanying teacher's guide has now firmly established itself as the principal instructional device.

## TEXTBOOKS AS PRODUCTIVE INSTRUCTIONAL AIDS

A common operational definition for a mathematics curriculum is one that uses a description of content items from one or more mathematical specialties. The implementation of such a curriculum is said to occur by the selection and use of particular textbooks. Should anyone seriously challenge this conceptualization, he need only reexamine any one of the popular curriculum descriptions called "scope and sequence charts" so often included as a part of a commercial textbook series in mathematics—or he could read either the syllabi by which state departments of education and school systems describe a mathematics curriculum or the "creative thinking objectives" of a mathematics series as they often appear in advertising copy describing the series. Statements of this sort appear

in experimental textbooks as well as in commercial textbooks. Each of these descriptions is estranged from the actual consumer—the learner and, unfortunately, chooses instead to focus on the inanimate presentation of mathematics.

Such a conceptualization of the mathematics curriculum is clearly unsatisfactory not because of what is said about the pursuit or organization of mathematics, but because of what is not said about the learner and what is absent as guidance for the instructor.

### Observations on Teacher's Manuals

The teacher's edition has come to be the source of the prescriptions for the strategies of instruction. If those who observe classroom teachers in action are objective about their observations, few teachers interpret the manual other than serving as an oral messenger. In the extreme, this set of prescriptions is an unalterable source of "The Truth" for the teacher and by repeated exposures eventually becomes such a source for the learner. In this insidious fashion children are transformed from learners into mimics. The textbook in such an instructional system determines the content and the order of content presentation for the learner and, of course, for the elementary mathematics teacher. Therefore, working within any instructional system, the development of a set of procedures for the selection of a textbook becomes a responsibility that must be assigned the highest of priority ratings.

### Development of a Functional Means of Describing Textbooks

How does one develop a functional description of a mathematics textbook which can be applied in the construction of an effective procedure for the selection of mathematics textbooks? The lesson to be learned from the history of mathematics textbook development is clear: the content and the instructional procedures have undergone continual change, and there is every reason to believe they will continue to change. The history of mathematics textbook development coupled with observations on the evolving character of the mathematics discipline argues strongly against the usefulness of developing textbook selection procedures based on the appearance of particular content or the presence of special vocabulary or the organization of material according to a recommended mode of instructional presentation. The dynamic state of the mathematics discipline and of the instructional strategies employed in the communication of the discipline renders the application of such criteria inefficient.

### Search for Feasible Criteria

But what, then, constitutes a feasible set of criteria? What element of description can be employed which is independent of any changes in content or instructional procedures and still provides an objective means of comparison? Quite a sticky question, indeed! The assessment and selection of mathematics textbooks is a persistent problem for school systems. The mathematics textbooks available today are quite versatile. There are texts for almost any learner audience one can imagine at the elementary or secondary school level. In fact, many of the large publishing houses have numerous mathematics textbooks competing for use among the same audience.

Suppose, just for the moment, that these conditions did not hold. Instead, imagine that all mathematics textbooks intended for a given age group of learners possessed these four characteristics: (1) contained a one-to-one correspondence between the content and the order of presentation, (2) were written in the same style, (3) used the same vocabulary to describe mathematical tasks, and (4) provided equivalent space for similar topics. That is, imagine conditions for which assessment and selection would be simple—it would be based upon reliably observable criteria such as the cost of the volume, the construction of the binding, the attractiveness of presentation, the weight of the paper, the

size of the print, and so on. Such a set of conditions, fortunately, does not prevail—fortunately, because content and organizational differences produce competition, and competition is one principal catalyst of innovation.

Why bother with such a contrived situation? Because of the clue provided by the characteristics of the items which did remain unaffected and do provide possible categories to be employed in the judgment. The categories are characteristics whose description yields to *reliably observable criteria.* Is it possible to employ an element of description that would yield to reliably observable criteria for the descriptions of textbooks, even though all texts are different by varying degrees? Clearly content is not a candidate for such an element, nor are any of the non-learner-related categories, but *reliably observable performance* does come through loud and clear as a likely candidate.

Suppose we agree to describe each textbook in terms of reliably observable performances the learner is expected to acquire from the instructional lessons. Further, suppose we also agree to provide descriptions of the situations that can be structured so as to maximize the learner's probability of acquiring these specified (and, in the author's opinion, desired) mathematical behaviors. What does this mean one would need to provide in a textbook description, and how would it be accomplished?

Four questions may be employed as a means of characterizing the requirements:

1. What (that can be reliably observed) will the learner be able to do *after* he has completed work with a particular instructional sequence that he was unable to do *before* he was exposed to it?
2. What is the evidence that the learners exposed to the particular instruction actually acquire the stated behavior?
3. Given the statement of an objective in terms of a reliable observable behavior, what capabilities must the learner already possess which will optimize his probability of acquiring the desired behavior?
4. What is the evidence that the stated behavioral prerequisites assist the learner in the acquisition of the behavior described by the objective?

The four questions are purposely learner-oriented and attempt to focus on the question of assessing success in terms of reliably observable outcomes.

## Advantages of Behavioral Description

Consider the consequences of accepting the responsibility of providing a detailed response to each of these questions. What advantages will accrue from accepting such a responsibility? In other words, why undertake such a task?

The advantages of having available a behavioral description for each collection of mathematics materials are especially dramatic for those charged with the responsibility of designing or selecting a mathematics curriculum for a school system. Such descriptions would provide the school personnel with an objective means of comparing various instructional programs in mathematics. The school system will be able to select a mathematics curriculum by means of a well-ordered sequence of events: (1) a collection of human capabilities that were desired as outcomes of an instructional program in mathematics for that school system would be identified by the local school personnel, (2) a systematic examination would be made of the behavioral descriptions and the supporting evidence of actual learner accomplishments published by each of the mathematics curricula being considered as candidates for adoption, (3) a sorting of the mathematics curricula that best satisfy the behavioral requirements specified by the school system in task 1 would be conducted, and (4) the selection of a particular instructional sequence would be made on the basis of experimentally verified responses to the examination, task 2 of the sequence, included as part of the package of instructional materials. Figure 3.8 traces this series of events diagramatically.

Behavioral descriptions of the intended out-

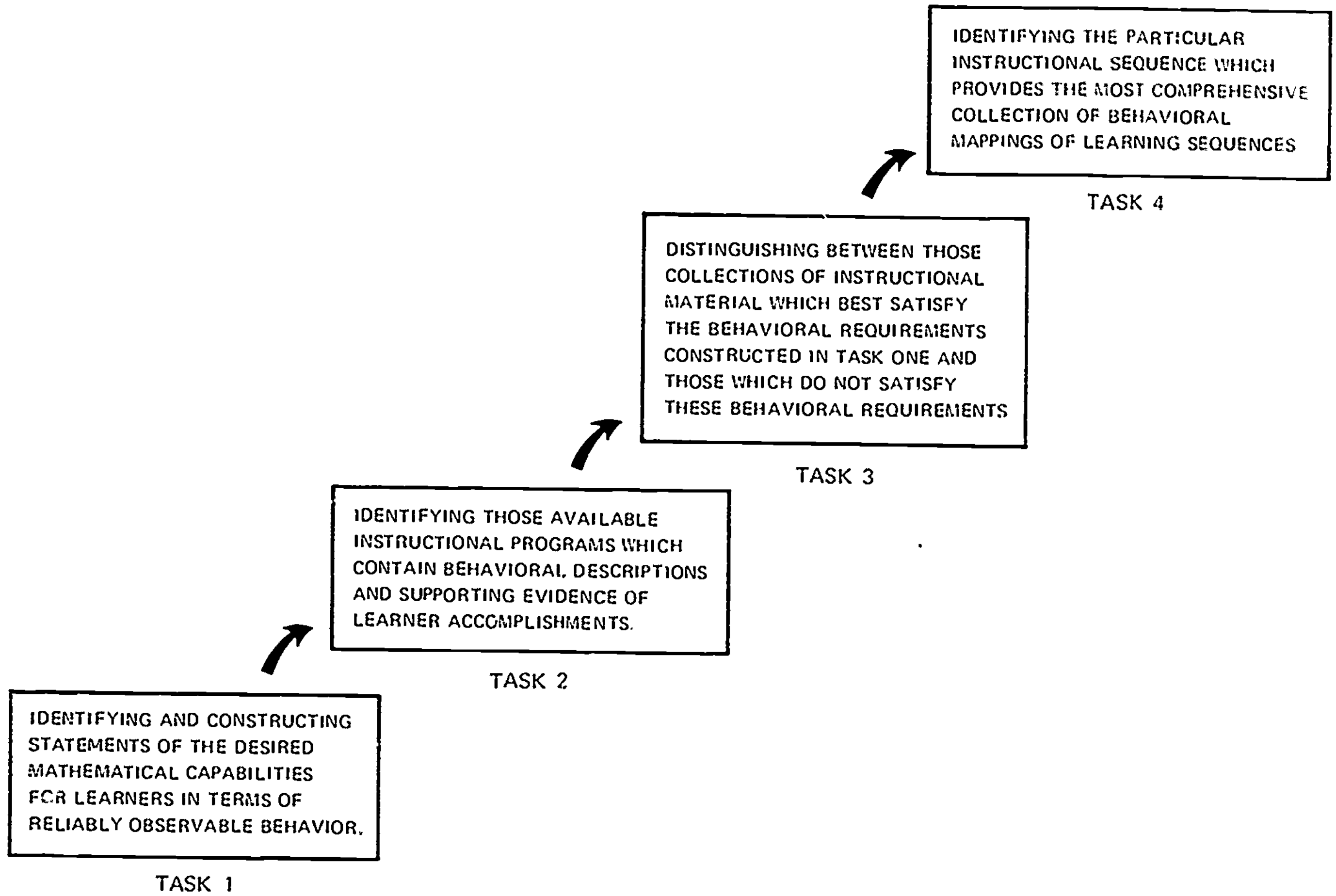



FIGURE 3.8. *Tasks in the selection of an instructional sequence*

comes of an instructional program in mathematics are as valuable to the classroom instructor as they are to those who must select mathematics programs. With this information the instructor is explicitly informed as to the desired outcomes of each instructional component within the entire mathematics curriculum, for which he is responsible.

So now you are saying, "OK, OK, I'm convinced, you've sold me on the possibility of using reliably observable behavior as the fundamental unit for describing instructional materials such as the mathematics textbook" (so pretend if you're not convinced or go on to the next chapter in the yearbook)—but now you ask, "What can I do about it?" If one attempts task 2 in Figure 3.8 using the mathematics textbooks commercially or experimentally available, one would probably find that he has identified the empty set of mathematics texts. A like outcome will be observed whether the volumes are intended for the elementary or the secondary school level. Textbook publishers simply do not provide such information about their products and do not appear to be interested in providing such information. It is an interesting exercise in futility to write to the publishing houses in the United States and request, for a particular textbook series, the evidence supporting the claims. The ambiguous nature of 'understanding," "mathematical literacy," and "appreciating" are seldom better illustrated than in the responses one receives as a result of such requests.

What can be done to alter the situation? One possibility, of course, is to refuse to purchase any textbooks that do not provide a behavioral description and supporting evidence of actual learner accomplishments and to encourage the funding agencies to refuse funding to "experimental" mathematics development projects that

do not provide similar descriptions and evidence. A drastic step, yes! But a necessary one if the desired changes are to be realized. In reality the professional community of mathematics educators would be asking for very little from the producers of instructional materials. If one writes instructional materials and expects others to use these materials, he surely should be able to identify successful learners of his material by the performances they exhibit. All the community of professional users would be calling for is that the author share this information. If, however, an author does not supply this information, we are being asked to use instructional materials for which even the author does not provide a means of distinguishing a successful learner from an unsuccessful one. Furthermore, without evidence of learner accomplishment we, the consumers of textbooks, are being asked to purchase materials whose worth is unestablished.

### *Recommendations*

At this juncture in the discussion it seems appropriate to pause and make three recommendations based upon the observations described up to this point.

RECOMMENDATION 1. The National Council of Teachers of Mathematics should construct, endorse, and publish a set of technical standards for mathematics curriculum evaluation.

RECOMMENDATION 2. The National Council of Teachers of Mathematics should be charged with the responsibility of reviewing all instructional material intended for elementary and secondary school instructional programs in mathematics. Each review published by the NCTM should include a statement of whether the material meets these criteria: (1) performance descriptions of objectives, (2) data on the acquisition of the behaviors described as objectives, and (3) a statement of where the performance list and data are available. The criteria established by the set of technical standards recommended above should be used in constructing the review.

RECOMMENDATION 3. A moratorium should be declared on the purchase of elementary and secondary mathematics textbooks or collections of instructional materials until the author and or publisher provides a description of performance objectives and evidence of accomplishment of the stated performances for the material. The evidence should be obtained from a sample of the intended population of users. Both the sample and the population should be clearly described.

### *Active Teacher Involvement*

The authors and distributors of instructional materials are not, however, the only ones in need of change. Consider the problems inherent in an attempt to accomplish task 1 in Figure 3.8. The most pertinent question would appear to be "How does one describe mathematical capabilities in terms of reliably observable behaviors?" Assuming that active learner participation is desirable, the following sequence was developed with the objective that after completing this sequence you, the reader, will be able to—

1. distinguish between a behavioral and a nonbehavioral objective;
2. describe one definition of a behavioral objective;
3. describe one procedure for constructing a working set of action verbs;
4. construct a behavioral objective from a given nonbehavioral objective.

## A SET OF PROCEDURES FOR DESCRIBING AND CONSTRUCTING BEHAVIORAL OBJECTIVES[1]

Just what purpose do statements of objectives serve? Do you use the statements of objectives found in textbooks or courses of study to plan your instructional program?

Yes or no? ______________

1. This instructional material is adapted from "Sessions 1 and 2" of volume 1 of the *Maryland Elementary Mathematics Inservice Program* (MEMIP) of the University of Maryland Mathematics Project (24), prepared under a grant from the U.S. Office of Education.

Be honest now, no one is going to collect your responses. Suppose you plan an instructional session from a teacher's commentary that contains the usual statements of objectives. Now suppose all the statements of objectives in your book were eliminated, for example, by covering them with tape. How would your instructional planning be observably affected? Would your planning be different?

Yes or no? ________________

There are few teachers from the inexperienced to the experienced who would say that the usual broad statements of curriculum objectives (as they are usually constructed) actually contribute to their planning for, or execution of, instruction. Why is this? Is it simply that objectives can serve no useful instructional purpose? Must curriculum objectives remain an instructional window dressing, or can they be made to assume a more functional role?

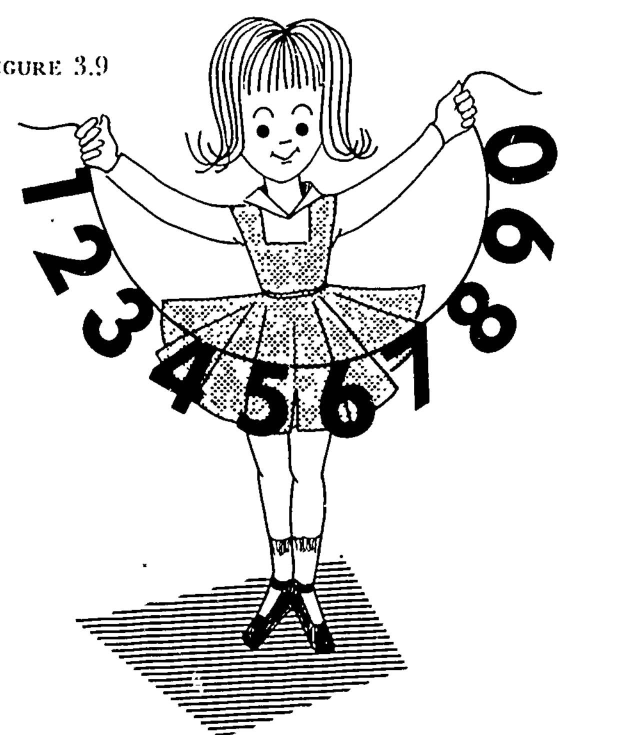
You are about to participate in an instructional program. The program is intended to aid the reader in the acquisition of certain competencies. Among these competencies are the ability to identify ambiguous descriptions of objectives and the ability to change ambiguous descriptions of objectives to instructionally functional descriptions of objectives.

For the remainder of this section you will be asked to respond at various intervals by writing on a response sheet. If you are to acquire the competency expected from this portion of the exercise, you must respond when asked to do so in the program. It is important that you be an active participant.

The objectives of any instructional materials should be stated in a clear, unambiguous manner. Certainly there are few who would refuse to acknowledge this as an important characteristic, applicable to all statements of curriculum objectives. Statements of objectives for mathematics programs usually do not satisfy this requirement of specificity and clarity.

Consider the illustration in Figure 3.9 with the statement of a familiar objective—one common to experimental and commercial mathematics curricula available today.

FIGURE 3.9




The learner will build an understanding of the system of whole numbers.

What characteristics would you ascribe to an instructional program in mathematics if it is attempting to achieve this objective? Is the statement of the objective phrased in such a manner that several other mathematics teachers, working independently, would arrive at the same interpretation of the meaning of this objective?

Yes or no? ________________

*Have you made your selection? Go ahead, write down a choice!*

The illustration in Figure 3.10 identifies a stated objective of numerous modern mathematics programs which you are almost certain to recognize.

FIGURE 3.10


The learner will acquire

an appreciation of the

STRUCTURE of mathematics.

What activities would be necessary to achieve this objective? Do you suppose other mathematics educators would identify the same components as necessary to achieving this objective?

Yes or no? ________________

Have you made a written response? Fine! Is the variety of possible interpretations for these two objectives surprising? Not at all, when one considers their lack of specificity. In fact, that which is truly remarkable is the skill textbook authors have demonstrated in constructing ambiguous statements of objectives! Perhaps the most startling observation, however, is not the widespread use of these statements, but rather that most teachers so complacently accept these statements! As teachers, we acknowledge, or at least tacitly accept, these statements as reasonable descriptions of our goals and use them as the basis for justifying the selection of certain instructional materials or the performance of particular instructional acts. Each of these decisions is made or action initiated even though there is this disagreement as to the meaning of these objectives. Consider the statement in Figure 3.11 of an objective for an instructional program in mathematics.

What specific instructional activities in mathematics would you design to achieve this objective? Do you suppose other mathematics instructors would reach a similar decision as to the meaning of this objective? Come on now, make a choice.

Yes or no? ________________

Certainly this third objective is unlike the first two in that it names a particular portion of mathematics, namely arithmetic skills. However, narrowing the content from all of mathematics to arithmetic skills is not a solution to the interpretation dilemma. A large number of varied interpretations can still be made. Such specification is useful but is not sufficient.

The three previous illustrations suggest that the description of an objective needs to be specific if there is to be any hope of attaining uniform interpretation. The need for each mathe-

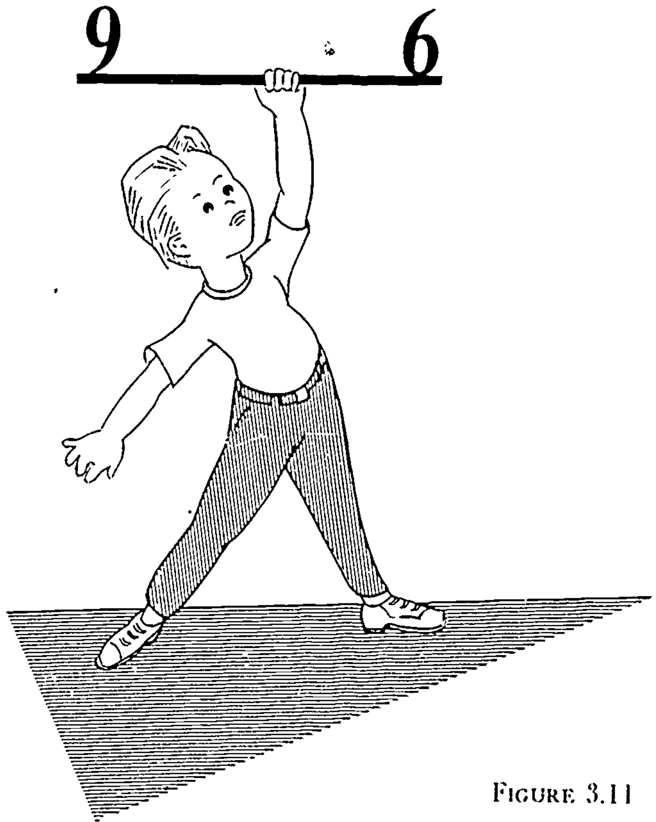



FIGURE 3.11

The objective is to strengthen his arithmetic skills by relating them to basic principles.

matics objective to be interpretable is especially important for those charged with the construction and/or implementation of an instructional program's objectives.

However, implementors of mathematics curricula are not alone in their acceptance of ambiguous objectives. Consider this statement (Figure 3.12) of an objective of contemporary mathematics textbook authors.

Now suppose you are one of four committee members charged with the task of independently observing students who have been exposed to instructional materials designed to aid the learners in acquiring this objective. Further, let's suppose that, based on these observations, you are to make a decision as to whether each student you observed had or had not attained the objective. Does the description of the objective in the previous illustration identify the specific performances which you would look for in your observations?

Yes or no? ________________

Just what performances one would be expected to observe in learners who had acquired a familiarity with these properties is not contained in the statement of the previous objective.

The description of an objective must identify the observable behavior that a learner who has successfully achieved the objective is expected to

FIGURE 3.12

The learner will acquire a familiarity with the properties of a field of numbers.

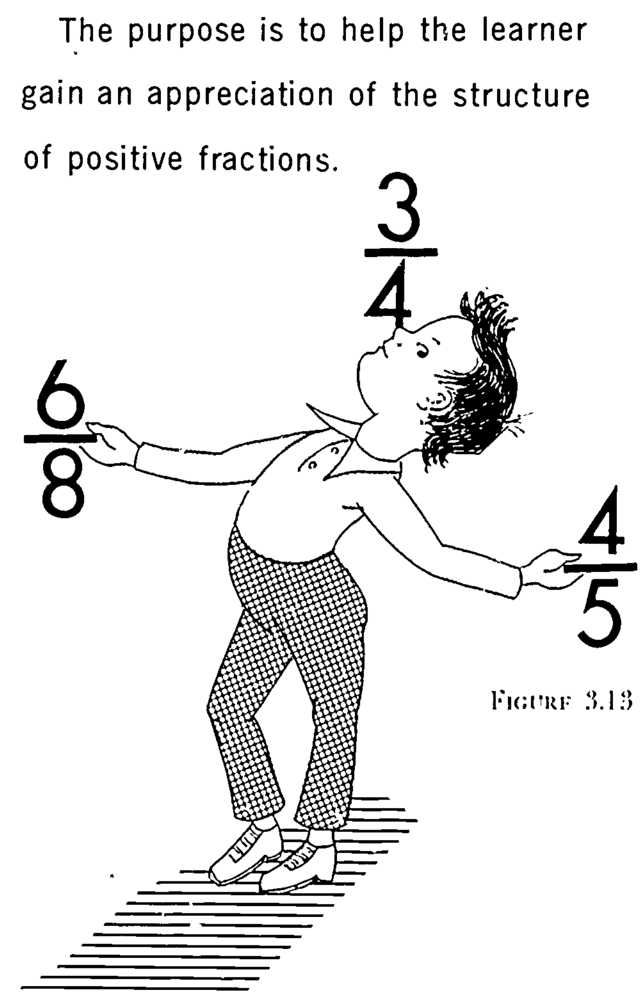



FIGURE 3.13

have acquired. Read the objective stated in Figure 3.13 with the purpose of identifying the observable behaviors a student should be able to exhibit if he has achieved the competency described by the objective.

Are there observable behaviors identified in the statement of this objective? Are there behaviors described in a way that would enable you to separate the successful from the unsuccessful ones?

Yes or no? __________

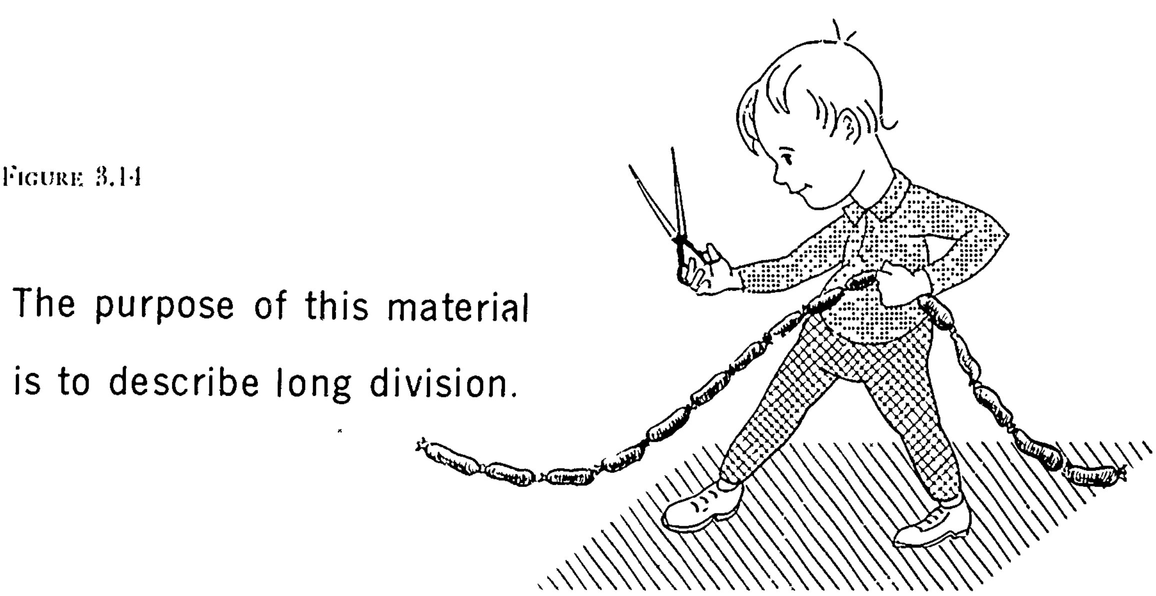
Since you decided yes, the description does identify the desired observable behaviors, and you can, of course, name them. Oh? You say you decided no? Good for you! The statement does not contain any such performance specification, and therefore the appropriate response is no.

The statement of an objective should describe desired learner behaviors. In order to be able to interpret an objective, these behaviors should be clearly described. The intent of an objective is reliably communicated by descriptions of observable behavior. Consider this next objective (Figure 3.14) in the context of how effectively the statement communicates the desired behavior of the objective.

Does this objective describe the behavior to be acquired by the learner?

Yes or no? __________

FIGURE 3.14



Does the statement in the previous illustration describe an environment that requires the presence of a learner?

Yes or no? ______

Have you made a written response? Don't read on until you have.

As you are likely to have concluded already, the description of the previous objective does not identify the behaviors the learner is to acquire. Nor, peculiarly enough, is a learner even necessary, since the instructor might describe without any learner's being present.

Read the description of the objective contained in Figure 3.15 and decide whether the learner is necessary to the achievement of this objective.

What did you decide about the necessity of a learner in the achievement of this objective? Is he necessary, or is he not necessary?

______

A learner is necessary to the acquisition of the behaviors described in the previous illustration. You don't agree? A detailed analysis might be given even though no learner is present.

Objectives must be constructed so as to be specific descriptions of what a learner is to do or say. Only by fulfilling this descriptive requirement of learner performance can objectives become functional for the innovator, planner, developer, teacher, and learner. Objectives must be constructed so as not to allow for the exclusion of a learner under any interpretation.

Ambiguity is often cloaked in the garment of prestigious phrases. A few of the most common of these phrases are shown in Figure 3.16 together with a ringer—one phrase that does not belong because it conveys specifically a desired behavior.

Identify the phrase that does describe an observable performance. Which one is it?

______

Select one. Don't hesitate, write it down. Now!

Did you select "builds an understanding"? It's not, you are on the right track. Perhaps you picked out the "appreciating" phrase, or the "feeling" phrase, or the "awareness" phrase, or the

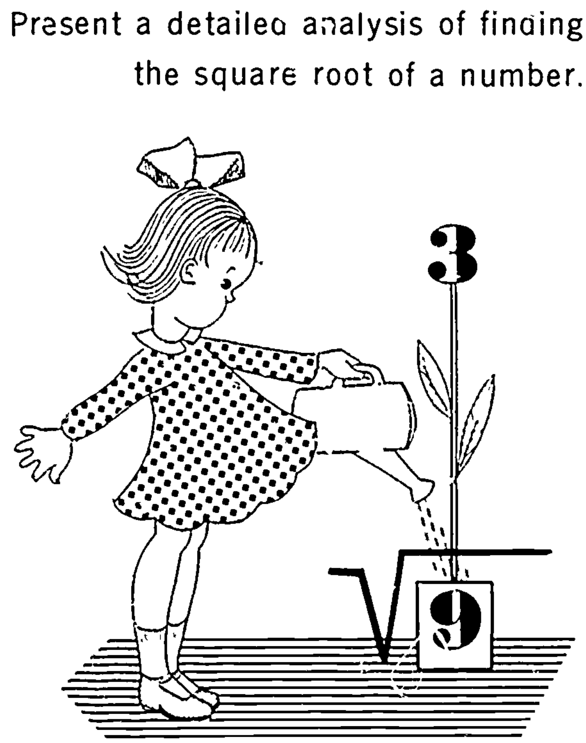



FIGURE 3.15

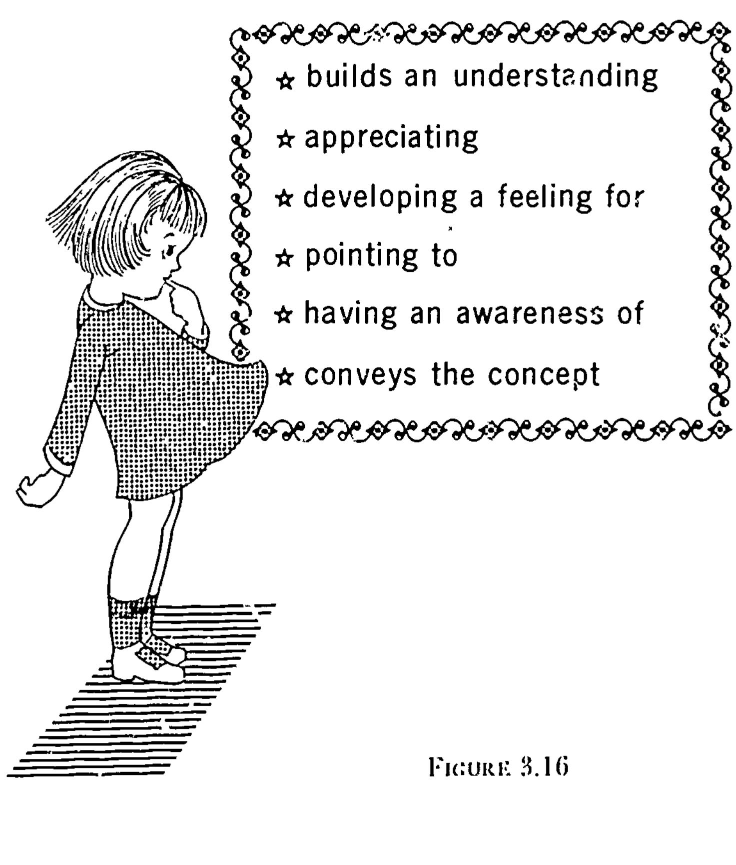



FIGURE 3.16

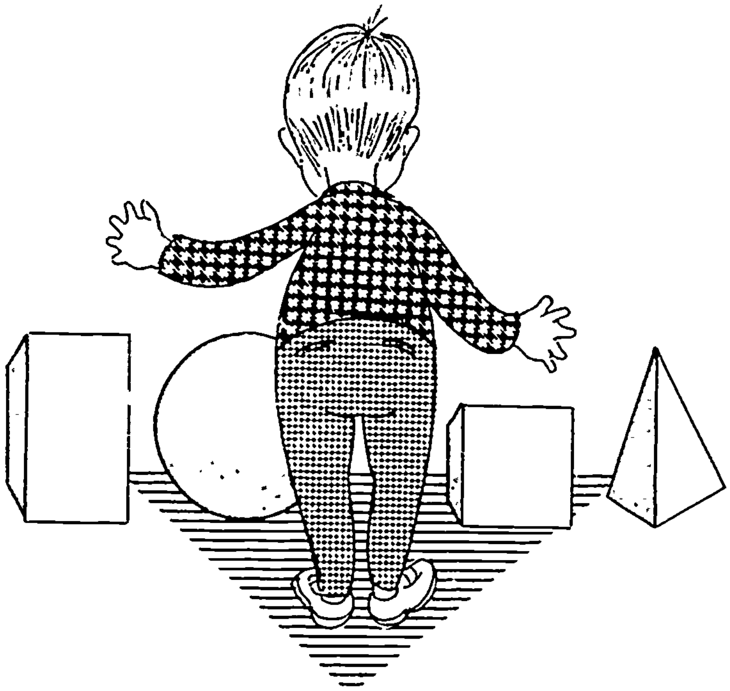


FIGURE 3.17

"conveys" phrase. No? Good. "Pointing to" is the appropriate choice and should have been identified without difficulty.

Suppose a variety of three-dimensional objects such as those in Figure 3.17 were placed in front of you.

Let's suppose you have been asked to identify the cube. Would identifying be interpreted as demanding some sort of an observable action on your part? Observable or vague?

______________

It requires an observable action, of course. You might carry out the identifying by pointing to an object, or by placing your fingers on an object, or by actually picking up an object.

In Figure 3.18, two objectives are described. Read them carefully and select the description of the objective which would have the less ambiguity.

Did you select statement A or B? ______. Because "fully understand" has many possible meanings, the more specific objective would seem to be statement B.

A

The learner will comprehend and fully understand the procedures used in the division of fractions.


B

The learner will be able to identify the closure property in finding the quotient of two fractions.

FIGURE 3.18

Examine the illustration in Figure 3.19, which contains two descriptions of objectives. Which of these objectives is the more specific description of desired learner performance?

A or B? _______________

I suppose that you decided that neither of these was a behavioral objective. No? Oh! You made a choice that only statement A was a behavioral description or that only statement B was a behavioral description. No? Good! Did you decide both A and B are behavioral descriptions of desired student performance? If so, you have acquired the behavior of being able to identify behavioral objectives.

Should an observable performance or action be identified in the description of a behavioral objective?

Yes or no? _______________

Even the most casual consideration of the necessary requirements of a performance description suggests that each description must contain some action word or phrase. What class of words most often describes action in English—nouns, verbs, adjectives, or what?

_______________

Of course, most often action is communicated by verbs. But which action verbs shall we use? All of them or just a few? Are there some particular tactics that are more effective?

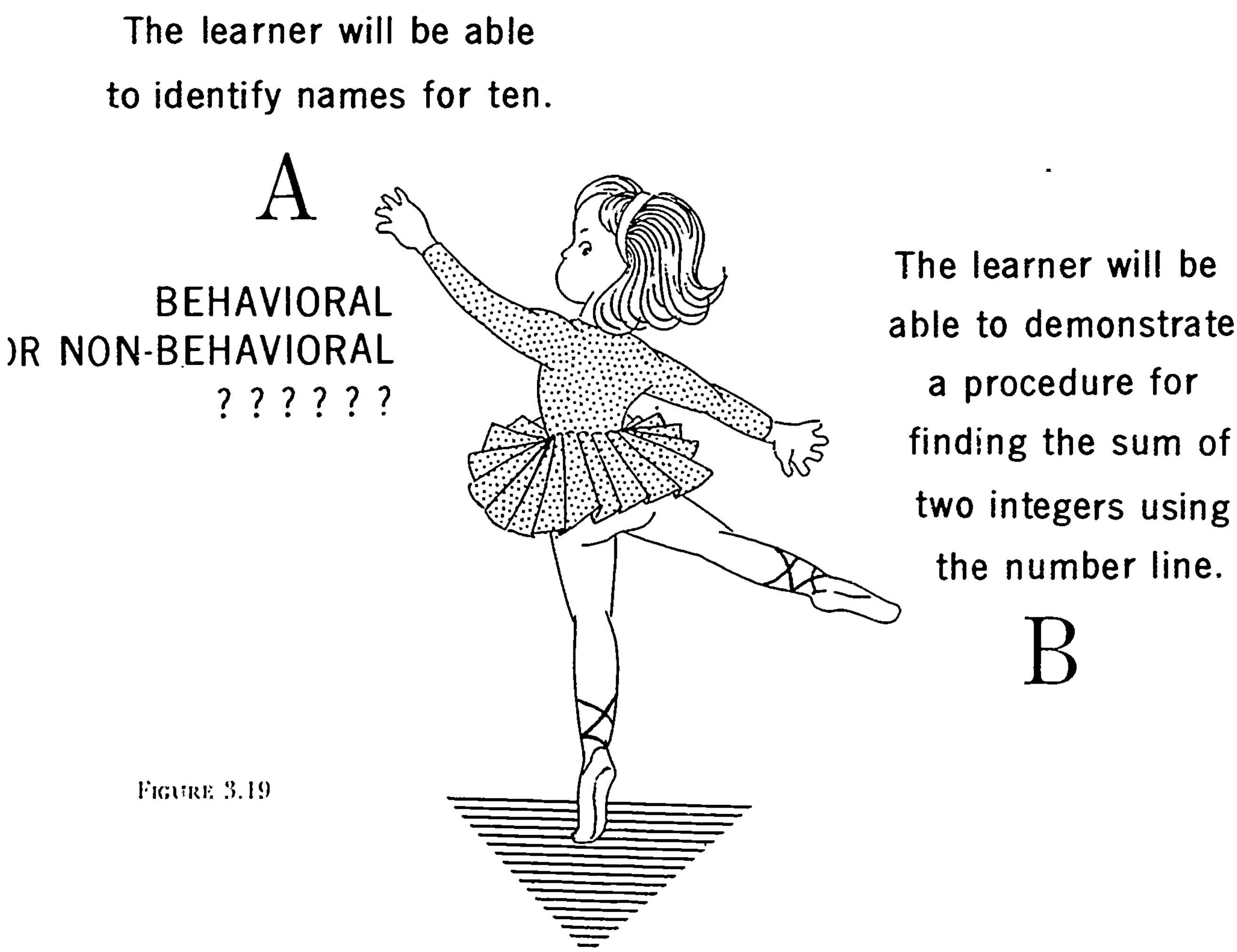



FIGURE 3.19

The following material describes one technique that has been used with satisfactory results. The tactic is to define each action verb in such a way that a class of performances is defined. The best way to discuss the technique is to have you participate in it. So let's have you do just that!

You will need the following materials for your participation.

Rectangles: red. 5 cm by 8 cm; white. 8 cm by 10 cm; blue. 5 cm by 10 cm

Triangles: red. equilateral: white. obtuse: blue. isosceles

Spherical piece of hard candy

One necessary component in the description of a behavioral objective is an action verb. But now just wait a minute! How many possible action verbs are there in English—a few or a great many?

__________

You could decide to use any of a large variety of action verbs. The variety itself, however, may contribute as much or more to maintaining the ambiguity than to facilitating clarity.

The problem would now appear to be one of reducing the number of possible action verbs used in the description of objectives without reducing the variety of learner performances being called for by the objectives.

Spread out the materials in front of you. You will be asked to make several performances. Carry out each task as best you can.

1. Pick up a blue rectangular region. (Go ahead. don't be bashful, pick it up. Put that object back with the materials.)
2. Now select a blue rectangular region. (Have you made a selection? Good! Put that object back with the materials.)
3. Identify a blue rectangular region.

Notice that three different action verbs were used in initiating the three performances you made. One of the action verbs was *select*. What were the other two action verbs?

__________

Did you write *pick up* and *identify*? Wonderful!

What are words that have similar meanings in English called?

__________

Think back. you are certain to have encountered the term before. Since words that have similar meanings are called *synonyms,* action verbs that elicit similar behavior are called *behavioral synonyms.*

Are the three verbs *identify, pick up,* and *select* behavioral synonyms?

Yes or no? __________

Of course they are!

Why use all three of these action verbs? If behavioral objectives are to be specific and describe observable behavior, would it seem sensible or not sensible to use one verb in place of all three?

__________

The sensible thing is to agree upon one action verb and use it as the name for a class of performances. Which one of the three verbs shall we agree to use? Since it does not seem to make much difference, let's agree to use *identify*.

Identify all of the triangular regions in the materials and place them off to the side. Do you have them all? Arrange the triangular regions from the one with the least area to the one with the greatest area. Now order the triangular regions from the one with the longest base to the one with the shortest base.

After identifying the two sets of objects, you performed two tasks. The instruction for each involved an action verb. One of the action verbs was *arrange*. Name the other action verb.

__________

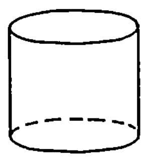
Were the performances you exhibited alike or different?

__________

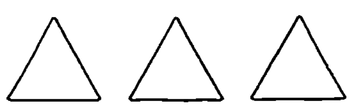
If you are reading this before you have written a response to the last question, you are not playing the game. Go back and try to write a response to the question. The conclusion that seems justified is that these two action verbs are *behavioral synonyms* (call for a similar performance). Let's agree to use *order* whenever such a behavior is called for in the description of a behavioral objective.

What do you call an object shaped as shown below?

What is the name of a three-dimensional object shaped as shown below?



Tell the number of triangles pictured here.



Are the performances required by these three tasks similar or different?

Similar, of course. What action verb would you use to describe these behaviors?

Any number of different action verbs are possible candidates. A few of these behavioral synonyms are *tell, specify, call for,* and *name.* Let's agree to use *name.*

The agreements about action verbs made up to now mean that when you describe a behavioral objective and the performance is—

1. "choosing the rectangles," you would write "__________ the rectangles";
2. "classifying the objects from the heaviest to lightest," you would say "__________ the objects from heaviest to lightest";
3. "telling the colors in this painting," you would write "__________ the colors in this painting."

If you're reading this before you have responded to the previous three tasks, go back and respond. Did you write *identifying, ordering, and naming?* That's a collection of acceptable responses. Now you're really catching on!

Look at the trapezoids in Figures 3.20 and 3.21. Show how you would decide whether the following statement is true or false:

Segment *m* is longer than segment *n*.

Demonstrate how you would decide whether this statement is true or false.

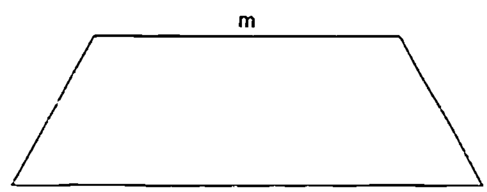



FIGURE 3.20

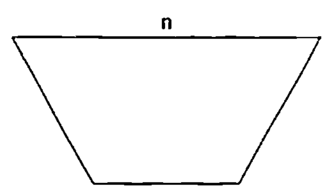



FIGURE 3.21

One of the action verbs used to elicit these two performances was *show* and the other was__________. *Demonstrate and show* are__________ __________ because they elicit the same type of performance. Let's agree to use *demonstrate* as the action verb for this class of performance.

Consider the spherical object among your materials. Suppose that someone in another city had an object just like yours, together with a number of similar objects. Your task is to identify and name a sufficient number of characteristics of your object so that the other person will be able to identify the object you are talking about. Start naming!

If you have identified and named only color and shape, your description is not adequate because it fits most of the objects the second person has in front of him. Add a few more characteristics. If you added mass, volume, diameter, and thick-

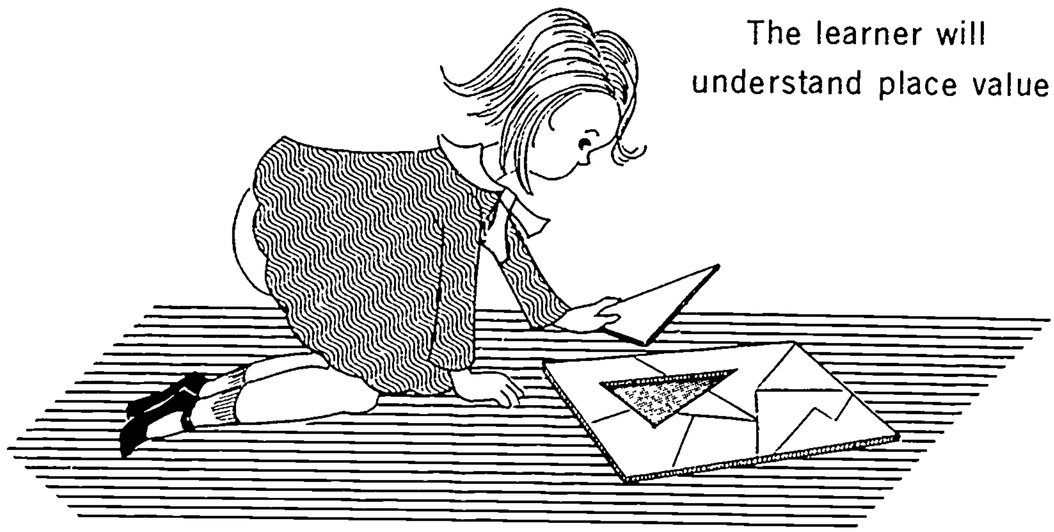



FIGURE 3.22

ness you would be much closer to a satisfactory description.

What shall we call such behavior? What action verb should we use? Could we use *identify*? ________ Could we use *order*? ________

Since *identify* requires the individual to select an object that has been named for him, this action verb does not seem satisfactory. In the same way, *order* does not seem an appropriate choice. The distinctive characteristic of this new behavioral class is that the learner identifies and names the characteristics or properties. More than one characteristic is usually included, and there must be a sufficient number of these characteristics so that a second individual will be able to identify what is being discussed.

How then shall we name this class of behaviors? Suggest a possibility.

---

Many choices could have been made. The particular action verb that seems most appropriate is *describe*. The *describe* behavior involves the individual's identifying a sufficient number of characteristics of an object or action so that a second person would be able to identify it without having it pointed out.

The preceding activities should suggest the procedure that might be followed in the construction of a working set of action verbs.

Now that we have agreed on the meaning for a few action words in the construction of behavioral objectives, let's turn our attention to the problem of constructing a behavioral objective. Read the objective in Figure 3.22.

Is the objective described in the illustration above behavioral?

Yes or no? ________

No, of course it is not. With the use of one of our agreed-upon action verbs rewrite the nonbehavioral objective described in the illustration above and make it a behavioral objective.

---

If you have not completed rewriting the objective, do not read this part. Go back and do it now. When you have completed the task of rewriting the objective, read it over to see whether you have: (1) used one of the action verbs, (2) described the situation in which the learner should exhibit this particular behavior, and (3) indicated the nature of the product the learner is to produce. Learner products may be quite varied: a sentence, a word, a drawing, a series of check marks, and so forth.

Figure 3.23 contains a few of the possible descriptions of behavioral objectives which would have been constructed from the nonbehavioral objective. Notice that it is not a difficult task to translate this particular objective into a behavioral objective. The translation is accomplished by stating what one would look for in terms of learner performance rather than in terms of the nonperformance description that the word *understand* conveys.

Examine the objective described in Figure 3.24. Rewrite this objective so that it is behavioral.

______________________________

When you have completed the task of rewriting the objective and making it a behavioral objective, read these next statements:

1. Did you use one of the action verbs we have agreed upon?
   Yes or no? ______________
2. Is the situation in which the learner is to exhibit this performance clearly specified?
   Yes or no? ______________
3. Is the nature of the output that the learner is to provide clearly specified, as well as any restrictions on that particular output?
   Yes or no? ______________

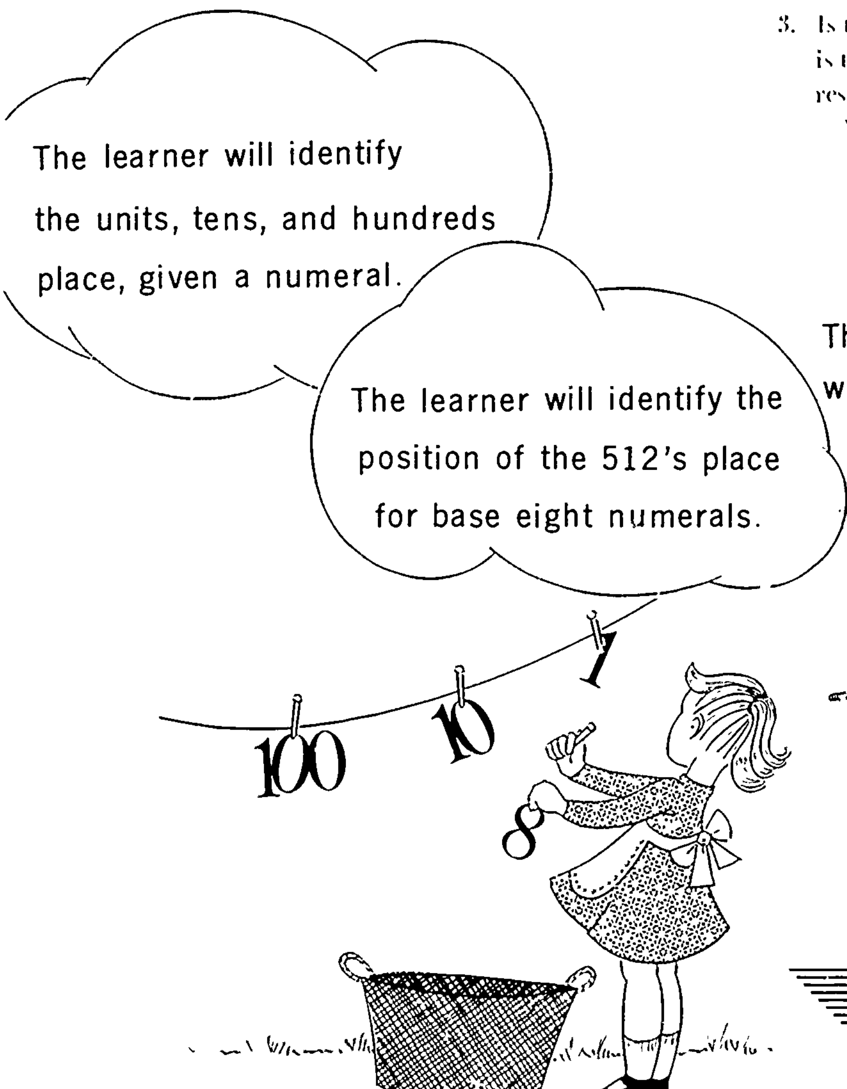



FIGURE 3.23


FIGURE 3.24

If you were not able to respond yes to each of the three questions, go back and correct whatever difficulties you have identified. Does the description of the objective in the illustration identify the specific performances you would look for in your observations?

Yes or no? ________________

The preceding material on behavioral objectives serves as an introduction to behavioral description. The instruction is incomplete. The reader should have acquired sufficient competence so that it is possible to pursue the relationship of behavioral description and instructional aids just a bit farther. One way to characterize the potential effect of behavioral objectives, evidence, and learning hierarchies on textbooks is depicted in Figure 3.25.

## INSTRUCTIONAL AIDS AND BEHAVIORAL OBJECTIVES

Instruction and instructional aids are intimately related to one another. In particular, instructional activities include the selection of instructional aids, the design of the instructional sequence, the identification of prerequisite behaviors, and the description of those observable performances that signal success and those that signal failure. The textbook is an excellent example of an instructional aid that exhibits these influences on instruction. The magnitude of the textbook's influence on mathematics instruction in the United States is clear from educational history. The textbook, more than any other single instructional aid, affects the content of instruction, the

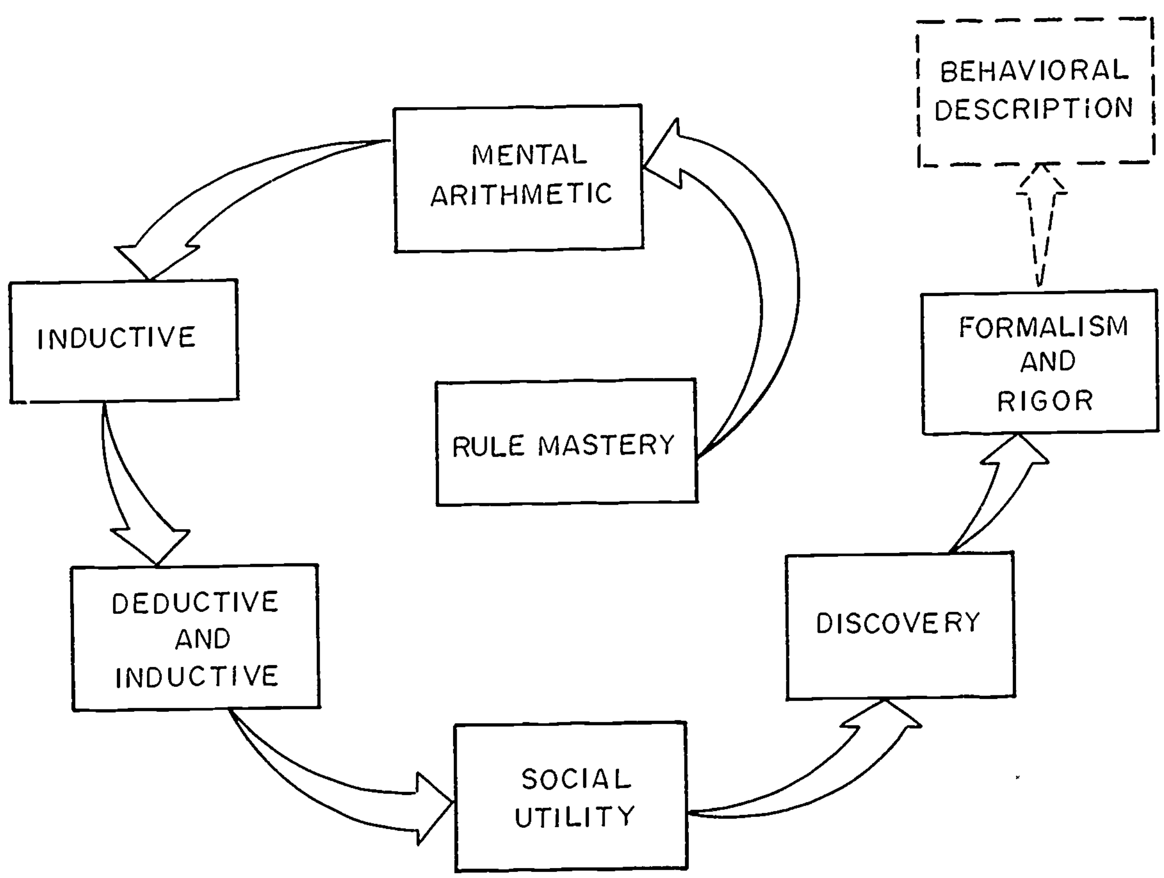



FIGURE 3.25. *The shifting emphasis of the mathematics textbook*

sequence of instructional activities, the level of expectation for the learner, and often even the instructional environment. To observe that the textbook currently occupies a position of singular importance in mathematics instruction is to border on understatement. This should not be and is not intended to be an endorsement of this state of affairs, it is merely an observation of the current reality.

The present activity directed at the study of teacher behaviors may ultimately bring about significant changes in the way teachers teach, but this has not yet occurred. The marvelous technology of the twentieth century may completely remold our everyday means of instruction as it has our means of communication, but this has not happened yet. The data on success or failure are not yet in the literature. And these realities leave us with the present, where the textbook remains the dominant instructional aid (and on occasion, the instructor as well).

The textbook communicates instructional purposes. Both the teacher and the learner are expected to receive and act on the purposes communicated by the textbook. Instructional purposes are expectations. Some would agree that certain of these expectations (the "really important" ones) cannot be translated into observable performances. The existence of such learning expectations is questionable. Because such desired outcomes cannot be translated into observable action, it is not possible to recognize who has achieved the objective and who has not. It would appear that such statements of purpose fill lines of print in a text but have no substance in the context of learner outcomes and, therefore, serve no instructional purpose.

The proponents of such objectives (those that cannot be translated into observable performances) espouse a most peculiar position. Something or other is a purpose of instruction, but the author of the instructional activity cannot tell the teacher how to recognize when the objective has been achieved. But don't be too critical of such stupidity: perhaps there is a place for vague, ambiguous statements of goals. A possible purpose for the description of expected outcomes that do not identify observable performances is to display an ignorance of desired effect without the penalty of public ridicule.

The conditions that allow the teacher to identify when he has succeeded and when he has failed are fundamental to the instructional act. Basic to this identification is the description of what behaviors (observable performances) are expected to be acquired by the learner. If the description of objectives for instructional activities in behavioral language is desirable, and it would seem that such is the case, then the teacher must be given assistance in acquiring the competencies necessary for the construction of behavioral objectives and learning sequences made up of behavioral objectives. A previous section of this chapter briefly describes one collection of procedures for constructing behavioral objectives (23; 39). There is a growing literature that deals with the acquisition of these competencies associated with the construction of behavioral objectives. The literature on learning sequence construction is small and less well known, but some representative work is provided by Gagné (16) and Walbesser (40).

The assumption now is that the reader is willing to accept (at least for the remainder of this chapter) these axioms: (1) instructional goals can be described in terms of reliably observable behaviors, (2) each instructional act has at least one behavioral objective associated with it, (3) a minimal set of behavioral objectives can be constructed for any instructional activity, (4) mathematical learning is subject to the accumulative acquisition of behavior, and (5) behavioral objectives can be sequenced into descriptions of accumulative learning sequences called learning hierarchies.

This set of assumptions affects five activities associated with instruction: (1) the selection of an instructional aid, (2) the organization of instruction, (3) the recognition of success or failure in instruction, (4) the revision of instruction, and (5) the design of instructional sequence.

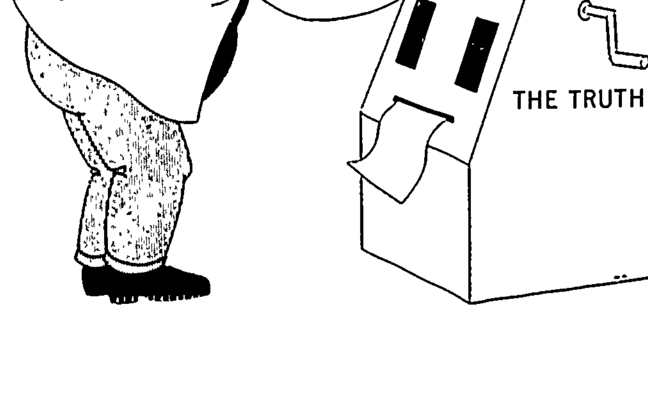
By way of illustrating the relation between

behavioral objectives and instructional aids, the textbook is chosen as the instructional aid under discussion. The selection of a textbook is dealt with in many published guides. Such guides may be lists of steps to be taken, or they may be detailed illustrations of the procedure being applied—or something between these two. These guides come in many varied formats. State departments of education, educational book divisions of publishing houses, local school systems, and professional organizations such as the NCTM are included among the sources issuing textbook selection procedures.

The procedures recommended vary to some extent, but not as much as might be expected considering the variety of sources of authorship. Some recommended procedures require the user to employ a list of objective criteria; some mix objective criteria with subjective value judgments. The deficiencies of these selection practices are well known and need not be enumerated here. Let it suffice to observe that the criteria commonly suggested are items such as these: time cost considerations, general attractiveness, judgment of the relevance of examples, level of vocabulary, graded variety of exercises from easy to difficult, variety of examples, authority (person, project, or school system) endorsement, publication date, durability, needed in-service training, and required equipment. Although such a list contains many characteristics relevant to the selection of a textbook appropriate to a particular collection of instructional tasks, the characteristics are incomplete. The characteristics identified are adjuncts of an effective selection procedure, but they ignore the fundamental question of learning outcomes. An effective selection procedure must contain some criterion for dealing with the question of what the learner will derive from the instruction. The reviewer must be provided a procedure that results in answers to these two questions: (1) *What will the learner be observably able to do after instruction with these materials that he could not do before instruction?* (2) *What is the evidence for any claims of learner acquisitions?* If the selection procedure fails to deal with the question of learner acquisitions, then the worth of the procedure is negligible.

There are certain conditions external to any list of procedures which often affect the selection of an instructional aid. Frequently, these conditions are the principal factors influencing the selection of a particular textbook. How often have you heard any one or more of the reasons illustrated in Figures 3.26–3.32 being advanced in support of one choice over another?

FIGURE 3.26. *Endorsement by Authority— "Well, this text is endorsed by so-and-so and so-and-so, who are well-known figures in mathematics. So the text must certainly be effective."*



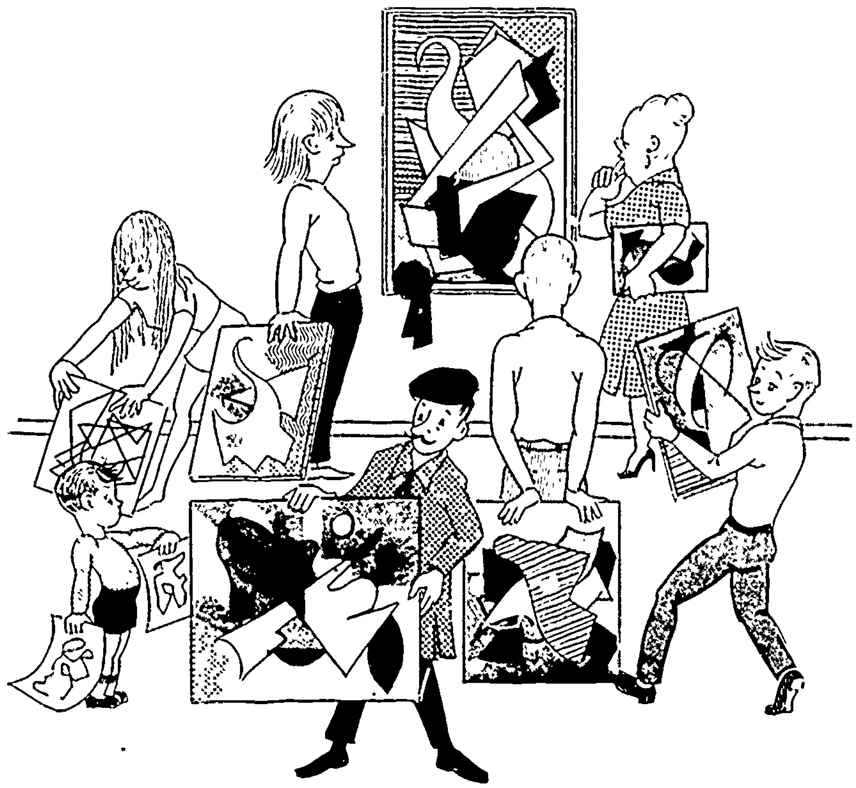


Figure 3.27. *The Bandwagon Effect—"Look at the list of school systems that have already adopted this text, it surely must be good."*

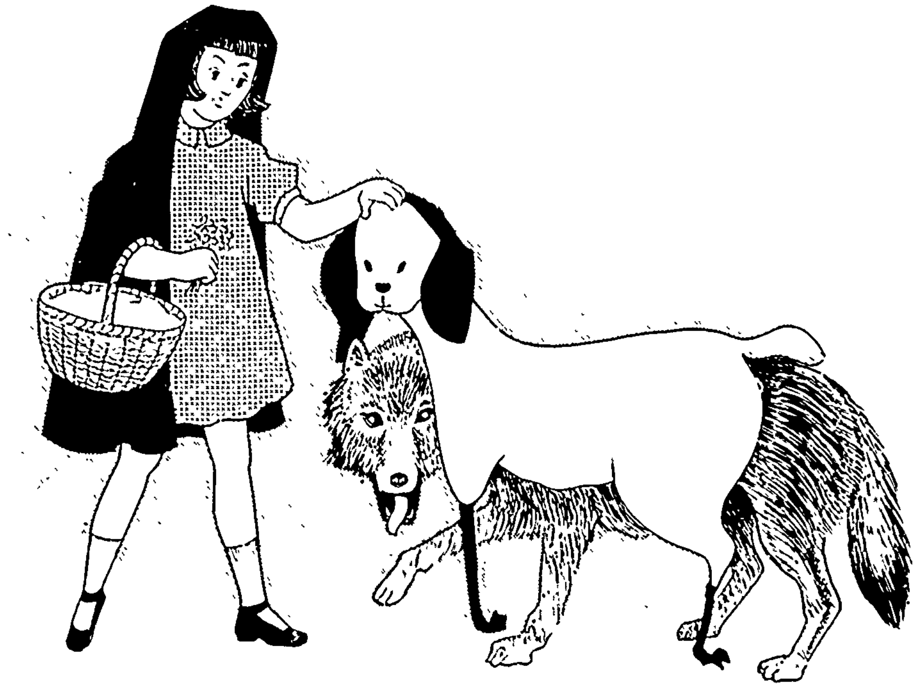


Figure 3.28. *The Advantages of High I. Q. (Identification Quotient)—"I like this series because these authors have always written good material."*


FIGURE 3.29. *Old Friends Are Best* "*We should adopt this textbook because the important, familiar ideas are still treated as they always have been.*"

FIGURE 3.30. "*The illustrations are magnificent and should really help motivate and retain interest.*"

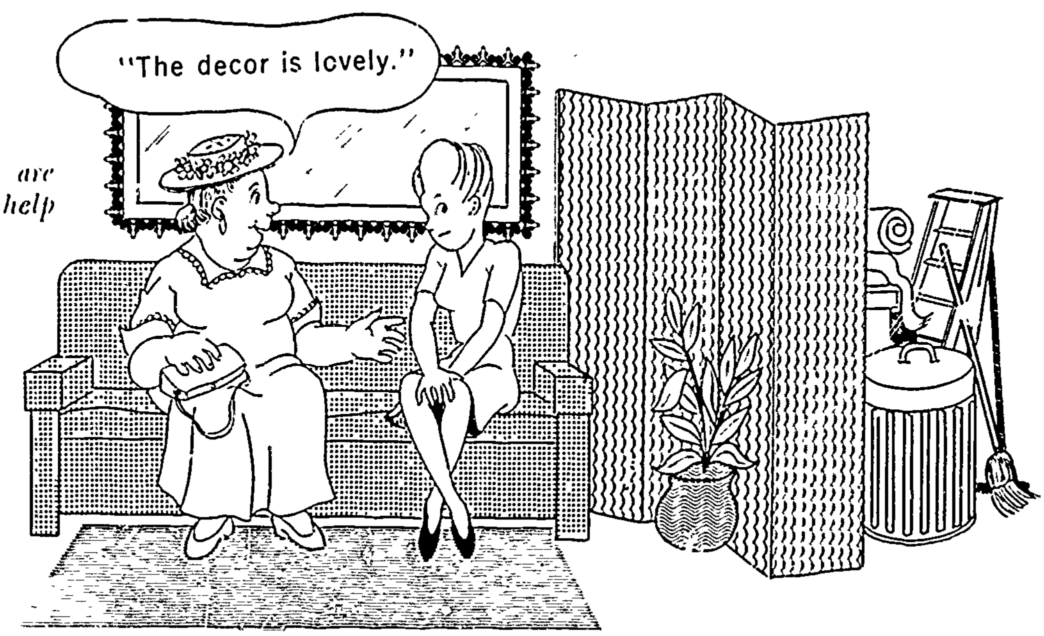



FIGURE 3.31. *Isn't Belief Wonderful! "The sales representative's presentation was detailed, well organized, thoughtful, and authoritative. He said that the sequence of exercises was carefully thought out and the vocabulary level is aimed right at our students."*

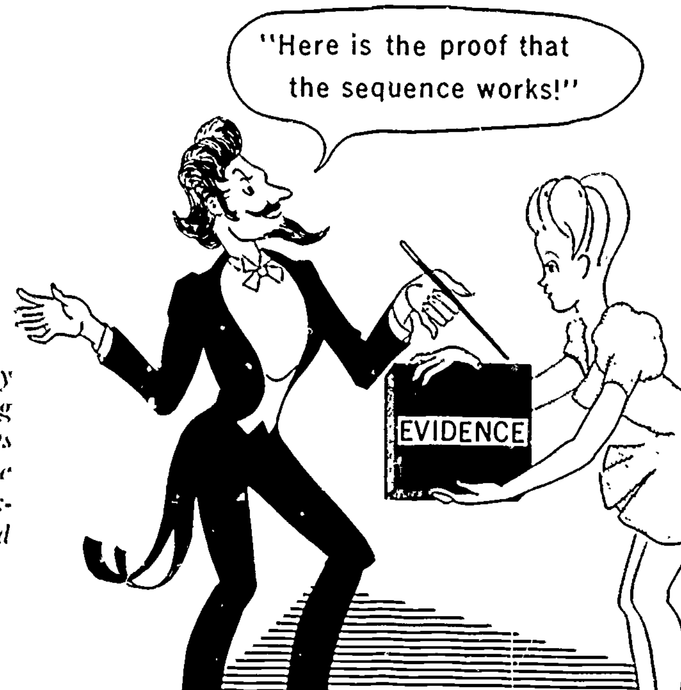



FIGURE 3.32. *"Why, this material is by someone who worked on the writing team that produced NEW. It follows their outline but doesn't have the typographical errors; it has more exercises, and the vocabulary is pitched at a more appropriate level."*

Now, of course, not all of these conditions prevail at each textbook adoption, and none may be present at some. The chances are, though, that some one or more of these are recognizable. It is deplorable that the selection of such a critical instructional aid might be influenced by such outside criteria. The question now becomes one of whether it is possible to organize a procedure of textbook selection which is objective, concentrates on instruction, focuses on expected learner outcomes, and is a procedure that a local school system can carry out.

## A POTENTIAL SELECTION PROCEDURE

Suppose a textbook selection committee has been given the charge of choosing arithmetic texts for grades K–6. What might be the sequence of procedures if it is agreed that instructional objectives are to be stated in performance language?

For a while, at least, the first step in the selection procedure is to see to it that each member of the selection committee acquires the competencies necessary to construct behavioral objectives and learning sequences based on the stated objectives. Once these competencies are acquired, the second step can be taken—the minimal performance expectations for each of the seven years of instruction can be constructed. The descriptions of these performance expectations for each year are statements of what a learner is expected to be able to do after the year of instruction that he could not do before instruction. These statements should, of course, be written. This reduces the likelihood of misinterpretation as well as facilitating memory.

The third step is to examine the behavioral objectives of each textbook considered a candidate for adoption. Those textbooks whose objectives match at least a simple majority of the objectives stated by the committee would be retained for further consideration, and those that do not meet this criterion would be discarded. The next step is to examine the evidence that the objectives of the selected textbooks are achieved. The evidence presented should at least describe the percentage of youngsters exposed to the material who acquire each behavior. A description of the population supplying the data should also be provided. If evidence is presented and the level of acquisition is acceptable for the population assessed, then the text should be retained for further consideration. If no evidence of learner acquisition is presented, the text should be discarded. If the level of acquisitions reported is considered too low, then the text should be removed from immediate consideration but not completely discarded. An acceptable level of acquisition will vary among systems. Perhaps a reasonable criterion can be that 90 percent of the exposed learners acquire 90 percent of the behaviors described by the objectives.

For those texts that have not been eliminated a content check should now be made. Each textbook should be examined in terms of the integrity of the content. More than 2 percent error (2 errors per 100 items checked) would eliminate the text from further consideration.

The learning sequences of each of the texts remaining after the content examination should be examined, together with the supporting evidence of learning dependency. Texts without such information would be eliminated from the list of possible choices. Remaining texts would be ranked for acceptability on the basis of how many of the desired objectives are part of the program and the efficiency of the learning sequence in assisting the learners to acquire the stated behaviors. Figure 3.33 summarizes the suggested procedure.

Because few (if any) currently available commercial or experimental mathematics textbooks contain statements of objectives written in behavioral terms, the suggested procedure would probably eliminate all of the possible textbooks at the step that calls for behavioral objectives. This would result in the outcome that no textbooks would be purchased for a short time. "A short time" is predicted because the profit motive will supply the necessary motivation to

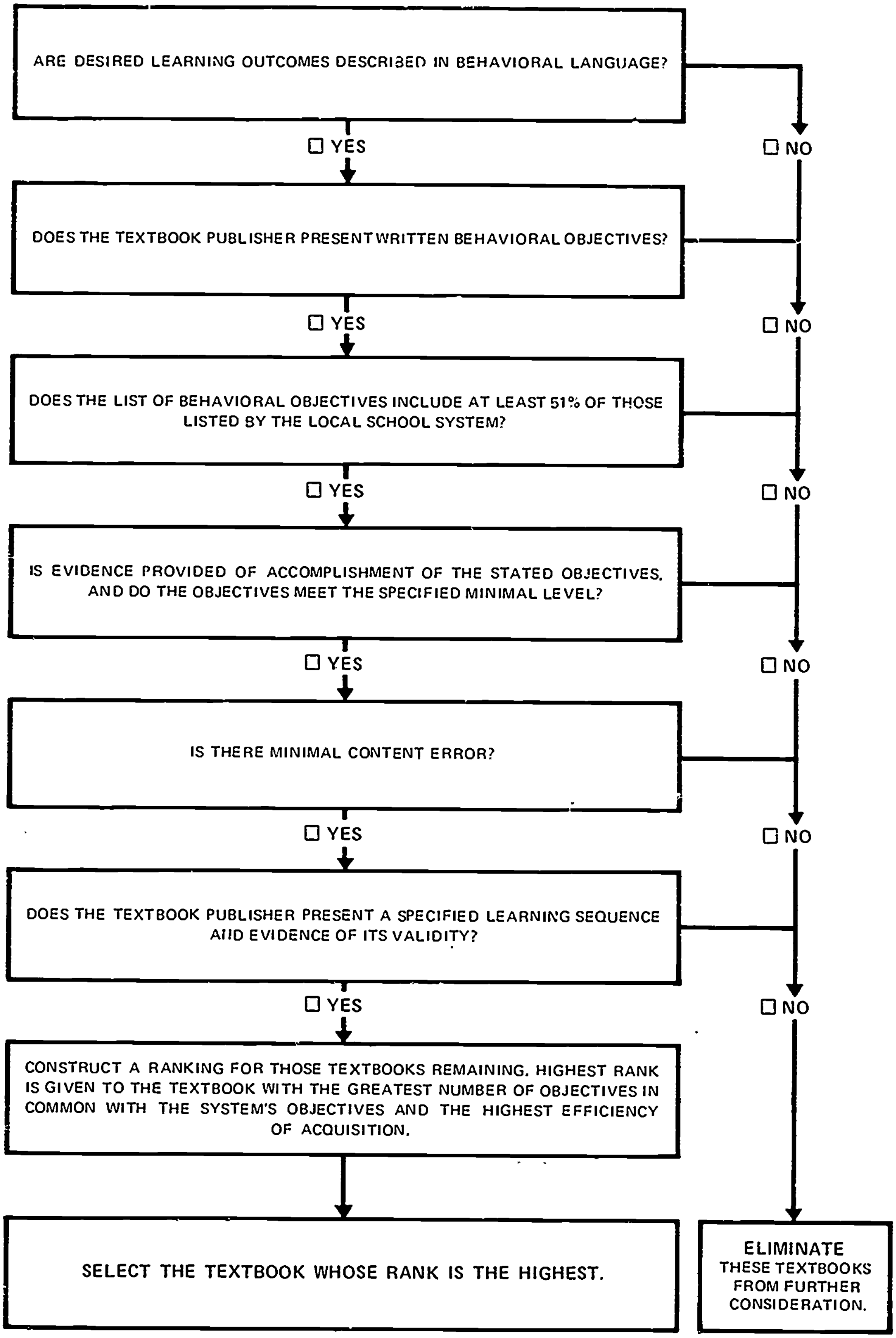



FIGURE 3.33. *Flow chart of procedure for the selection of a textbook, based on behavioral objectives and learning hierarchies. Input is all textbooks under consideration.*

cause such information to be provided.

*Compromise Procedures*

Should you wish to delay the advent of mathematics textbooks containing behavioral objectives and supplying evidence of the objectives accomplishment by a substantial portion of the exposed learners, you may have the selection committee try to second-guess the authors. The committee might try to describe the objectives for each potential selection. The limitations of this compromise procedure are obvious: (1) the guesses the committee makes may or may not be the objectives the author intended, (2) no evidence of learner accomplishment of the objectives is available, and (3) the time involved in constructing the behavioral descriptions is beyond a reasonable expectation for such a committee. However, if there is a pressing desire to change textbooks and no willingness to wait out the assumption of responsibility for objectives and evidence by the authors of textbooks, commercial publishing houses, and the development projects, then an attempt at the local level may be the best recourse available.

If the local school system undertakes this task the principal difficulty is the time required to produce behavioral descriptions for each available textbook. The service proposed by the Pennsylvania Retrieval of Information in Mathematics Education project is one example of how this task might be accomplished for elementary mathematics, grades K–6, and the results shared among school systems. The United States Office of Education's project ES 70 suggests another way in which behavioral description may be made available.

One of the school systems in the state of Maryland, when faced with this problem, adopted still another strategy worth describing. The Frederick County school system appointed a committee to select elementary mathematics texts for grades K–6. The committee invited the state supervisor of mathematics, Thomas Rowan, to assist them in the selection, and he accepted. This working group discarded the conventional procedures of textbook selection and decided to include behavioral objectives among their criteria to be applied in the selection of textbooks. Members of the committee recognized the lack of stated behavioral objectives and accompanying evidence of acquisition on the part of the available textbooks, but they decided some selection needed to be made. They also acknowledged that there was neither the manpower nor time available to write objectives for all the available texts, but they wanted to try the behavioral procedure. So it was decided that the behavioral objectives and learning sequences would be constructed for a few selected content themes they considered critical to the elementary mathematics curriculum of their community. Samples of the products produced by the committee and used in the comparison of three volumes are included next. The illustrative material deals with the description of behavioral objectives for measurement. Observe the differences in the number and quantity of objectives in the three textbook series, as well as the differences in sequential organization.

### TEXTBOOK A

GRADE 1

1. Demonstrating measurement of length using a nonstandard unit (does not introduce inch, foot, etc., and advises against these; no line segment; inequality signs used, but poorly related to concept of order)

GRADE 2

1. Naming and identifying line segments
2. Naming and identifying standard units of measurement
3. Demonstrating order on the number line using inequality signs (number line shown by ruler)
4. Naming the length, width, and height of an object
5. Naming closed figures

GRADE 3

1. Demonstrating measurement with linear measuring devices
2. Naming and identifying inch, foot, and yard
3. Identifying circles, squares, triangles, and parallelograms

GRADE 4

1. Demonstrating linear measurements using non-standard units
2. Ordering the results of linear measurements made with nonstandard units
3. Demonstrating linear measurements using the inch as a standard unit: using smaller units such as $\frac{1}{8}$ inch
4. Demonstrating the reading of a ruler with various degrees of accuracy
5. Naming, identifying, and describing a mile
6. Constructing a sum that represents mileage on a road map
7. Identifying and naming a region
8. Naming and describing the area of a region
9. Determining the area of a given rectangular region by counting the number of units it is subdivided into
10. Naming the number of subunits contained in a larger unit
11. Distinguishing between area and perimeter
12. Naming and identifying parallel lines

GRADE 5

1. Naming and identifying triangles, circles, squares, rectangles, and polygons
2. Constructing triangles
3. Identifying length, width, base, altitude, and height of figures
4. Finding areas by measurement and constructing 1-inch squares
5. Naming and identifying perimeters of squares, rectangles, and circles
6. Stating rules for finding perimeters
7. Identifying and naming square units
8. Distinguishing between counting a quantity and measuring
9. Identifying which unit of measurement to use
10. Identifying fractions on number lines

GRADE 6

1. Naming and identifying solid regions and closed surfaces
2. Constructing a volume
3. Interpreting volume
4. Computing volumes of solids
5. Stating rules for finding volumes
6. Identifying perpendicular and parallel lines
7. Identifying plane surfaces
8. Identifying and measuring a diameter
9. Identifying and measuring an altitude
10. Constructing a rectangle, triangle, and circle
11. Measuring the area of a rectangle, triangle, and circle by use of squares constructed within the figure
12. Demonstrating measurement of volume by use of cubic units
13. Stating the rule that all measurements should be in the same unit when used for finding volume
14. Applying rules for finding volumes of solids
15. Naming and identifying metric units
16. Identifying the order of linear units in the metric system of measurement
17. Demonstrating that rectangles and parallelograms of the same dimensions have equal areas

TEXTBOOK B

GRADE 1

1. Identifying, naming, and constructing line segments
2. Ordering line segments
3. Demonstrating and naming measurement of length using standard unit of measure (inch)
4. Applying rule that 12 inches = 1 foot
5. Identifying and naming triangles, equilateral triangles, circles, squares, and rectangles
6. Constructing and identifying rectangle, square, triangle, and equilateral triangle
7. Identifying inside and outside of closed figure
8. Applying concepts of inside and outside
9. Identifying sizes and shapes

GRADE 2

1. Applying rule that 12 inches = 1 foot
2. Identifying inside, outside, and on a closed figure
3. Applying concept of inside, outside, and on a closed figure
4. Identifying and naming perimeter
5. Constructing a figure for use in finding perimeter
6. Applying definition of perimeter by using standard unit of measure
7. Identifying and naming length, width, and area of rectangle
8. Stating rule that length $\times$ width = area
9. Naming and identifying the area of a surface by counting the number of units used to cover the surface
10. Demonstrating linear measure, using inches

GRADE 3

1. Identifying and naming length, width, and area of rectangle
2. Constructing a rectangle of a given area
3. Stating the rule that length $\times$ width = area
4. Applying the rule that length $\times$ width = area
5. Naming units of measure (feet, pounds, degrees, inches, etc.)
6. Identifying and naming an appropriate unit of measure, given an object to be measured and a choice of units
7. Naming $\frac{1}{2}$ inch on ruler
8. Applying rules for changing yards to feet and feet to yards
9. Applying rule for finding perimeter
10. Identifying and naming standard square

units of measure (square inch, square foot, square yard)
11. Constructing and naming line segments
12. Identifying and naming open and closed figures
13. Identifying inside, outside, and on a closed figure

GRADE 4

1. Identifying and naming line segments
2. Identifying and naming triangles, quadrilaterals, and pentagons
3. Identifying perimeter of closed figures
4. Applying rule for finding perimeter of closed figures using nonstandard units of measure
5. Identifying and naming inch and centimeter
6. Demonstrating measurement of length using inch and centimeter as units of measure
7. Demonstrating measurement of perimeter using inch and centimeter as units of measure
8. Identifying and naming open and closed figures
9. Identifying and demonstrating inside, outside, and on a closed figure
10. Stating rule for finding area of rectangle
11. Applying rule for finding area of rectangle using nonstandard square units of measure
12. Identifying and naming a rectangular prism
13. Stating formula for finding volume of rectangular prism
14. Applying rule for finding volume of rectangular prism
15. Identifying and naming a cube
16. Identifying and naming a cubic inch
17. Identifying and naming a cubic centimeter
18. Naming number of subunits contained in larger unit (inch, foot, yard, mile)
19. Applying rules for changing units of measure (rule is applied without being stated)
20. Constructing and identifying a circle, radius, and diameter
21. Identifying, on a ruler, $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ inch
22. Demonstrating linear measurements using the inch as standard unit and accuracy to $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, and $\frac{1}{8}$ inch
23. Identifying and naming right angles
24. Constructing and naming angles

GRADE 5

1. Identifying and naming line segments
2. Constructing congruent line segments
3. Constructing line segments, given ruler and compass
4. Identifying and naming congruent angles
5. Constructing angles, with use of compass, for purpose of comparing measures of angles
6. Identifying and constructing interior angles
7. Identifying and constructing equilateral, isosceles, and scalene triangles
8. Constructing lines of symmetry in triangles
9. Demonstrating lines of symmetry in triangles, using folded shapes
10. Applying rules for changing linear measures (small to large, large to small, involving inches, feet, yards, rods, miles)
11. Applying information about linear measure to solution of story problems
12. Demonstrating reading of ruler to the nearest $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ inch
13. Demonstrating reading a measurement to the nearest stated unit
14. Applying rule for finding area of closed figure
15. Applying rule for finding perimeter of closed figure
16. Constructing scale drawings of rectangular and triangular figures, given necessary related dimensions
17. Identifying and naming plane and space figures
18. Identifying and describing triangular, rectangular, pentagonal, and hexagonal prisms
19. Identifying and describing triangular, rectangular, pentagonal, and hexagonal pyramids
20. Identifying and naming spheres, cylinders, and cones
21. Stating rule for finding volume of rectangular prism
22. Naming the cubic inch
23. Applying rule for finding volume of rectangular prism
24. Applying rules for metric system of linear measure
25. Ordering metric units of measure
26. Identifying and naming right angles
27. Demonstrating and constructing right angles, using folded paper

GRADE 6

1. Identifying, naming, and constructing lines, rays, and line segments
2. Identifying, naming, and constructing congruent line segments
3. Identifying, naming, and constructing angles
4. Constructing circles and arcs
5. Constructing congruent angles
6. Demonstrating angle measure, using compass
7. Demonstrating measure of angle, using nonstandard unit of measure
8. Demonstrating measure of angle, using degrees on protractor
9. Constructing equilateral and isosceles triangles
10. Constructing a triangle, using straightedge and compass, given lengths of base and sides
11. Applying rules of linear measure for changing feet to inches, rods to miles, rods to

feet, feet to yards, and miles to rods
12. Identifying and describing square inch
13. Identifying square foot
14. Describing square foot
15. Applying rule for finding area in square units (foot, yard, mile)
16. Identifying acre
17. Applying rule for finding area of rectangle, using acres
18. Applying rule for finding area in square units to solution of story problems
19. Identifying and naming a cube
20. Identifying and naming cubic inch, cubic foot, and cubic yard
21. Demonstrating volume of rectangular prisms
22. Stating rule for finding volume of rectangular prisms
23. Applying rule for finding volume of rectangular prisms with given dimensions
24. Applying rule for converting all dimensions of closed figure to like units
25. Constructing rectangles with sides of given lengths
26. Constructing square given length of side
27. Constructing a triangle with sides congruent to given segments
28. Identifying units of measure in metric system
29. Ordering units of measure in metric system
30. Applying rule for changing large to small and small to large measures in metric system
31. Applying rule for finding area of a rectangle in metric units
32. Identifying and naming relationships between English units of linear measure and metric units of linear measure
33. Applying rules for converting English units of linear measure to metric units of linear measure
34. Constructing a circle, given the radius
35. Identifying and naming base and altitude of a triangle
36. Applying rule for finding area of triangle
37. Applying rules for finding area of a rectangle and area of a triangle to find areas of portions of rectangles and triangles
38. Identifying and naming an octagon
39. Applying rule for finding area of a circle
40. Applying rule of scale drawing using various objects

## TEXTBOOK C

### Grade 1

1. Identifying squares, circles, and triangles
2. Naming unit of measure–nonstandard unit
3. Demonstrating measurement of height using blocks
4. Constructing ruler with red and black scales (red for centimeter, black for inch)
5. Demonstrating measurement of height by two scales on same ruler
6. Identifying and naming an inch
7. Identifying and naming a centimeter
8. Demonstrating the measurement of length using an inch and a centimeter (on same ruler)
9. Demonstrating measurement of length that does not come out "just right," hence the identifying of approximation

### Grade 2

1. Identifying circles and polygons
2. Constructing ruler and pasting on cardboard, naming units on black scale inches and on red scale centimeters
3. Demonstrating measure of height by two scales on the same ruler
4. Identifying and naming an inch
5. Identifying and naming a centimeter
6. Demonstrating the measurement of length by using the inch and by using the centimeter (both scales on same ruler)
7. Demonstrating the measurement of length that does not come out "just right," hence the identifying of approximation
8. Constructing lines of given length
9. Distinguishing between inches and centimeters for a given measurement
10. Naming the unit along with the measurement
11. Demonstrating the measurement of a given length by using both scales
12. Demonstrating the measurement of liquids and of time

### Grade 3

1. Naming and identifying a foot
2. Naming and identifying a yard
3. Naming the length of an object
4. Naming and identifying the area of a surface by counting square units used to cover the surface
5. Naming and identifying the volume of solids by counting number of pieces of cake (blocks)
6. Demonstrating informally the three types of measurement: linear (wires), area (squares), and volume (blocks)
7. Demonstrating measuring of area by covering a surface with congruent shapes (squares)
8. Distinguishing types of measurement using examples from environment
9. Naming and identifying unit of length as a segment
10. Naming and identifying unit of area as a square
11. Naming and identifying unit of volume as a cube
12. Naming the length of an object

13. Naming and identifying the area of a surface by counting units used to cover the surface
14. Naming and identifying the volume of a solid by counting the units used to fill the solid
15. Demonstrating length by counting steps (segments) across classroom
16. Distinguishing between small, medium, and large steps
17. Naming the step as a segment
18. Demonstrating that the smaller the unit, the greater the number for a given length
19. Constructing individual units of measure (piece of string)
20. Naming and identifying a mile
21. Demonstrating an inch as approximately the length of a section of the forefinger
22. Demonstrating a foot as approximately the length of the forearm
23. Demonstrating the mile as approximately the length of a train with 120 boxcars
24. Demonstrating the yard as approximately the distance between the wrists when arms are outstretched forming a straight line
25. Naming an appropriate unit of measure given that which is to be measured and a choice of units (inch, foot, yard, or mile)
26. Demonstrating that a centimeter is shorter than an inch
27. Naming the length of pictured objects using both units, inches and centimeters (all answers are whole numbers)
28. Constructing centimeter ruler at least 13 centimeters long
29. Naming the measurement of length to the nearest inch
30. Naming and identifying half-inch as a "hinch"
31. Naming the length of pictured objects in "hinches"
32. Naming $\frac{1}{8}$ inch on a ruler
33. Naming $\frac{1}{2}$ centimeter on a centimeter ruler
34. Naming the measurement of length to the nearest $\frac{1}{2}$ inch
35. Naming the number of subunits contained in a larger unit (1 in. = 2.54 cm, 1 ft. = 12 in., 1 yd. = 3 ft., 1 yd. = 36 in.)
36. Naming and describing a square inch
37. Naming and describing a square centimeter
38. Naming and identifying the area of a variety of geometric figures by counting the number of units (squares)
39. Naming and identifying the area of a variety of geometric figures drawn on "squared" paper by counting squares
40. Demonstrating that two triangles (isosceles right) make a square
41. Constructing geometric figures containing a given number of square units
42. Demonstrating and naming an estimate of the number of square units in very irregular figures
43. Naming and describing a cubic inch
44. Naming and describing a cubic centimeter
45. Demonstrating and naming an estimate of the number of blocks in an array
46. Demonstrating and naming the volume of block arrays by counting the blocks
47. Naming and identifying the volume of a variety of geometric solids by counting the number of cubes

GRADE 4

1. Applying rule changing feet to inches, feet to yards, and yards to inches
2. Naming and identifying area
3. Naming and identifying volume
4. Naming and identifying terms used for measuring length (segment), area (square), and volume (cube)
5. Identifying and naming types of measurement with nonstandard units
6. Naming line segments
7. Demonstrating linear measurement using the inch as a standard unit and accuracy to the half-inch
8. Naming and identifying lengths of objects
9. Naming and identifying half-inch on a ruler and with objects
10. Demonstrating that a centimeter is shorter than an inch
11. Interpreting distance on a map
12. Interpreting scale drawing, using a map on which a centimeter represents a mile
13. Naming and identifying a square inch
14. Naming and identifying a cubic centimeter
15. Naming and identifying the area of various objects
16. Naming and identifying area by counting units of area
17. Naming area of a given figure using a grid
18. Demonstrating the perimeter (given the term), using a standard ruler and using a centimeter ruler
19. Estimating the differences in perimeter of various objects
20. Identifying a triangle
21. Constructing a square
22. Naming a length using a centimeter ruler
23. Applying the idea of using a ruler to find the perimeter
24. Distinguishing between volume and surface area
25. Naming a square centimeter
26. Applying a rule for changing a mile to yards
27. Naming the number of smaller units in a larger unit

28. Distinguishing the appropriate unit of measure—inch, foot, yard, mile
29. Identifying liquid measures

GRADE 5

1. Naming and identifying unit of length as a segment
2. Naming width of index finger as a segment
3. Constructing ruler with width of index finger as the segment
4. Demonstrating measurement of length by counting number of segments
5. Constructing half-segments on ruler (where segment is width of index finger)
6. Distinguishing which of two measurements on the hand is the greater
7. Demonstrating measurements with results to nearest half-segment
8. Constructing a "shoe" segment and dividing it into eight equal parts, each called a "toe"
9. Demonstrating measurement with "shoe" ruler
10. Demonstrating use of accepted units such as centimeter, inch, foot, and yard
11. Demonstrating measurement with standard units
12. Stating a rule for finding the perimeter of a polygon
13. Applying the rule for finding the perimeter of polygons the measures of whose sides are whole numbers
14. Determining the measures of the sides of a polygon by using a ruler and then applying the rule for finding the perimeter
15. Determining the measures of the sides of a polygon by rolling polygon along the ruler and thus finding perimeter
16. Naming and identifying circle, circumference and diameter
17. Demonstrating that both perimeter and circumference mean the distance around a circle
18. Naming and identifying area of rectangle by counting squares
19. Naming and identifying area of rectangle by multiplying two dimensions
20. Interpreting the formula for the area of a rectangle, $A = l \times w$
21. Naming and describing parallel lines and parallelogram
22. Demonstrating the formation of a rectangle from a parallelogram by cutting triangular section from one side and fitting it along other side
23. Interpreting the formula for the area of a parallelogram, $A = b \times h$
24. Naming and identifying surface area for space figures
25. Demonstrating surface area by counting squares in a plane figure
26. Constructing space figure (cube) from plane figure and again finding surface area by counting same squares as in item 25
27. Constructing rectangular solids and demonstrating surface area
28. Constructing a cylinder and demonstrating surface area
29. Demonstrating surface area with a variety of space figures, given their dimensions
30. Naming and identifying formally time ($t$), rate ($r$), and distance ($d$)
31. Interpreting the formulas $d = r \times t$, $r = d \div t$, and $t = d \div r$
32. Describing the use of the above distance formulas
33. Naming and identifying volume by counting cubic units
34. Naming and identifying volume of rectangular solid by multiplying the three dimensions
35. Interpreting the formula $V = l \times w \times h$
36. Applying the rule to find the volume of rectangular solids
37. Applying the rule for changing cubic inches to gallons
38. Applying the rule for changing cubic centimeters to liters
39. Demonstrating measurement of an angle by counting angle units
40. Constructing and counting measure of right angle using unit selected
41. Naming and describing standard angular units: radian and degree
42. Describing the protractor as a device to measure angles
43. Constructing a protractor and demonstrating its use
44. Naming the angle unit on the protractor as a degree
45. Applying the rule for changing one linear measurement to another
46. Demonstrating the conversion of smaller units into larger units (as a result of addition) or larger units to smaller (as needed for subtraction)
47. Demonstrating the reading of a ruler with various degrees of accuracy: $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, $\frac{1}{16}$ inch, $\frac{1}{2}$ centimeter, $\frac{1}{10}$ centimeter
48. Demonstrating relative lengths of two segments
49. Demonstrating measurement of sides of polygons to nearest $\frac{1}{4}$ inch and finding the perimeter
50. Naming and describing metric units: meter, decimeter, and millimeter
51. Describing use of decimals in measurement
52. Applying rule for changing one metric unit to another
53. Naming and describing the comparison of two sets of objects as a ratio

54. Describing distances from scale drawings
55. Naming a scale and drawing a map showing your home and school

GRADE 6

1. Naming and identifying unit of length as a segment
2. Naming and identifying sum of lengths of the sides of a polygon as the perimeter
3. Demonstrating measurement by use of inches and centimeters
4. Distinguishing between lengths by use of centimeters and inches
5. Demonstrating that the length of a given segment will have more centimeters than inches
6. Distinguishing between distances by use of miles and kilometers
7. Describing use of kilometers in various countries
8. Naming and identifying statute mile and nautical mile
9. Applying a rule and measuring lengths to find perimeters of various polygons
10. Naming the length of the side of a square when the perimeter is given
11. Naming the length of each side of a rectangle, given the length of one side and the perimeter
12. Naming the length of the third side of a triangle, given the length of two sides and the perimeter
13. Describing the perimeter of a circle inscribed in a square (whose sides are to be measured) and circumscribed about a hexagon (whose sides are to be measured)
14. Naming and identifying area of plane figure and surface area of space figure by counting square units
15. Naming and identifying area of rectangle by multiplying two dimensions
16. Interpreting the formula for the area of a rectangle, $A = l \times w$
17. Constructing space figures (cubes, rectangular solids) and finding surface area
18. Naming surface area, given space figures with dimensions
19. Interpreting word problems involving linear measurement, area, volume, time, rate, and distance
20. Naming and identifying hand (unit of measurement) and horsepower
21. Naming and identifying volume by counting cubic units
22. Naming and identifying volume of rectangular solids by multplying the three dimensions
23. Naming and identifying rectangular prism
24. Interpreting the formula $V = l \times w \times h$
25. Naming and identifying volume and surface area of various solids
26. Naming length, width, and height as dimensions
27. Interpreting resulting area of rectangle when one dimension is doubled: when both are doubled
28. Interpreting resulting volume of rectangular prism when one dimension, then two, and finally three are doubled
29. Interpreting resulting volume of rectangular prism when each dimension is tripled
30. Demonstrating and constructing the measurement of an angle by counting angle units
31. Demonstrating and identifying symbols for angle and degree
32. Naming size of given angles in terms of drawn protractor on which 18 equal divisions are used
33. Constructing above type protractor and demonstrating its use
34. Describing protractor with unit angle chosen as $10°$
35. Naming and describing standard angle units: degree and radian
36. Demonstrating measurement of angles and finding the sum of the measures of two angles
37. Demonstrating the measurement of the angles of a triangle and finding the sum of the measures of the three angles: same for quadrilateral, pentagon, and hexagon
38. Describing total degrees in angles of above polygons in terms of number of triangles times $180°$
39. Demonstrating complete rotation, $1\frac{1}{2}$ rotations, and so on, in terms of degrees
40. Interpreting word problems involving track records: prehistoric animals
41. Interpreting distances on a given map by scale
42. Interpreting distances in our solar system by using scale 1 yard = 93 million miles
43. Describing weight of diamonds (1 carat) in terms of grams where 1 cubic centimeter of water at 4° C weighs 1 gram
44. Interpreting word problems relating to weights of diamonds
45. Describing lengths of animals from pictures with indicated scales
46. Describing leaning tower of Pisa and work of Galileo
47. Describing and stating rule for falling bodies, $d = 16 \times t^2$
48. Naming and identifying base of triangle and right triangle
49. Naming and identifying area of rectangle; then taking half of it for area of triangle
50. Naming area of triangle, given base and altitude

51. Constructing squares on three sides of right triangle and finding area of each
52. Naming the hypotenuse and legs of right triangle
53. Stating and applying the Pythagorean theorem
54. Interpreting word problems involving measurement
55. Naming and identifying unit of length, meter; unit of weight, gram; and unit of capacity, liter
56. Naming and describing metric units
57. Interpreting word problems involving metric and English measures
58. Naming and identifying circumference
59. Constructing circle 3 centimeters in diameter, cutting it out, and measuring circumference by rolling it along a drawn line 30 centimeters long; repeated for circles with diameters of 4, 5, 6, 7, 8, and 9 centimeters
60. Naming special factor in each circumference by dividing circumference by diameter
61. Describing resulting factor as more than 3 and less than 3.3 and naming it $\pi$
62. Stating and applying the rule that circumference $= \pi \times$ diameter
63. Demonstrating procedure for estimating area of circle by counting number of square centimeters (this will be an estimate!)
64. Naming and identifying radius of a circle
65. Demonstrating area of circle as close to $3 \times r^2$
66. Stating and applying rule $A = \pi r^2$

The technique employed by the Frederick County textbook selection committee did result in some objective description of the various textbooks under consideration. Sequence deficiencies in terms of missing prerequisites were identified for the topics examined. Content errors were also identified for these topics.

The limitations of this procedure are apparent. The restricted number of topics examined may produce a distorted description of a text. However, for the topics examined the application of the procedure does yield an objective, detailed description.

Clearly the procedures adopted in the selection of instructional aids can be substantially altered by the use of behavioral descriptions of expected outcomes. The procedure detailed earlier in this chapter is one means of employing behavioral descriptions to just such a purpose. The advantages of moving to the use of a selection procedure based on behavioral descriptions are the change from a subjective base to an objective one, the availability of evidence that an instructional aid accomplishes its stated purposes, and the construction of instructional sequences from considerations of what is to be learned and the conditions under which it is to be learned. Under such a procedure the burden of proof that an instructional aid does accomplish its stated purpose would rest with the distributor.

Two recommendations concerning the selection of a textbook follow.

RECOMMENDATION 1. The procedure employed in the selection of a textbook should include (1) the description of the desired instructional outcomes in terms of behavioral objectives, (2) a comparison of the stated behavioral objectives in available textbooks with those stated by the selection committee, in connection with the procedure outlined in Figure 3.33, (3) examination of the textbook publisher's evidence that the behaviors described as objectives were acquired and by whom they were acquired, and, finally, (4) an examination of the learning sequence and the evidence to support the hypothesis that it is a sequence.

RECOMMENDATION 2. The selection of every instructional aid should include at least the first three steps as suggested in the previous recommendation for textbook selection.

## EPILOGUE

The advantages of behavioral descriptions and learning hierarchies extend beyond the selection of textbooks to include assistance in the design of instruction as well as providing guidance in the supervision of instruction. A bold proposal, indeed, that as teachers we specify the expected outcomes of our teaching. Imagine proposing that the measure of our competency rests with evidence of observable learner acquisitions!

The challenge is made. The next step is yours.

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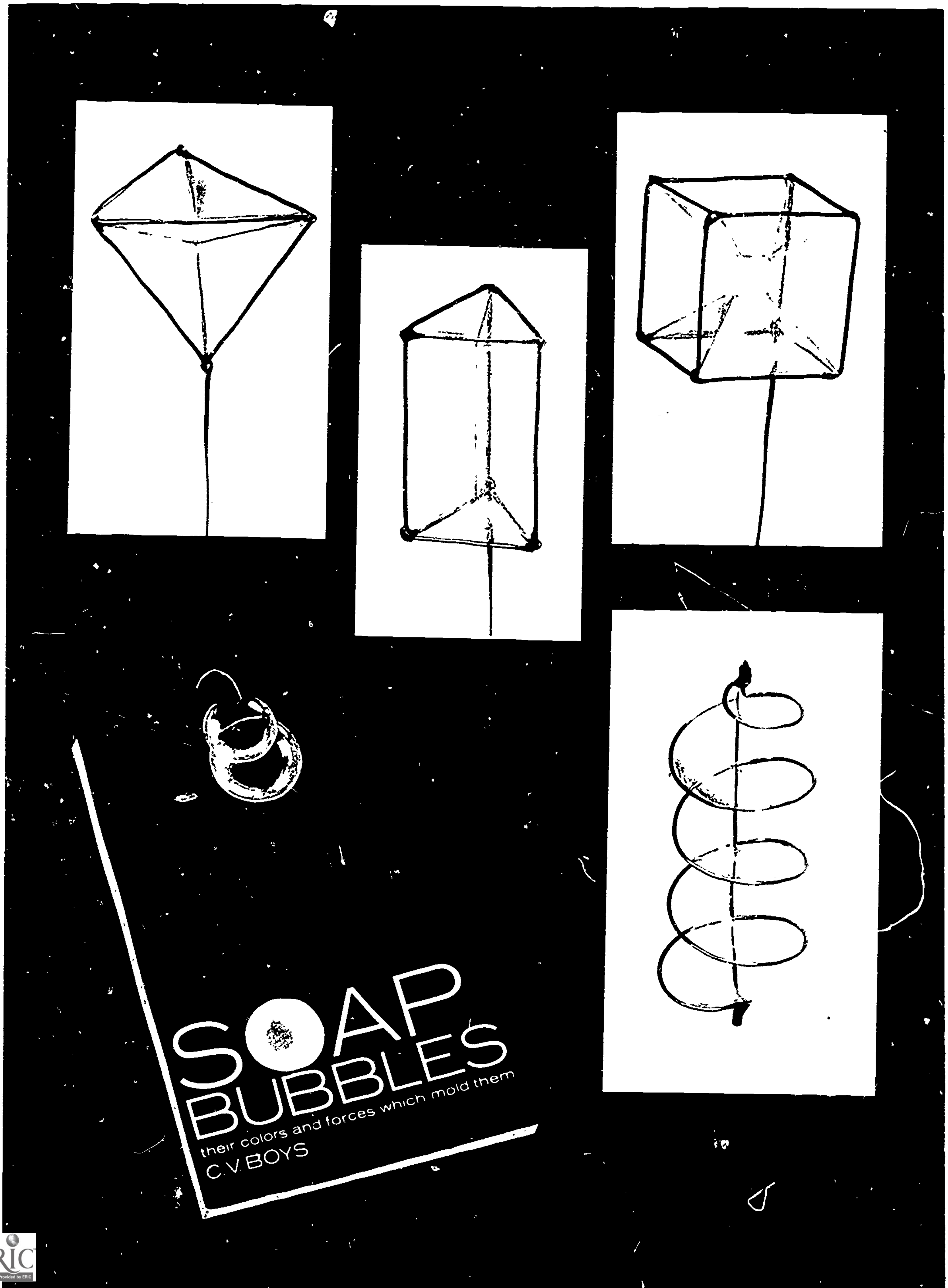
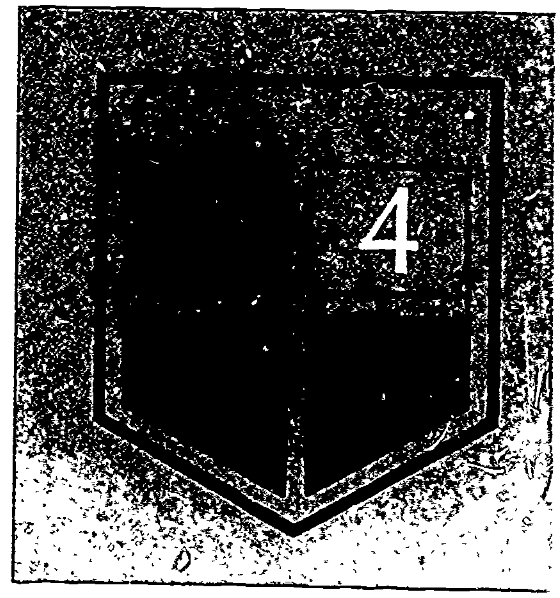
# 4. OTHER PRINTED MATERIALS


SOAP
BUBBLES
their colors and forces which mold them
C.V. BOYS

CHAPTER 4

# OTHER PRINTED MATERIALS




*by*
HILDE HOWDEN
Random House, New York, New York
JOHN N. FUJII
Merritt College, Oakland, California

A variety of printed materials that can be used to supplement a mathematics program is examined in this chapter. Reasons why such materials should play an important role in a well-rounded mathematics curriculum and specific ways in which they can be used effectively are also discussed. Fifty detailed examples illustrate exactly how a teacher can use a wide range of Other Printed Materials. Each example specifies an OPM, a function, a technique, and the grade level recommended for the introduction of these materials.

◄ A FILM THAT SPANS ANY CLOSED FRAME assumes a contour with minimum area. This property can be illustrated with soap film. When air drying plastic is used, the resulting film is permanent. The models shown in the picture were formed by dipping wire frames in a substance called Formafilm.

# 4. OTHER PRINTED MATERIALS

Have you ever wished you could vary the content and approach of the mathematics program you are teaching without abandoning its basic organization? Have you ever wished you could divide your time so that you might help students who need extra practice and at the same time renew the motivation of students whose interest is waning? Perhaps OPM (Other Printed Materials) can help you realize your wishes.

## WHAT ARE OPM?

In this chapter, OPM are considered to be instructional aids in mathematics consisting of pamphlets, library books, reference books, resource books, paperback books, skill kits, independent units, curriculum guides, charts, pictures, "free and inexpensive" materials, newspapers, periodicals, student journals, bibliographies, and so on. Standard and programed textbooks, prepared to be used for regular class work as basic instructional aids, are not considered to be OPM.

### *Why Are OPM Being Produced?*

Curriculum development in school mathematics is in a state of continuing change. Since it takes time for textbooks to be written and published, OPM, which can usually be produced more speedily, are being used to obviate the time lag.

Furthermore, until the experimental aspects of curriculum development are thoroughly tested, it is financially impractical to include them in standard textbooks. However, many of these experimental concepts and approaches can be published in OPM form and used in conjunction with a standard textbook until they are popularly accepted and integrated into the curriculum.

Another reason for the production of OPM is that authors of textbooks frequently follow curriculum guides produced by state and local school systems, and these do not include everything there is to learn about mathematics. As a result, much of the material that is not required by various curriculum guides is often published only in OPM form.

Some materials, particularly those whose primary purpose is recreation, application, or individualized drill and practice, are not adaptable to the standard textbook format. These can be presented most effectively as OPM.

In summary, OPM are produced to help keep the curriculum up to date and to give it balance. How OPM can be used to accomplish these goals is the most important concern of this chapter.

### *How are OPM used?*

To learn how OPM are actually used in today's schools, the Yearbook Editorial Committee surveyed nearly a thousand mathematics supervisors across the country. Here are illustrations of what the committee found.

In one junior high school, a teacher uses the local newspaper in general mathematics classes. Grocery ads are compared in preparing a shopping list. Advertised prices of various kinds of cars are compared with the total cost when purchase is made on an installment plan.

In another school, a teacher capitalizes on the enthusiasm of airplane buffs. Attention is turned from tossing paper airplanes out of classroom windows to reading *The Great International Paper Airplane Book* (OPM Example 3, described later). From this book, students who are interested in airplanes learn about the relationship between the center of gravity and the center of the lateral area of paper airplanes. Meanwhile, students who are not interested in the mechanics of airplanes are challenged by the problem of creating fabric designs that illustrate the renowned four-color problem, moiré patterns, or one of the network problems from topology. These concepts can be investigated in one of several OPM.

In another junior high school, teams of four students each competed for two months in an investment project. Each team began with an imaginary $500, which could be invested in any way the team members agreed on. Sales and new purchases were also agreed on by the team. After the initial organization in class, all investigation of news media and professional literature that led to agreements concerning the imaginary transactions and all record keeping were performed outside of class, except for a weekly updating of progress that was kept on a chart in the classroom. Interest ran so high that several of the teams continued the project for the school year and even longer.

These and other examples that could be cited are perhaps somewhat unusual. Others that were reported in the survey were more closely related to the curriculum under study, of shorter duration, or less demanding of the students. Most examples reported can be classified under one or more of the following seven functions:

1. Motivation
2. Discovery
3. Enrichment
4. Change of pace
5. Drill and practice
6. Application
7. Professional growth

The techniques for using OPM ran the gamut from occasional inclusion of interesting asides by the teacher to independent student research.

The following sections contain examples that illustrate how various OPM can be used to serve each of the seven functions listed above. To make the examples specific enough to use as workable models, each is based on a particular OPM and a particular technique. Where the OPM is used directly by the students, as in the case of workbooks and drill kits, directions for use are not detailed. They are given in the teacher's materials that accompany the OPM. For other types of OPM, the examples are explicit.

These examples are by no means exhaustive; they constitute a small sample of the available OPM and the many ways that OPM can be used. Since the function of an OPM is a basic consideration in its selection, the examples are classified according to function. Any one of the techniques (and many more than are described in the examples) can be combined with each function. Although many OPM are multifunctional, none was classified under more than one function in order to include as great a representation of OPM as possible in the fifty examples given.

## MOTIVATION

Motivation is a very personal thing: what motivates a particular student may be quite apart from the course content. Within the course content, however, motivation is generally more predictable. For the purpose of the examples, therefore, motivation will be considered the presentation of content intended to promote student involvement in learning.

### EXAMPLE 1

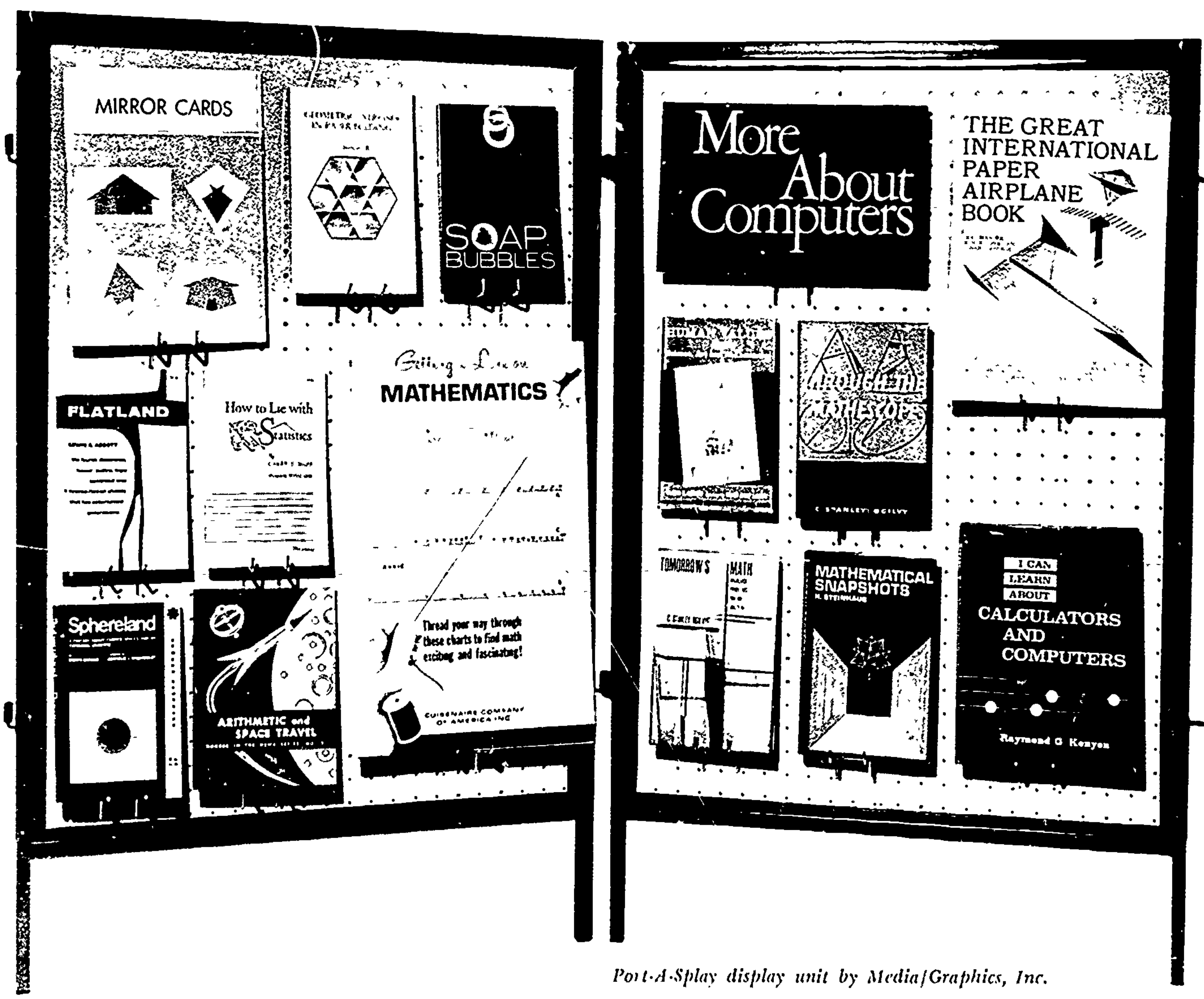
OPM: *Soap Bubbles*, 3d ed., by Charles V. Boys (New York: Dover Publications, 1959)
GRADES: 7–12
TECHNIQUE: Class demonstration (presented by two members of the class)

Refer the students to the OPM and ask them to experiment with the following problems in mind:

1. Why are bubbles that are blown from a pipe or flicked from a small metal loop always spherical in shape?
2. How do the areas of soap films compare when the film is bounded by a loop that lies in a plane and when it is bounded by a loop that does not lie in a plane? Why?
3. Can you predict the shape of the soap film that will be formed if the wire frame is made into various geometric forms such as a three-dimensional figure-eight loop or a spiral-shaped three-dimensional loop?
4. Can you predict the shape of the soap film when the wire frame forms a cube? A triangular prism? A tetrahedron?

Incidentally, although the OPM suggests formulas for soap solutions, the solution included in soap-bubble kits that can be purchased in any variety store or toy store works very well. Permanent models can be made by using Formafilm. (See the picture at the opening of this chapter.)

FIGURE 4.1. *Illustrations of motivational OPM*




*Port-A-Splay display unit by Media/Graphics, Inc.*

EXAMPLE 2

OPM: *Mathematical Snapshots*, 3d American ed., rev. and enl., by Hugo Steinhaus (New York: Oxford University Press, 1969)

GRADES: 9–12

TECHNIQUE: Introducing the next lesson

A few days before a topic is to be considered by the class, ask one member of the class to investigate a related "snapshot" and to report on it the day the topic is introduced.

As an introduction to the study of measure, have a student report on the following theorem:

> The area of any polygon whose vertices are points of a lattice is equal to one less than the sum of the number of interior lattice points and half the number of lattice points on the border.

This theorem is discussed and illustrated on page 96 of the OPM. As more conventional methods of finding areas of various polygons are studied, their results can be checked against the results found by using this theorem.

EXAMPLE 3

OPM: *The Great International Paper Airplane Book*, by J. Mander, G. Dippel, and H. Gossage (New York: Simon & Schuster, 1967)

GRADES: 6–9

TECHNIQUE: Contest

An active involvement in learning can be gained by proposing a paper-airplane contest to be held at some future date. The stage for the contest may be set by preparing a contest plan and drawing up a set of rules. A sample of such a set of rules follows:

1. Each contestant must make his own paper airplane.
2. An entry specifications sheet must accompany each airplane, describing as accurately as possible:
   *a)* The area of the lift
   *b)* The lateral area
   *c)* The center of the lift
   *d)* The center of lateral area
   *e)* The center of gravity
   *f)* The weight (determined with the aid of a good balance scale).
3. Each contestant must demonstrate the flight characteristics of his airplane at a designated time and place.
4. The winner will be judged on:
   *a)* The completeness and accuracy of the entry specifications sheet
   *b)* The workmanship and neatness shown in the airplane
   *c)* The flight characteristics of the airplane demonstrated at the designated time and place.

A few sample paper airplanes should be shown and demonstrated to the students. For each sample, have a prepared entry specifications sheet. In the earlier grades, it may be necessary for you to show the students how to find the information on the specifications sheet for each sample plane. Older students should be referred to the OPM itself or any other sources they may wish to use.

Set the contest date about a week from the time that the rules are given, and reserve part of a class period in the interim for clarifying the rules and answering questions. You may wish to encourage recognition of the participants in a newspaper story (school or local) or at a school assembly. An exhibit of the airplanes in the school display case would also be an appropriate form of recognition.

EXAMPLE 4

OPM: *Sphereland*, by Dionys Burger, trans. Cornelie J. Rheinboldt (New York: Thomas Y. Crowell Co., Apollo Editions, 1965)

GRADES: 7–10

TECHNIQUE: Question of the week

Select one of the books from a library or book-cart collection, and devise a question concerning a concept that is discussed in an interesting manner in the book. Post the question and the name of the book on the bulletin board. At the end of the week, discuss the answer briefly in class.

The following are a few sample questions concerning information contained in *Sphereland:*

1. Why is it that anyone who continues to travel due west will eventually approach his

starting point—from the east? (P. 61.)

2. Is the sum of the measures of the angles of a triangle always 180°? (Pp. 135–141.)
3. What is the shortest path between two points? (Pp. 163–70.)

EXAMPLE 5

OPM: *Geometric Exercises in Paper Folding*, by T. Sundara Row (New York: Dover Publications, 1966)

GRADES: 7–12

TECHNIQUE: Reviewing properties of polygons by paper folding

Distribute several sheets of irregularly shaped paper to each student in the class, and challenge the students to fold the paper so that the folds form a square. In order to form such a square, the students will need to recall the definition and various properties of a square. Depending on their age and ability, most students should be able to form a square, a rectangle, a rhombus, an isosceles triangle, and perhaps a regular hexagon and a regular octagon in the same manner.

EXAMPLE 6

OPM: *Flatland*, by Edwin A. Abbott (New York: Barnes & Noble, 1963)

GRADES: 9–12

TECHNIQUE: Making a movie

If your school has a photography club (or, better yet, if it offers a course in photography), a joint project of the photography and mathematics groups could prove exciting to both. The visual nature of its contents and the first-person style in which it is written make *Flatland* a natural subject for such a project. Making such a movie—writing the script and then directing and producing it—takes much planning, cooperation, and hard work; but it can be an unforgettable, involving experience.

EXAMPLE 7

OPM: *I Can Learn about Calculators and Computers*, by Raymond G. Kenyon (New York: Harper & Row, 1961)

GRADES: 7–12

TECHNIQUE: Mathematics club project

The differences in ages and mathematical maturity of club members limit the kinds of projects in which the entire club can participate. The explicit directions for making and using an oriental abacus, a set of Napier's bones, a "stepped-wheel" calculator, a digital computer, and an analog computer provide a challenge for every member. Club members can demonstrate the result—a working exhibit—in mathematics classes.

## DISCOVERY

If discovery is viewed as student investigation of concepts and problems that are new to him, then every student can have the satisfaction of discovery in mathematics whether he is led by a programed approach or left to his own devices. The techniques in the examples that follow may be varied to satisfy individual students or individual classes.

EXAMPLE 8

OPM: *Explorations in Mathematics*, by Robert B. Davis (Reading, Mass.: Addison-Wesley Publishing Co., 1966)

This discussion guide can be used individually by students, or it can be used by teachers as a source of novel ways in which students can explore a variety of mathematical concepts. The technique discussed below is based on pages 77–83.

GRADES: 4–8

TECHNIQUE: Guessing functions

Ask a volunteer who has made up a "rule" that describes the relationship between two variables to step up to the chalkboard. As members of the class suggest values for one of the variables, the volunteer lists them and uses his rule to find the corresponding values of the other variable, which he also lists. The object of the game is to guess the rule. To avoid having one or two students always guess the rule before the majority of the class has discovered it, any student who thinks he has discovered the rule should write it on a slip of paper and show it to the volunteer. If the rule was guessed correctly, the guesser is

recognized as having discovered it, but the rest of the students can continue the game until most (or all) of them know the rule.

EXAMPLE 9

OPM: *Shape and Size*, a Nuffield Mathematics Project publication (New York: John Wiley & Sons, 1968)

GRADES: 3–7

TECHNIQUE: Working with models

Divide the class into groups of two or three students each and provide each group with paper fasteners and strips of stiff paper punched near each end. Direct them to make frames of three, four, five, and six sides by joining the strips with the fasteners. If a frame is not rigid, additional strips (struts) should be fastened until the frame becomes rigid. One student in each group should record the information shown in the table.

| Number of Sides in Frame | Least Number of Struts Needed for Rigid Frame | Number of Triangles When Frame Is Rigid |
|---|---|---|
| 3 | 0 | 1 |
| 4 | 1 | 2 |
| 5 | 2 | 3 |
| 6 | 3 | 4 |

The number patterns suggested by the first two columns and by the first and third columns enable the students to make predictions for frames with any number of sides. These predictions can be tested by making more frames or by sketching polygons. Older students may be asked to express these patterns for a frame with $n$ sides; that is, to represent the number of struts by $n - 3$ and the number of triangles by $n - 2$.

Further experimentation leads students to discover that the sum of the measures of the interior angles of a polygon of $n$ sides is $(n - 2) \times 180°$.

EXAMPLE 10

OPM: *Number Patterns*, by William H. Glenn and Donovan A. Johnson (Manchester, Mo.: Webster Publishing Co., 1960)

One of the things a mathematician does all the time is to look for patterns. Searching for patterns can lead to the discovery of important new mathematical ideas. Some mathematicians feel that the study of mathematics is a search for patterns.

GRADES: 8–11

TECHNIQUE: Class activity

The activity outlined below is based on pages 16 and 17 of the OPM. It is particularly effective when studying multiplication of polynomials.

1. Multiply: $(1)(2)(3)(4) = 24$.
   Add 1: $24 + 1 = 25$.
   Note that 25 is a perfect square.
   Thus: $(1)(2)(3)(4) = 5^2 - 1$.
2. Repeat step 1 for $(2)(3)(4)(5)$.
   for $(3)(4)(5)(6)$.
   for $(4)(5)(6)(7)$.
   .
   .
   .
3. Generalize for $(n)(n + 1)(n + 2)(n + 3)$.
   $$\begin{aligned}(n)(n+1)(n+2)(n+3) &= [(n^2 + 3n + 1) - 1][(n^2 + 3n + 1) + 1]\\ &= (x - 1)(x + 1)\\ &= x^2 - 1, \text{ where } x = n^2 + 3n + 1\end{aligned}$$

If step 3 is omitted, the described activity is appropriate as an investigation or as motivation for drill in the lower grades.

EXAMPLE 11

OPM: *Laboratory Manual for Elementary Mathematics*, by W. Fitzgerald, D. Bellamy, P. Boonstra, J. Jones, and W. Oosse (Boston: Prindle, Weber & Schmidt, 1969)

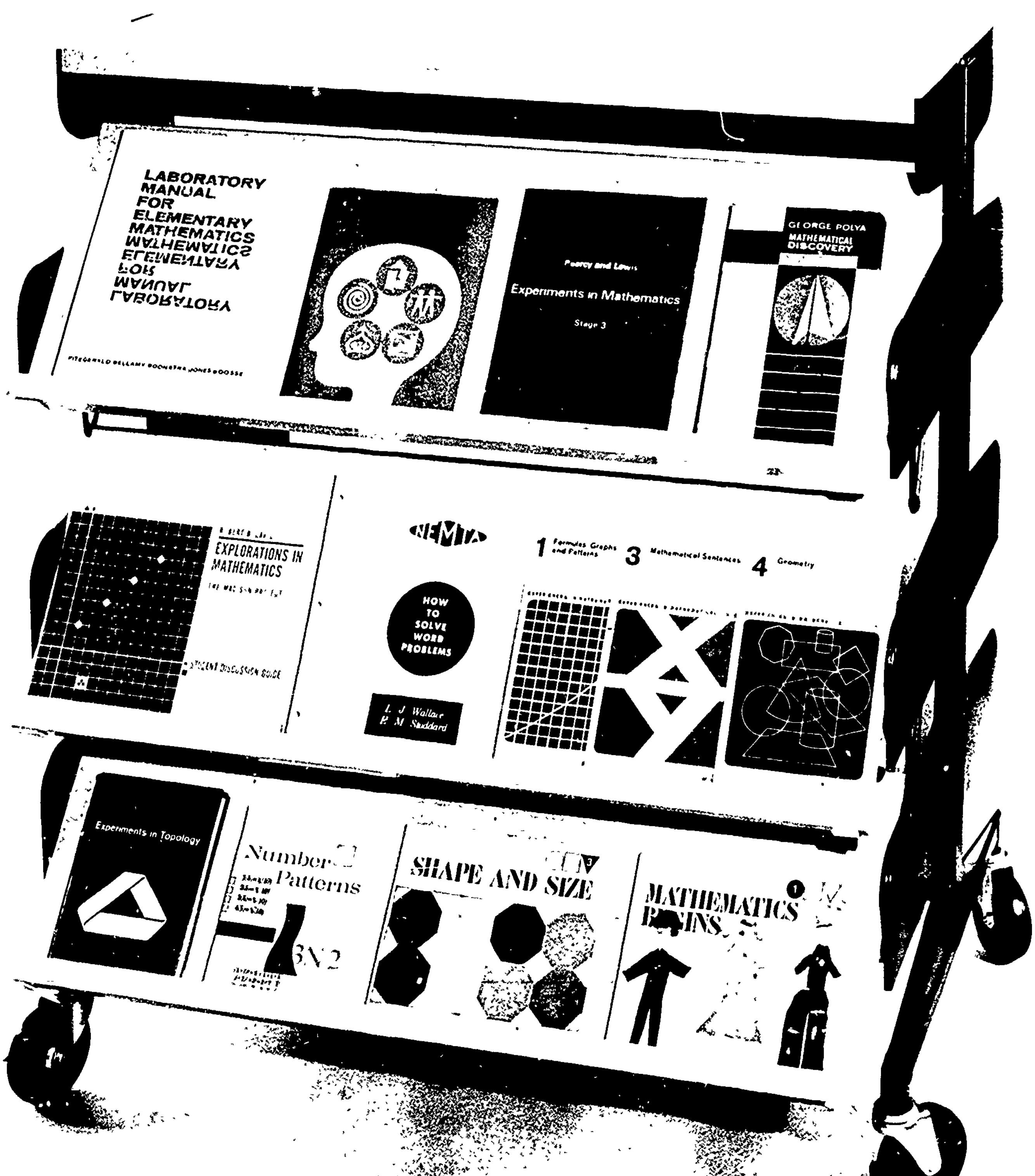
This OPM was designed to be used in college courses for elementary teachers or prospective teachers. However, the activities can be adapted for elementary and high school classes.

GRADES: 5–12

TECHNIQUE: Laboratory lesson

Have each student, on graph paper, outline square regions with one, two, three, four, five, six, and seven units on a side. The smallest figure contains one square region. The next figure contains five square regions, four of which have an area of one square unit each

FIGURE 4.2. *Illustrations of OPM for discovery activities*




*Roll-away resource center display unit by Smith System*

and one of which has an area of four square units. After recording the number of square regions contained in each of three or four of the smallest figures, the students should predict the number of square regions contained in the next larger figure. The prediction can be checked by counting the square regions. The results can be generalized as shown in the following table, with the first column representing the number of units on a side and the second column representing the number of component square regions.

| Number of Units | Number of Regions |
|---|---|
| 1 | 1 |
| 2 | $5 = 2^2 + 1^2$ |
| 3 | $14 = 3^2 + 2^2 + 1^2$ |
| 4 | $30 = 4^2 + 3^2 + 2^2 + 1^2$ |
| . | . |
| . | . |
| . | . |
| $n$ | $n^2 + (n-1)^2 + (n-2)^2 + \ldots + 1^2$ |

EXAMPLE 12

OPM: *Experiments in Mathematics, Stage 3,* by J. F. F. Pearcy and K. Lewis (Boston: Houghton Mifflin Co., 1966, 1967)

This OPM is one of a series of three manuals designed to be used by students in a laboratory situation. The experiments in each book vary in subject matter and difficulty. Many of them can be integrated into the curriculum and adapted for more traditional classroom use.

GRADES: 5–12

TECHNIQUE: Designing your own graph paper

Prepare a few examples of nonrectangular coordinate grids on acetate sheets. For example, one such grid may have equally spaced horizontal grid lines intersected at angles of 60° by equally spaced grid lines. Another such grid may have equally spaced vertical grid lines, while the distance between the horizontal grid lines doubles from line to line. For a third grid, space both the horizontal and the vertical grid lines at distances that double from bottom to top and from left to right, respectively. A fourth grid may have equally spaced vertical grid lines, while the intersecting grid lines all meet in a point.

After the students have worked sufficiently with graph paper consisting of square grids, project one of the other grids onto the chalkboard and ask a student to map a diagonal line segment from a square grid onto the projected nonsquare grid.

EXAMPLE 13

OPM: *Geometry,* Experiences in Mathematical Discovery, Unit 4 (Washington, D.C.: National Council of Teachers of Mathematics, 1966)

This is one of a series of ten self-contained booklets designed for use by ninth-grade general mathematics students. However, many of the concepts are appropriate for use with younger students.

GRADES: 5–10

TECHNIQUE: Class discussion

Each section of the OPM includes a class-discussion feature that is composed of sequential questions based on some activity the students perform. The questions lead the students to generalize the results of the activity and draw conclusions.

EXAMPLE 14

OPM: *Mathematical Discovery,* vol. 1, by George Polya (New York: John Wiley & Sons, 1962)

This OPM, written for the teacher, can be used as a source of problems and ways of investigating problems that teach students the methods of discovery.

GRADES: 8–12

TECHNIQUE: Special unit

On the first day, analyze the following problem in class:

1. A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits are there?

Identify the unknown, the given data, and the condition that relates the unknown and the given data. Then assign several problems such as the following for the students to analyze:

2. One pipe can fill a 600-gallon tank in 12 minutes: another pipe can fill it in 20 minutes: and a third pipe can fill the tank in 30 minutes. How long will it take all three pipes to fill the empty tank?
3. A dealer has two kinds of nuts: one kind costs 90 cents a pound. the other costs 60 cents a pound. He wishes to make 50 pounds of a mixture that will cost 72 cents a pound. How many pounds of each kind should he use?
4. If the area of a right triangle is 37 square inches and its perimeter is 30 inches. what is the length of its hypotenuse?

On the second day, use the groping method to solve problem 1. Have the students make a few stabs at the answer and tabulate the results.

| Hens | Rabbits | Feet |
|---|---|---|
| 50 | 0 | 100 |
| 0 | 50 | 200 |
| 25 | 25 | 150 |

From the table it is clear that there must be more hens than rabbits. Continue the table on this premise.

| Hens | Rabbits | Feet |
|---|---|---|
| 26 | 24 | 148 |
| 28 | 22 | 144 |
| 30 | 20 | 140 |

Thus, there are 30 hens and 20 rabbits.

Assign problems 2, 3, and 4 to be solved by the groping method.

On the third day, use the algebraic method. Begin by stating the problem as shown:

| In English | In Algebraic Language |
|---|---|
| There are a certain number of hens ........................................ | $x$ |
| and a certain number of rabbits .. .................................... | $y$ |
| The hens plus the rabbits have 50 heads ........... ................ | $x + y = 50$ |
| and 140 feet .................................. | $2x + 4y = 140$ |

Then solve the system of equations.

Assign problems 2, 3, and 4 to be solved by the algebraic method.

On the fourth day, compare the problems for similarities and differences.

## ENRICHMENT

Enrichment is usually considered to be any extension of the curriculum—either horizontal or vertical. As such, it is probably the most general of the functions of OPM. Many teachers like to encourage nonstructured enrichment opportunities by giving students time to browse in the mathematics section of the school library or by giving them time to select one of the books from the classroom collection. Some teachers, on the other hand, like to make definite assignments to meet specific instructional objectives.

Where a classroom collection is not practical because of change-of-room procedures and the need for many classes to share the collection, book trucks that can be moved easily from class to class have proved very successful (Figure 4.3). A schedule can be planned to coincide with class use, and students can be assigned the responsibility of delivering the book truck to the scheduled class.

### EXAMPLE 15

OPM: *Measuring Systems and Their History*, by the engineering staff of Ford Motor Co. (Dearborn, Mich.: The Company, 1966)

GRADES: 7–12

TECHNIQUE: Group activity as introduction to a unit on measure

Although the OPM can be used beneficially at any one of the grade levels suggested above, the following suggested activity is most suitable for grades 7–9.

Assign each class member to one of five committees—one committee for each unit in the booklet. Each committee should plan how best to report on the unit for which it is responsible. Maps, tables, drawings, and other visual displays should be prepared to illustrate an oral report to the class.

After the initial planning period, the work should be done outside of class. The oral reports can be given one each day for a week during the study of the unit on measure, and the displays can be used as bulletin-board exhibits.


FIGURE 4.3. *A collection of OPM that can be used for enrichment*

### EXAMPLE 16

OPM: *Men of Mathematics,* by Eric T. Bell (New York: Simon & Schuster, 1937; paper, 1965)

GRADES: 9–12

TECHNIQUE: Supplemental reading

This OPM can be read for enjoyment at any time. It is the kind of book that should be included in a collection of OPM that students may read in class after completing an assignment. Students whose interest is aroused by the IBM chart (OPM Example 25) will enjoy reading about the lives of mathematicians represented on the chart.

Reference should be made to the OPM in conjunction with the discussion of concepts that are studied in class. For example, when the Cartesian plane is introduced or when the equation of a circle is discussed, suggest that the students read Chapter 3, "Gentleman, Soldier, and Mathematician." When prime numbers are studied, recommend Chapter 4, "Prince of Mathematicians." When complex numbers are considered in class, recommend Chapter 19, "An Irish Tragedy."

### EXAMPLE 17

OPM: *The Gentle Art of Mathematics,* by Daniel Pedoe (New York: The Macmillan Co., 1963)

GRADES: 8–12

TECHNIQUE: Adding depth to classwork

During the class discussion of one-to-one correspondence, lead the class to describe how a one-to-one correspondence can be established between the natural numbers and various sets of numbers, such as the ones listed below.

1. The negative integers
2. The square numbers
3. The positive square roots of the natural numbers
4. The even natural numbers
5. The odd natural numbers
6. The natural numbers divisible by 3
7. The set of positive rational numbers that can be expressed with 2 as the denominator
8. The set of positive rational numbers that can be expressed with 7 as the numerator
9. The set of integers; that is,

   $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

   (e.g., when $n$ is odd, $n \leftrightarrow \frac{n-1}{2}$; when $n$ is even, $n \leftrightarrow -\frac{n}{2}$)
10. The set of powers of 2; that is,

    $\{\ldots, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$

    (e.g., when $n$ is odd, $n \leftrightarrow 2^{(n-1)/2}$; when $n$ is even, $n \leftrightarrow 2^{-n/2}$).

Point out the need to arrange the members of the sets in some order so that there is a first member, a second member, a third member, and so forth. The students should see that if the members of a set can be arranged in such an order, then they can be put in a one-to-one correspondence with the natural numbers.

Then ask the students to investigate whether or not the set of positive rational numbers can be put in a one-to-one correspondence with the set of natural numbers. After they have spent a sufficient length of time on the investigation, suggest that the interested students read Chapter 3 of the OPM and report to the class the next time the class convenes.

### EXAMPLE 18

OPM: *What Is Mathematics?* by Richard Courant and Herbert E. Robbins (New York: Oxford University Press, 1941)

GRADE: 12

TECHNIQUE: Question of the week

The following question, together with the suggestion that the OPM may be helpful as a reference, may be posted on a bulletin board.

Can you show that $e^{\pi i} + 1 = 0$?

To find the sum, the students should review the infinite-series definition of $e^x$, as given on page 449 of the OPM, and should verify that

$$e^{ix} = \cos x + i \sin x.$$

Page 478 may provide the hint necessary for this verification. Replacing $x$ by $\pi$ gives

$$\begin{aligned} e^{\pi i} &= \cos \pi + i \sin \pi \\ &= -1 + 0. \end{aligned}$$

EXAMPLE 19

OPM: *Elementary Mathematics: Enrichment*, by Lola J. May (New York: Harcourt Brace Jovanovich, 1966)

These materials were prepared to be used by students. They are self-explanatory and were designed to extend the average curriculum for each designated grade.

GRADES: 3–6

TECHNIQUE: Individual activity or class project

Individual activity is suggested (for better students while less able students are concentrating on drill) when class sets are available and for a class project when class sets are not available. Such a project for a fourth-grade class can be centered around "calendar squares," which are discussed on pages 12–15 of the OPM.

If actual calendar pages are available, supply one for each student and direct each student to outline any 3 by 3 array of numerals on his page with a square. Then tell the students to find the sum of the numbers in the center row, the center column, and each diagonal of the square. Although the sums will vary for different students, each student should find the same answer for each sum. Furthermore, the center number should equal one-third of this sum.

Similar results can be found for any 5 by 5 calendar square. Interested students should be encouraged to investigate 2 by 2 and 4 by 4 calendar squares.

EXAMPLE 20

OPM: The Franklin Mathematical Series (Pasadena, Calif.: Franklin Publications, 1968)

This series consists of the following eleven books. The first seven are case-bound, and the remainder are paperbound. The series was designed specifically to extend and enrich the mathematics curriculum. Each book is activity-oriented and directs the students to construct, experiment, analyze, and generalize.

*Learning about Measurement*, by Sylvia Horne (grades 3 and 4)

*Mathematics around the Clock*, by Margaret F. Willerding (grades 5–7)

*Patterns and Puzzles in Mathematics*, by Sylvia Horne (grades 5–7)

*From Fingers to Computers*, by Margaret F. Willerding (grades 6-8)

*Probability: The Science of Chance*, by Margaret F. Willerding (grades 7 and 8)

*Mathematics: Man's Key to Progress*, by Richard A. Denholm (book A, grades 6–8; book B, grades 7 and 8)

*Learn to Fold—Fold to Learn*, by Janet S. Abbott (grades 3 and 4)

*Mirror Magic*, by Janet S. Abbott (grades 3–5)

*Paper and Pencil Geometry*, by Susan Roper (grades 4–6)

*Making and Using Graphs and Nomographs*, by Richard A. Denholm (grades 5 and 6)

GRADES: 7 and 8 for the technique that follows

TECHNIQUE: Studying in small groups composed of students who do not require help in a current unit

When less able students require extra help in fundamental skills or on the current unit, students who do not need such help can study *From Fingers to Computers* (fourth book of this OPM series) in small groups. The membership of each group should remain constant during the study of each of the five parts of the book.

The study group should function independently with almost no help from the teacher. During the study of Part 1, "Finger Computation," the students should study the descriptions and diagrams and then master the computations by performing the exercises. Discussion and group criticism should be part of the learning process.

Although Part 2, "The Abacus," can be studied from the diagrams in the OPM, a commercial abacus—or one the students in the group make—would be even more satisfactory.

In Part 3, "Napier's Rods," and Part 4, "The Slide Rule," the students are directed to build instruments and use them in computing.

Part 5, "The Electronic Computer," may be separated into several smaller units to allow the membership of the group to change during the time it is under study.

### EXAMPLE 21

OPM: *Continued Fractions,* by C. D. Olds, New Mathematical Library, bk. 9 (New York: L. W. Singer Co., 1963)

Although concepts discussed in this OPM can be presented to students as soon as they can add fractions, the OPM is not intended for independent reading by students below the ninth or tenth grades.

GRADES: 9–12

TECHNIQUE: Extension of classwork

When studying the solution of quadratic equations by factoring, some equations are found that cannot be factored in the usual manner. Ordinarily, this situation is followed by the immediate development of the quadratic formula. You may wish to introduce an intermediate method, as described on pages 5–7 of the OPM. The equation

$$x^2 - 3x - 1 = 0$$

may be expressed as

$$x = 3 + \frac{1}{x},$$

providing $x \neq 0$.

Since $x = 3 + \frac{1}{x}$, this value may be used to replace $x$ on the right side of the equation, giving

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{x}}.$$

Continuing to make the replacement gives

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{x}}}}}.$$

Note that this continued fraction gives successive approximations for $x$:

$$3,\ 3 + \frac{1}{3},\ 3 + \cfrac{1}{3 + \cfrac{1}{3}},\ 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3}}},\ \ldots .$$

Evaluating these fractions suggests that the sequence converges to the value

$$x = \frac{3 + \sqrt{13}}{2} = 3.302775\ \ldots .$$

found by using the quadratic formula.

### EXAMPLE 22

OPM: Mathematics in the Making (Boston: Houghton Mifflin Co., 1967, 1968)

This OPM consists of the following booklets. The booklets were designed to be used by the students. If classroom sets are not available, many of the activities described in the booklets can be adapted for use by individual students.

1. *Pattern, Area and Perimeter,* by Stuart E. Bell
2. *Binary and Other Numeration Systems,* by Stuart E. Bell
3. *Looking at Solids,* by Stuart E. Bell
4. *Rotation and Angles,* by Stuart E. Bell
5. *Curves,* by Stuart E. Bell
6. *Scale Drawing and Surveying,* by Stuart E. Bell
7. *Transformations and Symmetry,* by Stuart E. Bell
8. *Networks,* by Stuart E. Bell
9. *All Sorts of Numbers,* by Stuart E. Bell
10. *Sets and Relations,* by Stuart E. Bell
11. *Graphs,* by G. Long and K. A. Hides
12. *Statistics,* by Irene Campbell

GRADES: 3–8

TECHNIQUE: Laboratory lesson on drawing curves

Adapt the directions outlined in *Curves* (bk. 5), pages 22–25, 31 and 32, for classroom use.

### EXAMPLE 23

OPM: *Theory of Equations,* by Cyrus C. MacDuffee (New York: John Wiley & Sons, 1954)

(Actually, any textbook on theory of equations could be used.)

GRADES: 11 and 12

TECHNIQUE: Independent research

After studying methods for solving polynomial equations, ask the students to suggest ways to solve the cubic equation

$$x^3 + 3x^2 - 15x - 127 = 0.$$

When the familiar methods of factorization and trial by synthetic division fail, suggest substituting $y - 1$ for $x$. The result gives

$$y^3 - 18y - 110 = 0.$$

Then let $y = z + \frac{6}{z}$ to obtain

$$z^6 - 110z^3 + 216 = 0.$$

Since the resulting polynomial equation is in quadratic form, it can be solved by using the quadratic formula to give

$$z_1 = 3\sqrt[3]{4} \quad \text{and} \quad z_2 = \sqrt[3]{2}.$$

To relate $z_1$ and $z_2$ to the original equation, substitute the values of $z_1$ and $z_2$ into $y = z + \frac{6}{z}$ to give

$$y = 3\sqrt[3]{4} + \sqrt[3]{2}.$$

Thus, $3\sqrt[3]{4} + \sqrt[3]{2} - 1$ is a solution of the original cubic equation.

Ask the students to verify this solution and then to generalize the solution procedure for

$$ax^3 + bx^2 + cx + d = 0.$$

Interested students may wish to compare their results with those in the OPM.

EXAMPLE 24

OPM: *Numbers Old and New*, by Irving and Ruth Adler (New York: John Day Co., 1960)
GRADES: 3–6
TECHNIQUE: Student preparation of a classroom chart

This activity is particularly suitable for days when absence is very high because of bad weather, a religious holiday, or high incidence of illness. Any new work introduced on such days would need to be retaught later; but students who are present should be able to participate in a meaningful learning experience.

An opaque projector can be used to project the numerals of the ancient Greek numeration system, as they are given on page 14 of the OPM, onto a sheet of tagboard. By adjusting the distance of the projector from the tagboard, the chart can be enlarged to cover the available space. Individual students can be assigned to trace parts of the projection. If the students use grease pencils or felt-tipped pens with broad tips, the resulting charts can be easily read from a distance and can be made part of a permanent collection. The Ancient Hebrew numerals, shown on page 15 of the OPM, could be used to make a companion piece for the chart described above.

Such charts, made by members of the class, will be more than classroom decorations and are likely to be studied by students before and after class.

Many other suitable materials from the public domain can be copied in this manner without violating copyright agreements. *Arithmetic Charts Handbook* and *Charts for the New Math*, by E. Dumas, C. F. Howard, and J. E. Dumas, provide collections of charts prepared to be reproduced by means of an opaque projector. *Making and Using Charts* (rev. ed.), by A. Liechti and J. Chappell, is another source of information on charts. These three books are published by Fearon Publishers, Palo Alto, California.

EXAMPLE 25

OPM: *Men of Modern Mathematics, a History Chart of Mathematics from 1000–1900* (Armonk, N.Y.: International Business Machines Corp.)

This 150 by 24 inch chart illustrates the development of mathematics on a timetable background of important historical events in all fields.

GRADES: 4–12
TECHNIQUE: Wall-chart reference

This chart can be a valuable addition to any instructional space and particularly to small group spaces. One of the easiest ways to use the chart is to make frequent reference to it as one of its entries is mentioned during a class discussion. Students should be encouraged to consult it frequently and to do research on the lives and work of mathematicians listed on the chart.

The amount and level of research inspired by the chart will vary with the grade in which the chart is used, but the feeling that mathematics is a vital part of history and that it is created by flesh-and-blood men and women can be fostered at all grade levels.

## CHANGE OF PACE

There comes a time in every class when a change of pace is needed. A change of pace can take many forms such as a quickie puzzle at the end of a class period, recreational material during the last class period before vacation, a competition between class teams, or a special unit.

### EXAMPLE 26

OPM: *Coordinated Cross-Number Puzzles*, by William H. Crouch (Cincinnati: McCormick-Mathers Publishing Co., 1969)

This OPM is an excellent source of many carefully selected cross-number puzzles. Each student in the class should be provided with a copy.

GRADES: 3 and 4

TECHNIQUE: End-of-class-period activity

Have students start a puzzle about ten minutes before the end of a class period. This will provide time to give them a good start and to generate interest in finishing the puzzle outside of class.

### EXAMPLE 27

OPM: *Mathematical Models*, 2d ed., by H. Martyn Cundy and A. P. Rollett (New York: Oxford University Press, 1961)

GRADES: 4–10

TECHNIQUE: High-absence-day activity

Discuss tessellations—sometimes called tiling patterns—with the students and let them form their own. There are various types of tessellations. Regular tessellations consist of patterns of congruent regular polygons, all of the same kind. Semiregular tessellations contain two or more kinds of congruent regular polygons arranged so that the same combination of polygons meet at each vertex in the pattern. Other tiling patterns are formed by congruent nonregular polygons and by congruent curved regions. Examples of each type are illustrated and described in detail on pages 59–68 of the OPM.

### EXAMPLE 28

OPM: *Mathematical Challenges* (Washington, D.C.: National Council of Teachers of Mathematics, 1965)

This OPM is a compilation of 140 selected problems from the *Mathematics Student Journal*, together with their solutions. The problems are organized into geometric, algebraic, inferential, and miscellaneous categories.

GRADES: 7–12

TECHNIQUE: Class opener

Write one of the short, simple problems presented in the OPM on the chalkboard so that students who arrive before the class bell rings can begin thinking about the problem. Use the first five minutes of the class period to discuss the problem and its solution. The students should then be ready to settle down to the lesson for the day.

### EXAMPLE 29

OPM: *We Built Our Own Computers*, ed. Albert B. Bolt (New York: Cambridge University Press, 1966)

GRADES: 9–12

TECHNIQUE: Using logical diversions

Either during or after the study of truth tables, ask the students how they can use the methods of logical reasoning to solve the murder described on pages 16 and 17 of the OPM. Incidentally, the complete solution is given in the OPM.

In the next class period, challenge the students to a game of tick-tack-toe under the following rules:

1. The teacher always plays first.
2. The teacher always opens with 0 in the lower left-hand square.

During the third or fourth game, begin to question the reasoning behind the various moves. Lead the students to express the reasons for the moves as implications. With the students working in pairs, have them analyze the various possible plays in a game and express each move in logical form. In order that results can be compared, number the squares from left to right in each row, starting with the top row.

Examples of various logical conditions used in the game are shown in Chapter 8 of the OPM, and a computer-wiring diagram for the game is given on page 71.

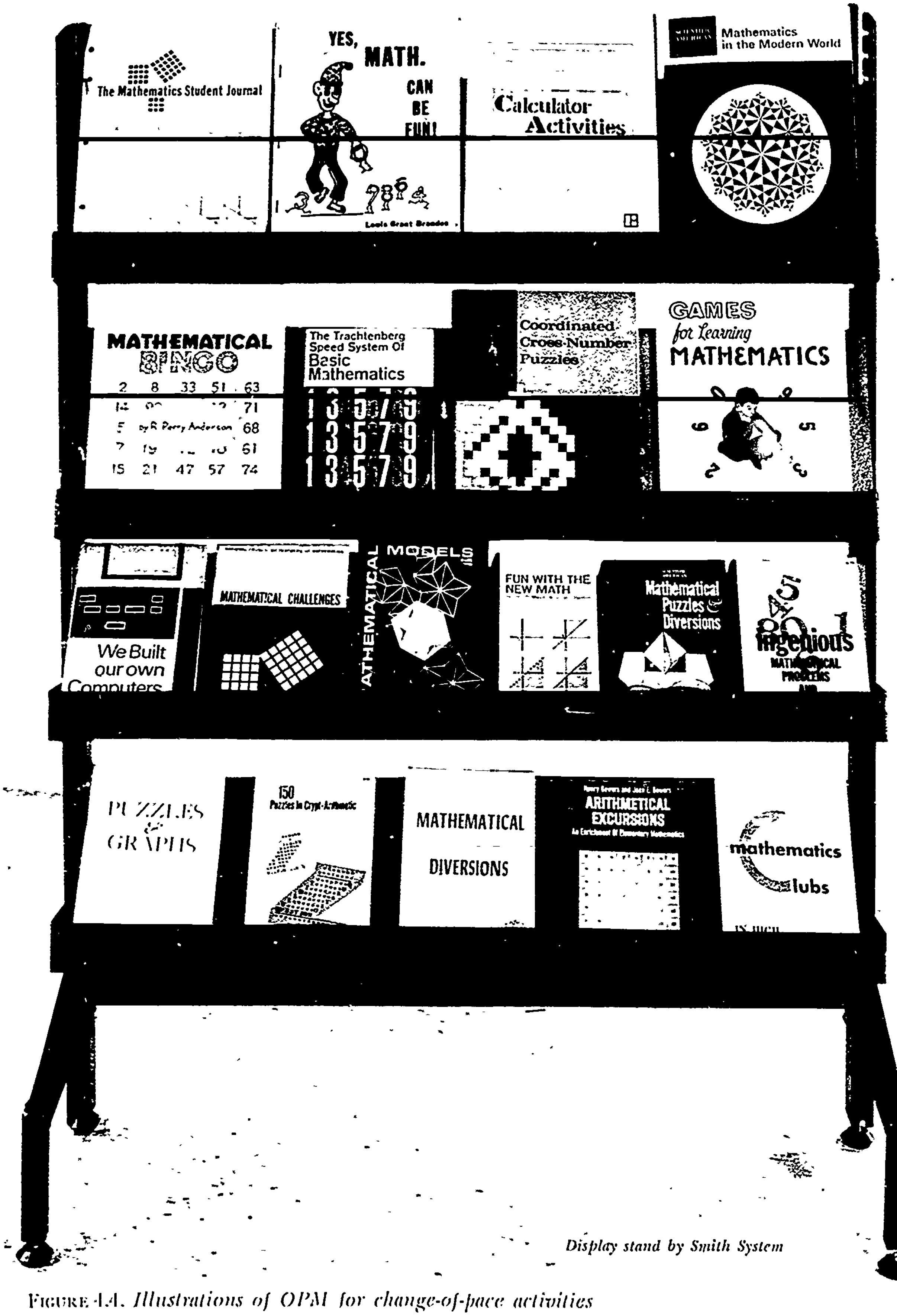



FIGURE 4.4. *Illustrations of OPM for change-of-pace activities*

EXAMPLE 30

OPM: *Games for Learning Mathematics*, by Donovan A. Johnson (Portland, Maine: J. Weston Walch, 1960)

This OPM contains descriptions and directions for seventy games that involve arithmetic, algebra, and geometry. Most of them are recommended for use by mathematics classes. However, one chapter contains party games suitable for mathematics club meetings or class parties.

GRADES: 5–9

TECHNIQUE: Quiz panels for a chapter review period

Two panels compete in identifying a subject. The panels take turns asking questions to which the moderator (teacher) can answer either yes or no. Five *no* answers is the limit per panel. Rules may vary to accommodate special circumstances.

EXAMPLE 31

OPM: *The Scientific American Book of Mathematical Puzzles and Diversions*, by Martin Gardner (New York: Simon & Schuster, 1964)

GRADES: 6–12

TECHNIQUE: Prevacation activity

Provide the students with strips of paper (construction paper works well) I¸ inches wide and approximately 32 inches long. If [illegible]cher aide is available for such tasks, each strip should be prefolded into 19 adjacent, congruent triangular regions, as described on page 4 of the OPM.

After making the hexahexaflexagons in class, the students can form designs by coloring the faces, as described on pages 8 and 9. These designs can form the basis of an amusing diversion for younger students as well as the impetus for serious research for older students.

EXAMPLE 32

OPM: *Puzzles and Graphs*, by John N. Fujii (Washington, D.C.: National Council of Teachers of Mathematics, 1966)

Although this OPM could be used as a complete change of pace unit, time does not ordinarily permit such diversions. However, using ideas from it occasionally in class may influence students to read it independently.

GRADES: 8–12

TECHNIQUE: Going off on a tangent

After discussing the definition of a polygon, define a complete graph as one in which each pair of vertices is connected by an edge. Then ask the students to determine the greatest number of vertices for which a complete graph can be drawn with its edges intersecting at some point that is not a vertex. The answer to this question is illustrated on page 7 of the OPM.

The OPM contains many other subjects suitable for tangential consideration, including bonus puzzles at the back of the book.

## DRILL AND PRACTICE

Whether it is in spite of the current emphasis on understanding basic principles or because of it, the fact remains that the need for drill is an ever-present one. How to vary the routine nature of drill and how to individualize it for the maximum benefit of students are, likewise, ever-present concerns.

Cross-number puzzles, variations of magic squares, number sentence flash cards, computational skills kits, and practice problem kits can all be used as variations of drill.

EXAMPLE 33

OPM: *1, 2, 3: A Book to See*, by William Wondriska (New York: Random House, Pantheon Books, 1959)

GRADES: K and 1

TECHNIQUE: Tell a story

Have the children build a story around the objects that are the characters of the book. Number and numeral recognition (from 1 to 10) and order become an integral part of the story. After the book has been used several times, consider starting at the back and working forward, or working from start to finish and then back again for a given story.

EXAMPLE 34

OPM: *Individualized Mathematics: Drill and Practice Kits*, by Patrick Suppes and Max Jerman (New York: L. W. Singer Co., 1969)

Each of four kits contains 500 different prac-

tice lessons, 20 different pretests, 20 different posttests, 4 sets of posttest answer cards, 30 student-profile cards, and a teacher's guide. By varying the order of the blocks of lessons, each kit can be adapted to fit almost any elementary mathematics curriculum.

GRADES: 3–8

TECHNIQUE: Independent student activity

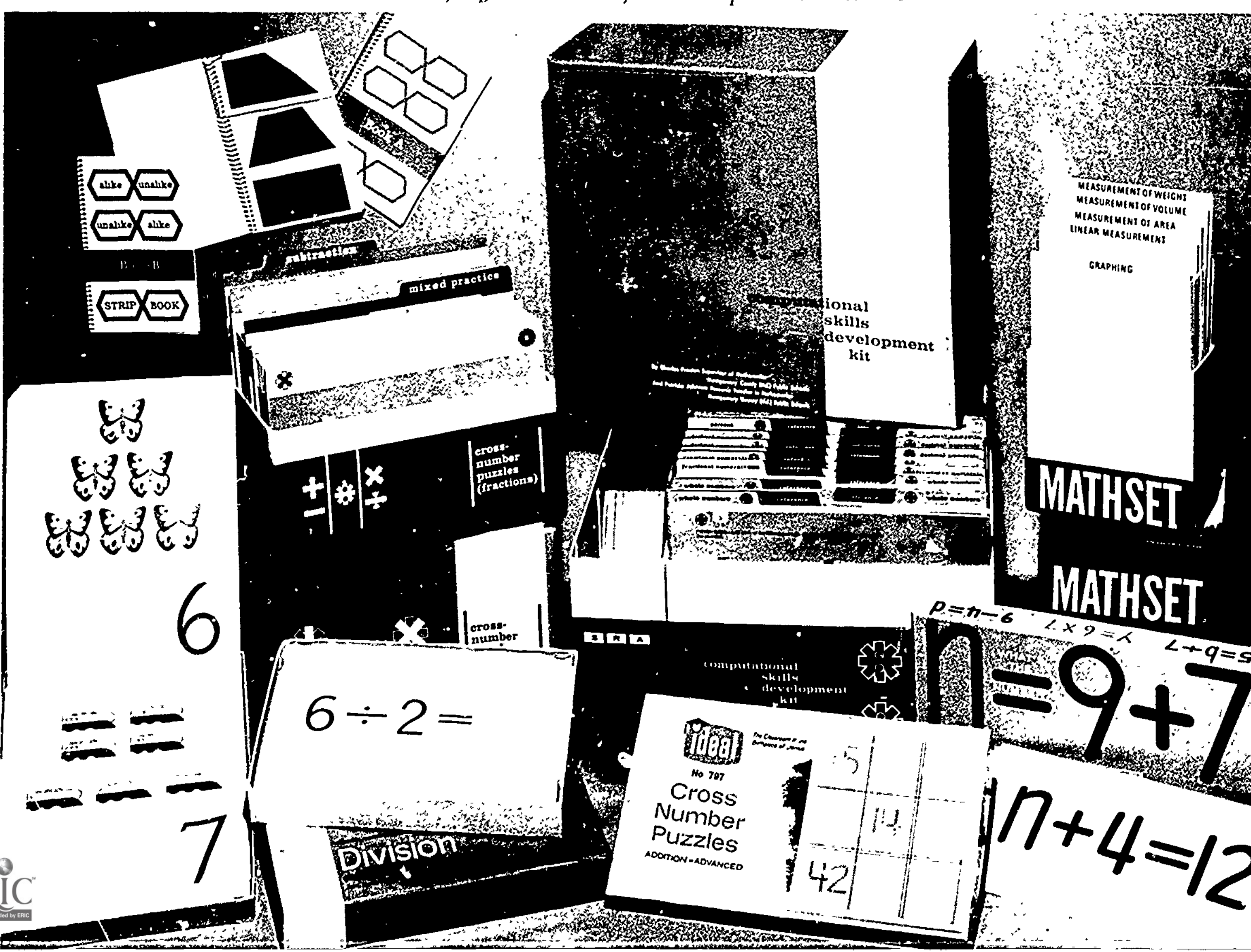
On the basis of his initial pretest score, the student is directed to a specific lesson card. He then works independently—administering, correcting, and scoring his own lessons. His score on each lesson determines whether he will continue on the same level of difficulty or move up or down a level. After the fifth lesson, the student is directed to the posttest. He records this score on his profile card, from which the teacher can evaluate the student's progress at any time.

EXAMPLE 35

OPM: Alike Unalike Strip Books, set 1, by P. B. Radnore (London: Philip & Tacey, 1964); distributed by Math Media Div., H and M Associates, Danbury, Conn.

A series of four books, each of which stresses recognition of shape and size. The strips in any one of three horizontal sections into which the pages are cut may be flipped independently until

FIGURE 4.5. *A collection of different kinds of drill and practice materials*



the shapes and sizes of all three exposed figures are matched.

GRADES: K–5

TECHNIQUE: Small discussion groups

In grades K–2, a teacher or teacher aide should work with each group. When necessary, the adult can ask questions that lead the children to recognize similarities and differences in the shapes and sizes of the various figures.

In grades 3–5, groups of students should work independently. One member of each group may be designated to sketch each group of matched figures.

EXAMPLE 36

OPM: *What's the Number?* (Pasadena, Calif.: Franklin Teaching Aids)

In Box 1, each of 50 cards (4½ by 9 inches) contains an addition or subtraction combination in equation form. The answer and the related subtraction or addition combination is given on the reverse side.

GRADES: 3–6

TECHNIQUE: Contest

Separate the class into teams A and B, and display the first card. If the first person on team A gives the correct response, show the reverse side and discuss the answer and the related combination. If the correct response is not given, call on a volunteer from team B. If he gives the correct answer, discuss the reverse side of the card. If he does not give the correct answer, insert the card at random in the unused pile of cards. Record a point for the team whose member supplies the correct answer.

Reveal card 2 for the first person on team B and repeat the procedure.

EXAMPLE 37

OPM: *Cross-Number Puzzles* (Oaklawn, Ill.: Ideal School Supply Co.)

This OPM consists of cards that list operations to be performed and individual workbooks in which the students record their answers in the form of cross-number puzzles.

GRADES: 4–7

TECHNIQUE: Individual drill

Each student can proceed through the workbook at his own speed. Because of the nature of cross-number puzzles, every puzzle contains a built-in check on the student's answers. That is, if all the answers that are to be recorded horizontally are entered first, the blanks in the puzzle columns are automatically filled and should be the correct answers to the "down" operations. If they are not, the student is alerted to the inaccuracy of one of the "across" operations.

## APPLICATIONS

Not all students need to see frequent applications of what they study, but all students get a sense of accomplishment from applying the mathematics they learn to out-of-school daily life situations.

EXAMPLE 38

OPM: HFC Money Management Program (Chicago: Money Management Institute, Household Finance Corp., 1968)

GRADES: 6–12

TECHNIQUE: Special unit

This program is a self-contained unit available in both English and Spanish which includes twelve booklets, with a teacher's guide for each, and five film strips. It can be used in place of, or in conjunction with, a chapter on related content in the text.

EXAMPLE 39

OPM: *The World Almanac and Book of Facts*, ed. Luman H. Long (Garden City, N.Y.: Doubleday & Co.)

The statistics concerning heights of mountains, areas of states, average temperatures, and so forth, make excellent information for bar graphs, circle graphs, histograms, and broken-line graphs.

GRADES: 4–8

TECHNIQUE: Class reference

EXAMPLE 40

OPM: Catalog of Sears Roebuck & Co., Montgomery Ward, or any other large mail-order house

GRADES: 5–8

TECHNIQUE: Use of activity sheet

If a classroom supply of fliers is available, students can work independently. When only a few copies of a catalog must be used by the entire class, students may work in small groups.

Since a catalog is an item that is not ordinarily kept after its expiration date, it would be easy to accumulate a classroom supply. This would make it possible to compute the percent of increase (or decrease) in the price of comparable items from year to year and, in any one catalog, to compare shipping charges for different items.

A sample activity sheet is given below.

Name ____________________ Date __________

BUYING A REFRIGERATOR

The refrigerator in your home needs to be replaced. It must contain a freezer unit, and it must fit into a space 30 inches wide and 68 inches high. It may be either white or coppertone.

1. Select a refrigerator that meets these requirements and record its price.__________
2. What is the shipping weight of the refrigerator? __________
3. What is the shipping point of the refrigerator? __________
4. How far do you live from this shipping point? __________
5. Use the freight information table in the catalog to find the shipping charge.__________
6. What is the total cost of your purchase? __________

Questions concerning sales tax and installment buying may also be included for the appropriate grades.

EXAMPLE 41

OPM: Individual income-tax forms (Washington, D.C.: Internal Revenue Service)

GRADES: 9 and 10

TECHNIQUE: Special unit

In preparation for filling in the tax forms, decide with the class on vital questions concerning an imagined family, including the number of persons, kind of housing, and so forth.

Assign groups of students to prepare information necessary for filling out the forms. One group can be responsible for information concerning wages, including all the deductions. Another can be assigned to home expenses, such as interest on a mortgage if the family is buying a home. Other groups can be responsible for medical expenses, contributions, and so forth. When all the information is assembled, it can be duplicated for each member of the class.

Work on the forms can be done individually, as a class activity with the help of an opaque projector, or by small groups.

EXAMPLE 42

OPM: Map of your state, available from various sources

GRADES: 3–6

TECHNIQUE: Special project—planning an automobile trip

This project can be as extensive as you wish to make it. After the itinerary is decided on in class, students may write for hotel, motel, and other information on which to base expenses. All expenses, such as cost of meals, rooms, tolls, car expenses, car repairs (if it is decided by the class that those will be necessary), and admission to museums and theaters should be carefully considered in conjunction with a predetermined budget.

EXAMPLE 43

OPM: *Better Homes and Gardens Cook Book* (Des Moines: Meredith Publishing Co.)

GRADES: 4–8

TECHNIQUE: Special project—planning a class picnic menu, complete with recipes

Lead the students to include some canned foods, such as baked beans; some foods measured by weight, such as hot dogs and marshmallows; and some foods to be prepared from recipes, such as potato salad and cookies.

Use the tables of weights and measures and the table of can sizes on the inside back cover of the OPM in planning the quantities needed. Have the students select recipes and then adjust the quantities of the ingredients as necessary to feed the entire class.


FIGURE 4.6

*OPM that have applications in out-of-school daily life situations*

## PROFESSIONAL GROWTH

Too frequently a teacher's professional growth is thought of exclusively in terms of course work. This, of course, is not the case. Professional growth should take place with the preparation of every lesson. Each time the mathematics to be taught is reconsidered, new insights, new methods of approach, and new applications can be found. But professional growth should also be more extensive. Meetings of professional organizations, curriculum study groups, textbook evaluation committees, workshops, seminars, and many other similar activities should foster professional growth. Of course, individual study should not be neglected. A few hours of reading and study each week can lead to a more satisfying and better teaching career.

### EXAMPLE 44

OPM: Yearbooks of the National Council of Teachers of Mathematics (Washington, D.C.: The Council)

*The Learning of Mathematics: Its Theory and Practice*, 21st Yearbook. 1953.

*The Growth of Mathematical Ideas, Grades K-12*, 24th Yearbook. 1959.

*Evaluation in Mathematics*, 26th Yearbook. 1961.

*Topics in Mathematics for Elementary School Teachers*, 29th Yearbook. 1964.

*Historical Topics for the Mathematics Classroom*, 31st Yearbook. 1969.

TECHNIQUE: Reference library

Every mathematics teacher's personal reference library should contain relevant volumes of NCTM yearbooks. The representative volumes listed above could be used as the nucleus of such a library. Periodic refresher browsing, as well as a search for answers to specific questions that arise in the classroom, results in the kind of professional growth that contributes to satisfaction in teaching.

### EXAMPLE 45

OPM: *Random Essays on Mathematics, Education and Computers*, by John G. Kemeny (Englewood Cliffs, N.J.: Prentice-Hall, 1964)

TECHNIQUE: Reading for fun

In these essays Kemeny examines such problems as outmoded teaching methods that squelch the excitement of mathematics, the education of the well-rounded man, and the "vanishing teacher."

### EXAMPLE 46

OPM: State curriculum guide for mathematics

TECHNIQUE: Informal study group

Most state and local curriculum guides for mathematics reflect the concerted efforts of leaders in mathematics education to adapt current trends to local needs. The guides ordinarily contain an outline of the content for each course or grade, course objectives, and suggestions on approach.

Unfortunately, many teachers are not as well acquainted with such guides as they should be. Academic-subject-matter supervisors or department heads might encourage the formation of informal groups to examine the local guide and discuss ways in which its aims can be met in individual classes.

### EXAMPLE 47

OPM: *An Introduction to the History of Mathematics*, 3d ed., rev., by Howard Eves (New York: Holt, Rinehart & Winston, 1969)

TECHNIQUE: Lesson preparation

Historical references concerning subject matter being studied make lessons more interesting and give students a greater understanding of the overall development of mathematics. For example, when studying the sum and product relationships of the solutions of a quadratic equation, consider demonstrating Euclid's geometric solution of quadratic equations as illustrated and explained on pages 73–74 of the OPM (pp. 67–69 of the 1964 edition).

Such a demonstration will review geometric-construction techniques and foster a greater understanding of the relationship between the solutions of a quadratic equation. It will also make the students appreciate the convenience of algebraic notation and help them to see algebra and geometry as branches of the tree of mathematics.

EXAMPLE 48

OPM: *Mathematics: The Man Made Universe*, by Sherman K. Stein (San Francisco: W. H. Freeman & Co., 1963)

TECHNIQUE: Reading for extended horizons

Teachers of mathematics in upper elementary through senior high school will find this OPM a source of many fresh mathematical ideas that can be shared with their classes. Furthermore, the interrelations and unusual applications of number theory, topology, set theory, geometry, algebra, and analysis that are presented in the OPM will provide many hours of pure enjoyment for anyone who is interested in mathematics.

EXAMPLE 49

OPM: *Major Topics in Modern Mathematics: Set and Group Theory*, by Donald E. Mansfield and Maxim Bruckheimer (New York: Harcourt, Brace & World, 1965)

TECHNIQUE: Independent study

This text was written expressly for mathematics teachers in England who are caught up in curriculum reform and need to judge the mathematical content of the many new courses being suggested for use. The authors never forget their readers—they anticipate the kinds of questions that a teacher would raise when faced with new ideas, and they include approaches and exercises that he can use in his teaching.

The content of the OPM includes work with sets, equivalence relations, mappings, cardinal numbers, groups, isomorphisms, homomorphisms, geometry as a study of properties invariant under groups of transformations, and "higher" structures such as rings and vector spaces.

EXAMPLE 50

OPM: *A New Look at Geometry*, by Irving Adler (New York: John Day Co., 1966)

Its clarity, style, and content make this OPM quite versatile—it serves as a challenging study for the amateur mathematician, as a reference for the high school mathematics teacher, as supplementary reading for the very able high school student, and as a text for the high school teacher and future teacher.

TECHNIQUE: In-service course

As the content of the traditional high school geometry course continues to be criticized and altered, it is important for high school teachers to be familiar with geometries other than Euclidean geometry. These may soon be part of the high school course. In this OPM the author examines many geometries: pre-Euclidean, Euclidean, metric, vector, transformational, non-Euclidean, and projective.

## SUMMARY

In recent years the revolution in mathematics education has resulted in many changes in the curriculum. Despite the rapidity with which these changes occur, it still takes three or more years for a textbook to be written and published. However, OPM provide a means of keeping the curriculum up-to-date and balanced.

There are a number of bibliographies that can be used as source lists of OPM. The publications list of the National Council of Teachers of Mathematics is an obvious source. The NCTM also publishes *The High School Mathematics Library*, by William L. Schaaf, in which about 800 entries are classified by subject and annotated (this is now in a 1970 edition) and *Mathematics Library—Elementary and Junior High School*, by Clarence Ethel Hardgrove and Herbert F. Miller, now in a 1968 edition. Another very useful compilation is *Bibliography of Mathematics for Secondary School Libraries*, by Robert A. Rosenbaum and Louise J. Rosenbaum, published in 1964 by Wesleyan University, Middletown, Connecticut. *Mathematical Booklist for High School Libraries* is published and kept current by Mu Alpha Theta, the National High School and Junior College Mathematics Club, University of Oklahoma, Norman. It lists 10 periodicals and 170 books that constitute a well-rounded mathematical library on a limited budget. The annotation for each entry contains the price and the publisher's address. *The Teacher's Library: How to Organize It and What to Include*, which is a joint publication of the

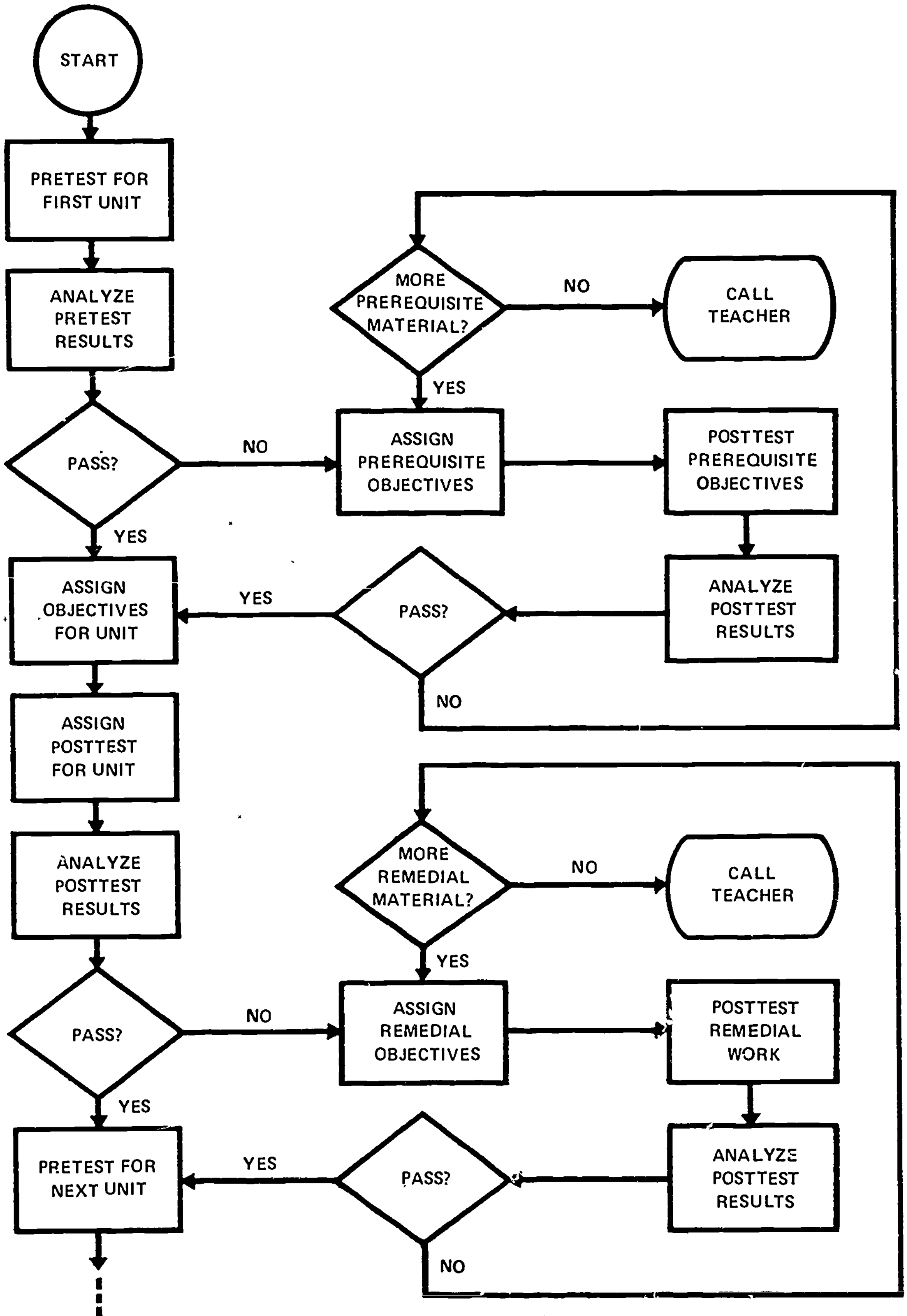
American Association of School Librarians and the National Commission on Teacher Education and Professional Standards, contains a list of 50 books and 5 journals considered most essential in a library for teachers of mathematics. It is available from the NEA Publication-Sales Section, 1201 Sixteenth Street NW, Washington, D.C.

It would be impossible to compile a list of all the available OPM. Useful items whose primary purpose is not related to mathematics instruction might be overlooked in preparing such a list with its unlimited possibilities. Such a list would include Continental Can Company ads that feature puzzle-type problems, the financial section of the *New York Times,* a finance-company information flier that shows a table of loans and payments, an AMTRAK railway timetable that shows the distances between stations, and many other similar items.

Many teachers have found card files, organized by subject and cross-referenced for function, to be of great value in using OPM effectively. Files of laboratory lessons and activity sheets are also very useful. Without such records, the work involved in preparing for each use of an OPM is lost; and, understandably, enthusiasm for their use wanes.

No chapter or article can tell you how you should use OPM; it can only make suggestions. The when, where, and how of using OPM must of necessity vary with the individual school, teacher, and class. However, the better acquainted you are with the available OPM, the more you will use them and the richer your classes will be.

# 5. TEACHING MACHINES AND PROGRAMED INSTRUCTION




A BASIC PROGRAMED INSTRUCTION PROCEDURE

CHAPTER 5

# TEACHING MACHINES AND PROGRAMED INSTRUCTION

5

*by*

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Chapter 5 reviews the history and development of programed instruction and teaching machines, and it discusses some uses and abuses of these instructional aids in the mathematics classroom. The production of PI and criteria for its selection are sketched so that teachers may better evaluate the materials they encounter.

# 5. TEACHING MACHINES AND PROGRAMED INSTRUCTION

A sequence of items, each requiring a response from a student which he can then compare with a given answer or which leads him to another item in such a way as to increase the efficiency and effectiveness of learning, is basic to programed instruction. The need to control such sequences led to the development of devices called *teaching machines*. Meanwhile, the preparation of learning sequences has made a significant contribution to twentieth-century education by focusing the attention of authors of texts on the goals of instruction and the specification of those behaviors a student should exhibit to demonstrate mastery. By structuring, or *programing*, learning experiences, a student may be led from whatever skills and abilities he possesses to whatever new skills and abilities he should possess as a result of his interaction with the instructional module. Programed instruction, then, is a method of learning and teaching geared to individual capabilities and using new and different methods which involve new concepts of preparing instructional materials, new methods of presenting these materials to students, new aspects of control of the learning situation, and new techniques for the determination of subject matter mastery. Programed instruction is based, in part, on three concepts agreed upon by psychologists and educators:

1. Students best learn at a rate that best suits their capabilities.
2. Students best learn if they are responding to and interacting with the subject matter actively and appropriately.
3. Students best learn when they receive immediate confirmation or correction for each response they make.

How programed instruction got started, how it developed, what technological advances have been made, and how these forces combine to enhance mathematics instruction is the concern of this chapter. A further goal of this chapter is to suggest some criteria for the selection and use of programed materials for mathematics and to provide further sources of information about programed instruction and the devices that have been developed to present materials to students.

In 1958 B. F. Skinner wrote, "There are more people in the world than ever before, and a far greater part of them want an education" (24). The decade that followed the publication of Skinner's article witnessed a revolution in educational technology with many of its roots in psychological laboratories not unlike that which Skinner operated at Harvard.

A CAVEAT: *Bear in mind that we are still in the Model T stage of this instructional medium; for, while teaching machines have evolved from scroll-fed viewers to complex computerized units, and programs have passed from sentences with properly selected words omitted to intricate branching patterns involving both student-constructed response items and multiple-choice items, it is still not clear what direction this evolution will take.*

## HISTORICAL SKETCH

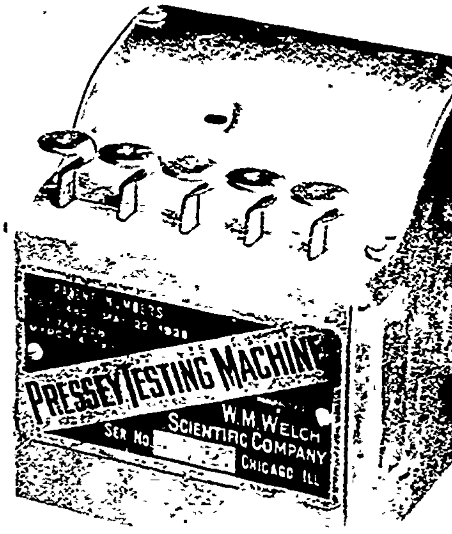
The Socratic method, recorded by Plato in his dialogue *Meno*, is often cited as a forerunner of the conversational method on which programed instruction is based. Such history is interesting but of little real value to us, for if the responses made by the student are indicators that behavior is being shaped from more or less random responses toward something sought as "terminal," the *Meno* offers a poor example.

In 1926 Sidney Pressey, of Ohio State University, developed a device by which his students might be tested and shown the results immediately (10). This device is pictured in Figure 5.1. Pressey's work remained relatively unknown and of little consequence in educational circles until the principle of immediate confirmation was exploited in the military trainers used during World War II. For such devices to work, it was necessary to have the questions prepared in

a multiple-choice format. This characteristic spurred a renewed interest in statistical probabilities and stochastic processes.

Meanwhile B. F. Skinner and his colleagues, after years of behavioral research on infrahumans (pigeons, mice, etc.), suggested that students would learn if they were rewarded for the construction of a correct response to a question. By sequencing the questions (hence the term *programing*) and gradually withdrawing the clues, behavior could be shaped. An example from an early constructed-response program is shown in Figure 5.2.

In the example we see the required repetition of a desired response. The claim made for such repetition is often based on the assumption that if 90 percent of the students construct the correct response 90 percent of the time, learning




*Courtesy of Sargent-Welch Scientific Company*

FIGURE 5.1
*Sidney Pressey's testing machine*

FIGURE 5.2. *Example from an early constructed-response program*

| | | |
|---|---|---|
| No. | 1. | The numbers we use to count with are called 'counting numbers' or 'natural numbers'. 0, 1, 2, 3, etc. are natural numbers. 1/2 is not a natural number. Is 3/4 a natural number? [ ] *(Yes or no?)* |
| natural | 2. | 0, 1, and 2 are natural numbers. 3, 4, 71, and 10,001 are also [ ] numbers. |
| natural | 3. | 0 is a [ ] number. |
| 7/8 | 4. | Which of the following is not a natural number? [ ]<br>0, 12, 627, 2+3, 7/8, 5−1 |
| 8001 | 5. | Which of the following is a natural number? [ ]<br>1/2, 10.2, 1 1/4, 8001, 1091/4279 |
| Yes. | 6. | 2 is a natural number; 5 is a natural number. If you add the two natural numbers 2 and 5, is the sum a natural number? [ ] *(Yes or no?)* |
| Yes. | 7. | Pick any two natural numbers; add them. Is the result a natural number? [ ] *(Yes or no?)* |
| No. | 8. | Is 1/2 a natural number? [ ] *(Yes or no?)* |
| No. | 9 | Many years ago men used only natural numbers. But they found eventually that natural numbers were not adequate for all their needs, such as expressing a part of a whole. For example, if a baker divided a loaf of bread into two equal parts and gave one part to each of his two friends, did each friend get one whole loaf of bread? [ ] *(Yes or no?)* |

*From* Modern Mathematics, a Programed Textbook, Book 1 *by Lewis D. Eigen, Jerome D. Kaplan, and Ruth Emerson.*

has taken place. A program consisting of a set of short, sequential, single-response, student-constructed items was often prepared on a scroll for presentation in a simple viewing device, in which each item (called a *frame*) would appear in an aperture and the correct response would remain concealed until the student had completed his response and advanced the scroll (Figure 5.3).

The student would then compare his response with the answer shown, which, it was hoped, would reinforce his decision. Such devices and the programs written for them precluded extensive skipping, reviewing, or branching and hence were termed *linear*. They were also referred to as *Skinnerian*, after B. F. Skinner, who initiated this format.

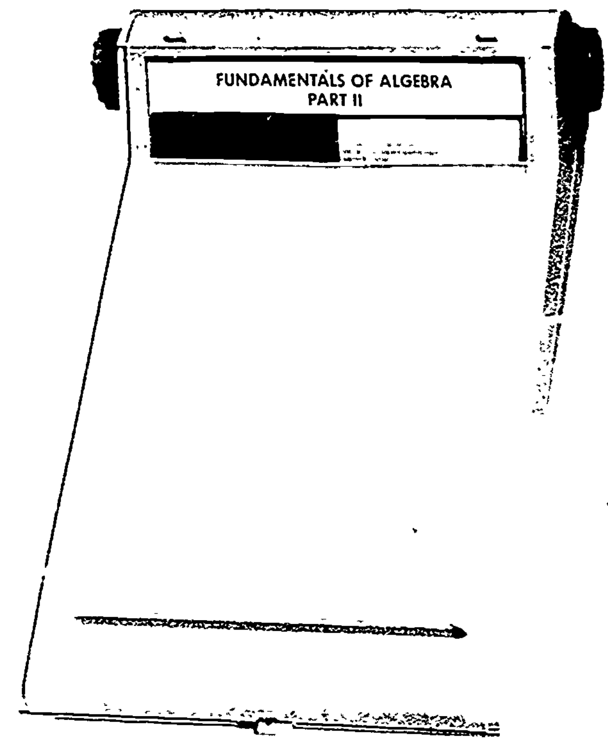



FIGURE 5.3
*MIN MAX III teaching machine*

*Courtesy of Teaching Machines Corporation*

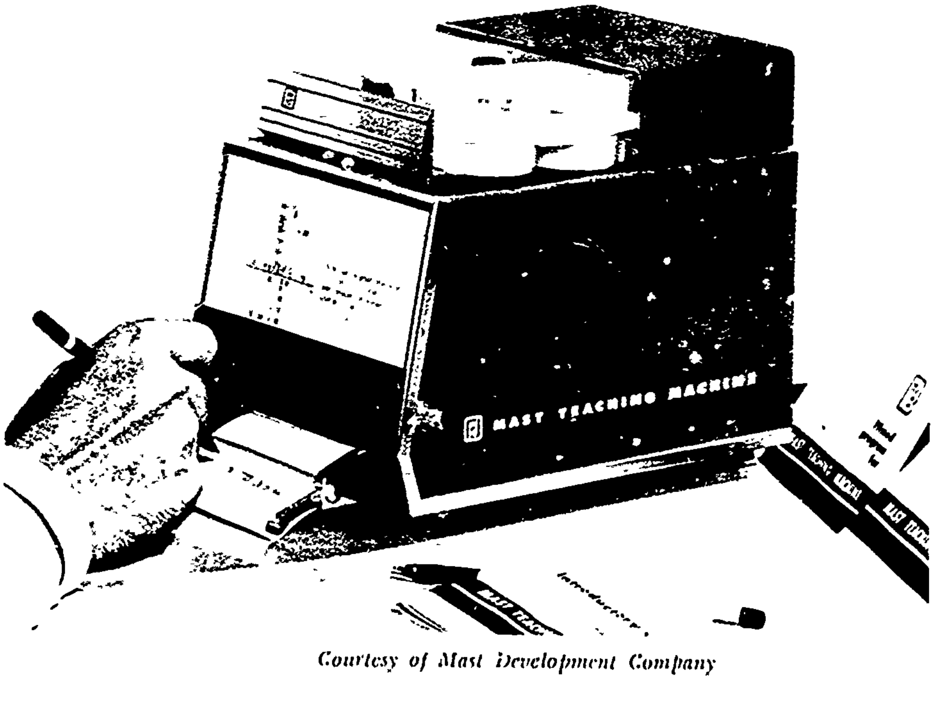



FIGURE 5.4
*MAST teaching machine*

*Courtesy of Mast Development Company*


FIGURE 5.5. *Cyclo-teacher learning aid—linear program, constructed-response device. This machine is manually operated. Students learn at their own pace, and handwritten responses are recorded on a separate answer sheet. The programs, on circular paper sheets, are reusable.*

The multiple-choice format required by Pressey's early device was used in the 1950s for troubleshooters and military trainers. Although the student might use only a small portion of the text, his incorrect choices afforded the author of the program the opportunity to correct bits of misinformation and to provide for a wide range of student ability. An example from an early branching program is shown in Figure 5.6.

In this example the bit of text is followed by a multiple-choice item, a list of alternatives, and a set of directions for finding the next bit of information. The format suggests, and was often incorporated into, a mechanical device utilizing a microfilm viewer under precoded mechanical control in order to facilitate the locating of the appropriate instructional media (Figure 5.7). The text version, with page-turning performed by the student, was known as the *scrambled book* program or as the *Crowderian* program, after Norman Crowder, who suggested this format.

In the early history of programed instruction, numerous studies comparing the relative merits of linear and branching programs were conducted (4, pp. 116–17). The controversy raged, and the results tended to show no significant differences (11: 23). That is, when the two programing methods were compared, it was seen that the degree of student mastery tended to be independent of the program used.

As positive programed-instruction research findings mounted, in the early 1960s, it was no surprise that both UICSM and SMSG undertook programing efforts. Both groups, pioneers in mathematics curriculum revision, used subject-matter specialists, teachers, and psychologists in cooperative writing efforts. The results favored a

**28**
[*from page 33*]

YOUR ANSWER: $\frac{a}{a} = 1.$

You are correct.

You should notice the exception of 0 from this statement. The expressions $\frac{0}{0}$ and $\frac{a}{0}$ are not defined. That is, they are never used. The reason is that all kinds of foolishness can be proven if we assign any value to the result of a division by zero.

In algebraic expressions, when two letters or a number and a letter are written side by side, they are meant to be multiplied. Thus, $2a$ means 2 times $a$, $ab$ would mean $a$ times $b$, and so on. For clarity, the numbers or letters to be multiplied are sometimes placed inside parentheses. Thus $(n)(1)$ means $n$ times 1.

Now, here are a number of very simple statements about numbers. All but one of these statements are true. Which one is not true?

$n - n = 0.$ **page 32**

$(n)(1) = n.$ **page 37**

$\frac{n}{1} = n.$ **page 40**

$n + 0 = n.$ **page 42**

$(n)(0) = 0.$ **page 45**

$n + 1 = n.$ **page 46**

FIGURE 5.6. *Example from an early branching program*

*Reproduced, by permission, from N. A. Crowder and G. C. Martin.* Adventures in Algebra. *© 1960 by Doubleday & Co.*

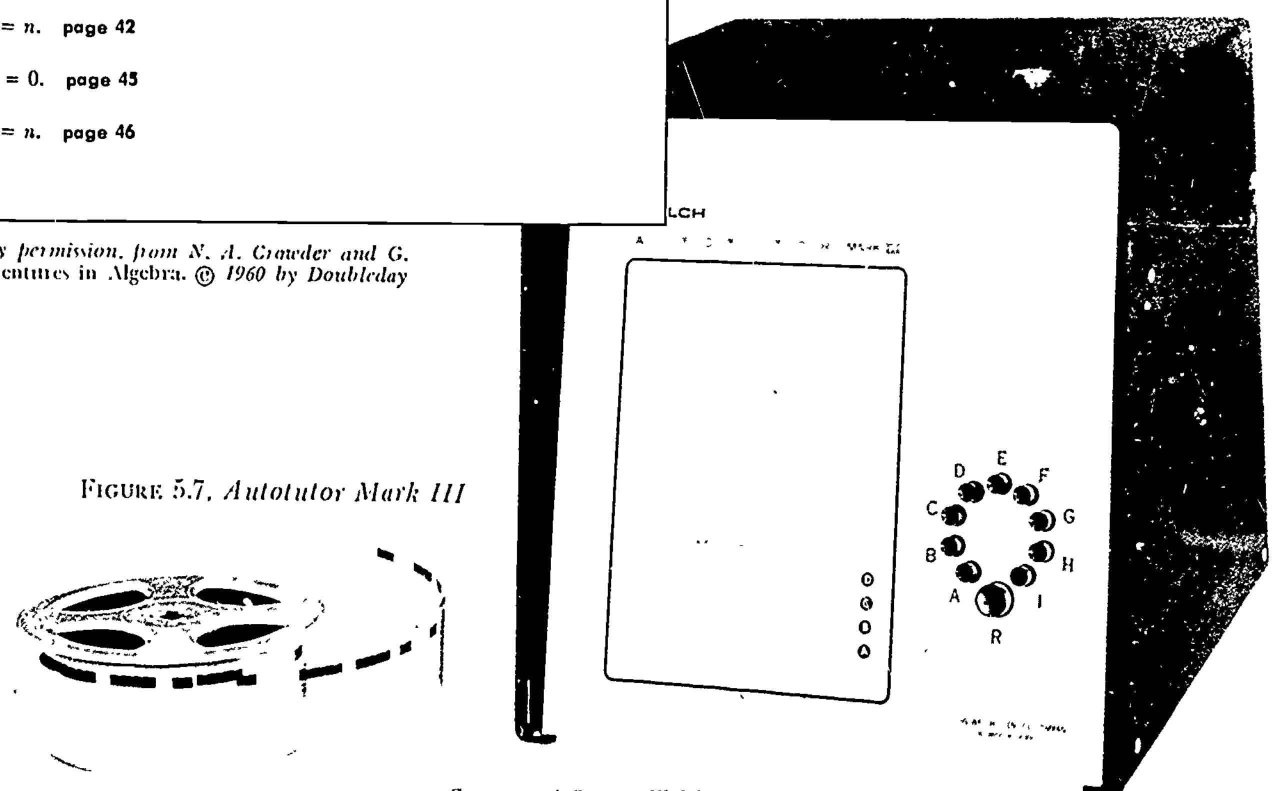



FIGURE 5.7. *Autotutor Mark III*

*Courtesy of Sargent-Welch Scientific Company*

## 7-5. The Additive Inverse

We have pointed out that 0 is the identity element for addition. This is simply another way of expressing the addition property of 0: the sum of any number and 0 is equal to the number.

Suppose we have two numbers whose sum is 0. These two numbers are related in a special way.

| | | |
|---|---|---|
| 1 | What number when added to 3 yields the sum 0? ____ | $-3$ |
| 2 | What number when added to $-4$ yields the sum 0? ____ | 4 |
| | In general, if $x$ and $y$ are real numbers and if $x + y = 0$, we say that $y$ is an <u>additive inverse</u> of $x$ and that $x$ is an additive inverse of $y$. | |
| 3 | Since $(-2) + 2 = 0$, 2 is an ______ inverse of $-2$. | additive |
| 4 | $(-\frac{1}{2})$ is an ______ ______ of $\frac{1}{2}$. | additive inverse |
| 5 | If $y$ is an additive inverse of $x$, then $x + y =$ ____. | 0 |
| 6 | An additive inverse of $(-\frac{5}{8})$ is ______. | $\frac{5}{8}$ |
| 7 | An additive inverse of 0 is ______. | 0 |
| 8 | If $t$ is an additive inverse of $s$, is it also true that $s$ is an additive inverse of $t$? ______ (yes,no) | yes |
| 9 | Given two real numbers, each is the additive inverse of the other if their sum is ______. | 0 |

10 Consider the following pairs of real numbers. Which pairs are pairs of additive inverses?

$$-7 \quad \text{and} \quad |7|$$
$$\frac{4}{3} \quad \text{and} \quad -\frac{4}{3}$$
$$0 \quad \text{and} \quad 0$$
$$4 + (-3) \quad \text{and} \quad (-4) + 3$$
$$(-5) \quad \text{and} \quad -(-5)$$

[A] all but one [B] all [C] all but two

Since for each pair we can verify that the sum is 0, [B] is the correct choice. If you answered incorrectly, convince yourself that the sum of each pair is 0.

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FIGURE 5.8. *Example from a hybrid program*

mixture of linear and branching items, called *hybrid* or *eclectic* programs (27). A typical page from the SMSG hybrid program is reproduced in Figure 5.8 (26, p. 252).

Note that the text is followed by constructed response items and then a multiple-choice item. Variations in this pattern prevail throughout the text, making it difficult to present the program in a scroll-like device. However, efforts have been made to execute and present hybrid programs using such devices. One example is shown in Figure 5.9.

Meanwhile, William Uttal, then a psychologist at the IBM Research Center, suggested that "automated instruction may be looked upon as a problem in simulation, in which the goal is to most completely imitate the conversational interaction between the student and his human tutor" (29). To imitate or simulate the complex patterns of which the human mind is capable clearly leads to the introduction of the computer.

The computer has opened up the potential for research on learning and teaching and has, itself, become a sophisticated teaching machine. Stored with materials of the sort we have seen (conventional programed instruction), the computer may operate in a tutorial mode, capable of comparing, verifying, supplying clues, branching, and reviewing, yet maintaining a record of student actions at all times. The tutorial program, insofar as it simulates the actions a teacher can perform, might be extremely difficult to prepare were it not for author languages that require minimal knowledge of computers on the part of the subject-matter specialist.

The potential of the computer as an instructional aid is evolving from an industrial research environment to a school-university research environment where federal monies are available to underwrite development costs. Suppes at

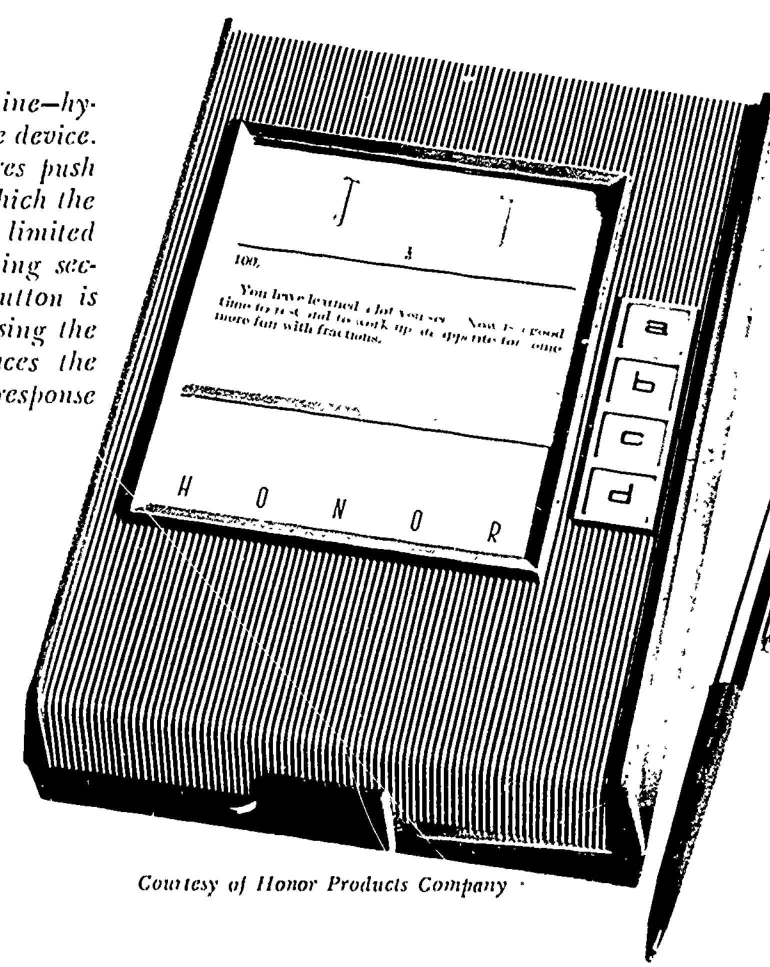



FIGURE 5.9. *Honor teaching machine—hybrid program multiple-choice response device. This battery-powered machine features push buttons to advance programs from which the student learns. The device has a limited branching capability. In the branching sections of each program a different button is listed for each possible answer. Pressing the correct button automatically advances the scroll past the explanations given in response to the incorrect answers.*

*Courtesy of Honor Products Company*

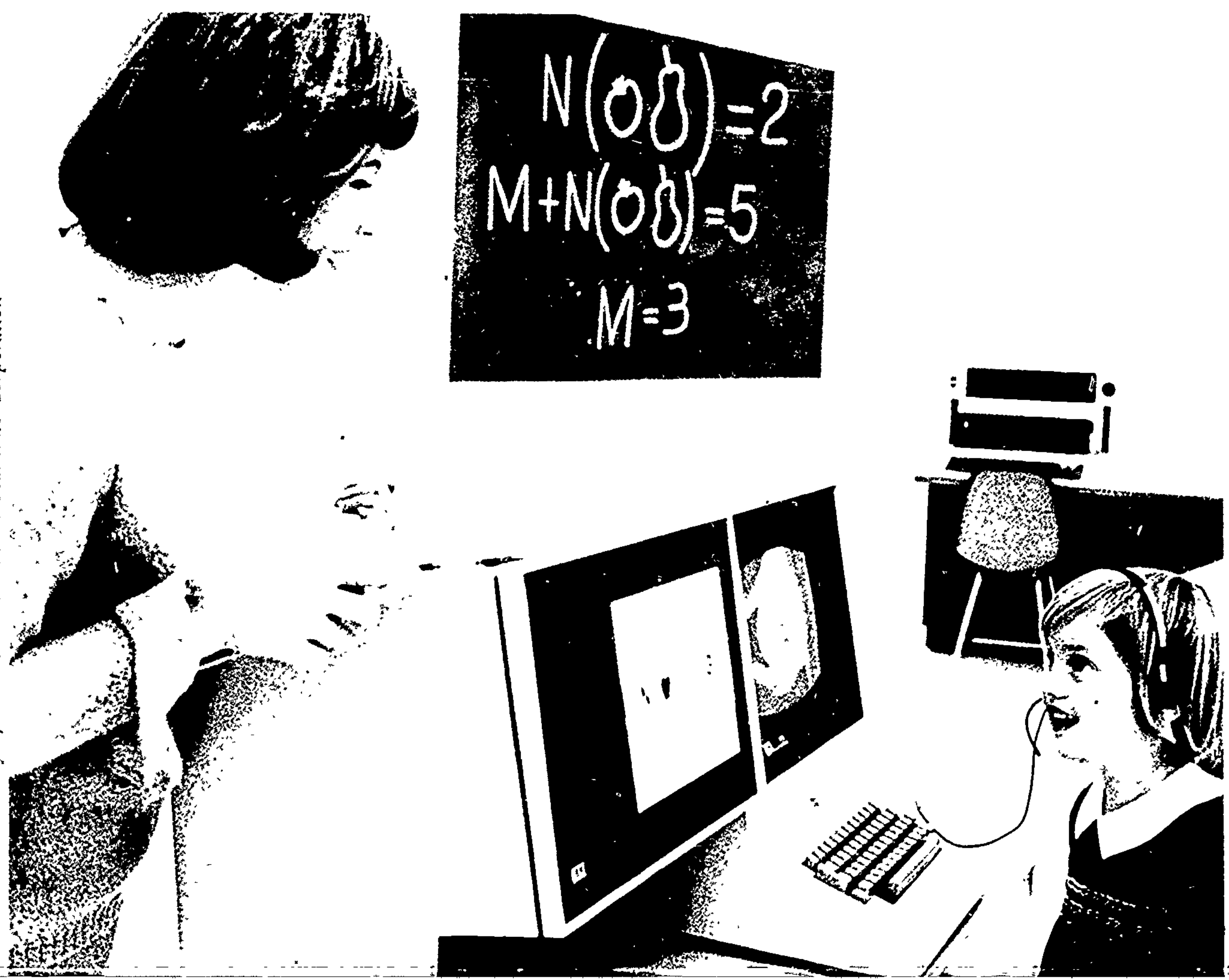
Stanford, Mitzel at Pennsylvania State, and Lambe at State University of New York represent a few of the university researchers in the emerging field of computer-assisted instruction.

Experimentation at the Brentwood School in Palo Alto, California, has shown the computer capable of controlling a sequence of drill and practice exercises; experimentation in the Board of Cooperative Education of Westchester County, New York, has demonstrated the computer as a simulator for use in games with elementary and secondary school students (30); and experimentation in several New England preparatory schools has proved the feasibility of the computer as a problem-solving tool. The research goes on (14), and the impact of programed-instruction techniques is felt permeating the development of a new generation of materials of instruction (4).

As early as 1961, research psychologists at System Development Corporation reported the construction of a "Computer-based Laboratory for Automated School Systems," called CLASS (8). While there appear to be many advantages in such a total system, the widespread acceptance and use of such systems has been delayed by factors of cost, technical development, and the lack of instructional materials prepared in this way.

A new era emerges; a new level of sophistication for programed instruction and teaching machines appears; and a new application of technology in the field of education enters the school situation by way of research. The developments in this field are rapid and exciting, holding much promise for the future of in-

FIGURE 5.10. *IBM 1500 computer-assisted instruction terminal*




*Courtesy of International Business Machines Corporation*

struction. Further information about these developments is given in Chapter 6, "Electronic Computers and Calculators."

## DEVELOPING PROGRAMED INSTRUCTIONAL MATERIALS: AN OVERVIEW

General references on program development abound. Those who seek to develop programed materials will find much general information in Brethower (2), Brethower et al. (3), Fry (10), Glaser (13), Lumsdaine and Glaser (17), Lysaught and Williams (18), and Taber et al. (28). In this section the characteristics of programed learning will be sketched; the subsequent section will provide more specific details of programing based on guidelines used by the SMSG writing efforts in mathematics (27).

Crucial to the development of programed materials is the specification of a set of instructional requirements, goals, needs, or problems. Mager has written extensively on the identification of objectives (19), and Briggs et al. have addressed the question of setting objectives as a vital function in the design of an instructional system (4). SMSG (27) includes explicit references to mathematics goal setting in the *Manual for Programers* which was used in producing the SMSG *Programed First Course in Algebra* (26).

After the instructional requirement is recognized and specified in terms of student behaviors, one must ask the question "Is programed instruction an appropriate medium?" At least four alternatives are available at this point:

1. Another medium is more appropriate.
2. A noninstructional system is "better."
3. Programed instruction is part of the optimal solution.
4. Programed instruction is, per se, the "best" choice.

Such alternatives then demand attention to economic, management, administrative, and curricular constraints, as well as the technological considerations concomitant with the use of programed instruction.

Once the decision is made to utilize programed instruction as a component of the instructional system (the third or fourth alternative above), the next question is "Are programed instruction materials available that will meet the instructional requirement?" Bibliographic sources of available programs include those of Fenton and Gilbert (9), Hickey and Newton (15), and Schramm (23). We have included in this chapter excerpts from a few of the mathematics programed materials available.

The question that then arises is "On what bases should a program be selected?" We shall address this question more fully in a later section of this chapter; but it should be borne in mind that if programed materials do not exist, the preparation of such materials necessitates a thorough empirical validation, as detailed in the following section. Such validation is, or should be, an integral part of the development of programed instruction materials. It is a time-consuming and difficult task, but one without which programed instruction may fail.

## DEVELOPING PROGRAMED INSTRUCTION MATERIALS: SOME DETAILS

Based on the experience of constructing programed instruction materials in mathematics, the authors propose the following steps:

STEP 1. Specify the objectives of the program in terms of behaviors that are to be performed by the student. The author of a program must be able to answer the question "What should the student be able to do as a result of completing the program?" The answer must be explicit—for example, "He must be able to factor expressions like $4x^2 - 64$." Gagné has suggested that such objectives require the author to go one step further and state those behaviors prerequisite to the specified goal; to wit, "He must be able to factor expressions like $4x - 12$ and $x^2 - 4$" and "He must illustrate the meaning of factor" (12). As tedious as this seems, such a

hierarchy may enable the author to address his audience more appropriately.

STEP 2. Prepare a set of criterion items that test attainment of the specified goals.

STEP 3. In terms of steps 1 and 2 above, prepare an initial instructional sequence. At this point the author must accommodate both cognitive style and desired end result. That is, the author must be alert to the ways in which a student learns (from the manipulation of objects, from visual images, or from verbal or symbolic statements) and the terminal behavior of the student. If the student is to discriminate between integer squares and nonsquares, the items may be of multiple-choice response type; if the student is to construct the squares of given integers, the items may be of constructed-response type. Usually, it is some combination of items that will work best.

STEP 4. Try out the instructional sequence on a small number of students, with an observer to note student difficulties. Ask students to react to the selection of wording, the length of the sequence, and the amount (more or less) of material needed to communicate the notion. At all costs, let the student be frank!

STEP 5. Revise the sequence. Take into account the student reactions, the effect of verbal change, and the mathematics being taught. While most research indicates little difference between overt (written) and covert (mental) responses, overt response has an initial advantage in the fact that it can be assessed (4, p. 123).

STEP 6. Try out the revision on more students who are of the audience to be addressed by the program. This trial will result in further suggested revisions and will tend to refine the program.

The process suggested here is not easy. It requires intellectual honesty and strains the patience of an author. The efforts can be justified if the result is student achievement. Some questions that may help authors avoid frustration follow:

1. Are the goals clearly defined in behavioral terms?
2. Does the content suggest a pedagogic strategy?
3. Who are the students for whom the program is written?
4. Have prerequisite skills been acquired?
5. Is the question clear? Too long? Too short?
6. Is the answer to a constructed response unambiguous?
7. Does the response require a change in behavior?
8. Are multiple-choice responses realistic? Distracting? Obvious?
9. Is provision made for review? Indexing? Help?
10. What will a student be able to do after completing the program?

Reacting to the above questions will not make a good author of a bad one. The questions represent only a few of those that occur; but they are not to be easily dismissed. While questions 1 and 10 appear to ask the same thing, it is worth noting the difference. Question 1 addresses the program goals while question 10 addresses student attainment.

There is no substitute for writing, revising, testing, and rewriting. A thorough knowledge of content and methods of teaching are prerequisites which "those expert in programing but not in mathematics" are inclined to forget.

## SELECTION AND EVALUATION OF PROGRAMED INSTRUCTION MATERIALS

Most teachers will find neither the time nor the energy to develop programed instruction materials other than short instructional sequences on discrete topics. In this section, therefore, some guidelines for the selection and evaluation of existing programed instruction materials will be sketched. Teachers of mathematics who contemplate using programed materials should read Kalin (16) and view the NEA slide/tape presentation (22). The article by Kalin relates specifically to selection of mathematics materials.

The first step in selecting mathematics materials, programed or nonprogramed, is the determination of an instructional need. This should include consideration of the content to be taught; the frequency of the offerings; the number of students to be served; the requirements of the students, faculty, and administration; and the availability of materials.

Let us consider the situation where a number of available programs do exist and can potentially meet the instructional requirements previously defined. The task then becomes one of selecting one or more for evaluation. Data should be collected about each program and its administration. Judgments must then be made about the level of each program, the time required to complete it, the mode of presentation, the manpower required to administer the program, and its economic feasibility. Some data may be obtained from the authors or producers of the various programs, but some must be gathered empirically. It is the empirically gathered data that will, in the end, determine the local value of each program. Some questions to be resolved include the following:

1. Are the authors known to be expert in the content and pedagogy?
2. On what population was the program validated?
3. Is the local population similar to that used in the validation?
4. Will the program satisfy local needs?
5. Are the authors' goals consistent with those of the local school?
6. Are pretests and posttests available to assess student achievement?

While it appears trite to ask if the content is appropriate, it is worth considerable effort to determine the results of using a program on a small number of students before its widespread use is contemplated. A very important step in program evaluation and selection is a trial run of a few students before extensive use is considered.

The teacher of mathematics who contemplates using a set of programed instruction materials should work through the material as a student before selecting it for use. Selecting a text, the teaching of which relies heavily on the teacher, may require less time for evaluation than selecting a set of programed instruction materials, where much of the teacher's control is assumed by the program.

Selection of programed instructional materials, then, must take the following into account:

1. *Student needs:* to satisfy course requirements (a course or part of a course being taught); to preclude course deficiency (preventive medicine); to improve course performance (raise a passing but low level of achievement); to augment a course of instruction (supplement or complement a course being taught); to provide new information (establish a core level of understanding for a course to be taught)
2. *Instructor needs:* to complement instruction (replace a portion of conventional instruction); to assist instruction (provide instruction in addition to conventional instruction); to augment instruction (provide material that is not or cannot be provided by conventional instruction)
3. *Administrative needs:* to extend offerings beyond local capabilities; to provide instructional control (uniformity of exposure where wide teacher variation has resulted in students receiving unequal teaching); to meet curriculum demands in areas where loads may be exceeded by student needs.

After selecting a program that meets the content demands *and* the student, faculty, and administrative needs, the evaluation begins. The programed materials should be compared with other media and presentation modes: films, mass lectures, special classes, and so forth.

Variables within a program which bear attention include: the relevance of responses; the variation of pedagogical patterns; the use of simulation; the use of induction; the appropriateness of criterion items; the allowance for review and practice; and the overall consistency of style in the program. Variables that are more

difficult to assess include: student attitude over an extended period of time; retention of material; teacher attitude and bias; administrative problems (how to grade work done, how to schedule classes, etc.); and hidden costs of devices and materials.

As more effort is expended in the evaluation and selection of programs, less will be spent in designing administrative solutions to real time instructional problems. The dictum "If the student fails, the instructional system has failed" will require continuous evaluation of programs and possible redefinition of goals of instruction.

In brief, the steps in selecting programed materials are these:

1. Define the instructional problem.
2. Identify possible solutions.
3. Work through the program (like a student).
4. Try the program on a few students.
5. Based on results, continue or select an alternative.

## USE AND ADMINISTRATION OF PROGRAMED INSTRUCTION

The same program may bode ill to one teacher and good to another. Smith (25), May (21), and Schramm (23) cite uses and abuses of this medium in research and practice. Several suggestions follow which may serve as a guide to teachers of mathematics who decide to try programed-instruction materials in their classes. The suggestions are arbitrarily classified into six areas: preparing students, student-program interaction, establishing guidelines, supervision, measurement and evaluation, and time.

### *Preparing Students*

A perceived need on the part of students involved in the instruction must be present in the target population. Even though a real need exists, if it is not perceived by the students, subsequent activity to relieve the need may be severely limited. The program must become a vehicle by which a need is satisfied. The goals of the teacher, the program, and the student himself should be clear. If the medium is new to students, a sample portion of the program may be extracted as a starter. Record keeping, response techniques, and methods of individual study should be covered in detail early in the course. The goal here is to be sure the student is prepared to begin learning mathematics without being distracted by the medium. he should have the right attitude, prerequisites, and materials to increase his probability of success.

### *Student-Program Interaction*

If the student does not interact with the program, nothing may be learned. Although this statement sounds obvious, its truth is often ignored by many who work with programed materials. Programs do not come packaged with contagious human excitement. The instructor and the educational environment must provide this.

Student-program interaction can be enhanced in many ways. The teacher who supervises programed study may reinforce the interaction during student interviews; other instructors may show an interest in student progress; and parents may support self-study and student participation.

CAUTION: *Program performance records (of time spent, frames completed, errors, quiz scores, test results) are important by-products but must not become whips by which students are held to a program. On one hand, they can give guidance, a measure of progress, a sense of achievement; on the other hand, they can become ends in themselves. If the objective is learning, it should be remembered that many paths will lead to it. A multiple-choice path may provide enrichment or clarification among its alternatives. Thus the shortest route (fewest responses) may not necessarily be best for each student.*

### *Establishing Guidelines*

An instructional system is an organization of materials of instruction, equipment, and faculty, together with a set of procedures that govern them, to satisfy specifiable student goals. It is important for both students and teachers to be well informed about those procedures that are imposed by programed instruction materials.

Knowledge of a program's objectives, materials, methods, and characteristics may be communicated through printed materials, although many other methods exist, such as group meetings, seminars, individual counseling, audiotape or videotape sessions, and small group discussions. A student's guide may facilitate the learning of the student. A teacher's guide may facilitate the administration of the program and alert the instructor to possible student difficulties in either content or use of the program.

A useful teacher's guide for a program should include at least the following eleven items:

1. A brief nontechnical history of the development of the program, including a rationale or philosophy underlying the creation of the program
2. A specification of objectives

   These should appear in behavioral terms to facilitate observation of student performances. Mager suggests one approach to preparing such objectives (19).
3. A statement of prerequisites required as entry behavior (absolutely essential)

   Such knowledge and skill requirements should be specific enough to enable the diagnosis of student deficiencies.
4. A statement of attitude, aptitude, and interest levels that may be essential to completion of the program
5. A statement of program validity

   Noted should be such data as a description of the field-test population, achievement-test scores, and statistical data to indicate central tendency, dispersion, and significance. Test reliability should be considered as well as conditions under which field testing was carried out.
6. A statement of program effectiveness

   How good is the program? Questions of cost, time, effort, and manpower efficiency should be addressed. The degree of outside supplementation needed should be mentioned.
7. A statement of how the program *was* used and *should be* used (vital)

   Ramifications of using the program may include follow-up studies, retention studies, requisite pretesting and posttesting, and observations that may recur as a result of using the program.
8. A statement of references to other curricular materials
9. A time plan to guide assignment and sequencing in the program
10. An indication of the goals to be met by the students in each unit
11. Test and problem-set answers

Some other helpful items that might be included in the teacher's guide are supplementary problems and tests, work sheets, glossaries, diagrams, and work and flow charts.

In addition to the teacher's guide, a student's guide may be useful. It should include features that will make the program enhance learning. The following types of information could be included:

1. A succinct statement as to why this program and this method are being used

   Share some of the teacher's thinking with the student.
2. A description of what programed instruction is and how it should be given to students with different backgrounds

   Some students will have been exposed to programed material before; others will be seeing it for the first time.
3. A statement of objectives and a sample of expected behavior

   These should enable the student to see what he is about to encounter.
4. A sample sequence on how to use the program

   This will result in fewer questions later. The student should see what records are to be kept and how they relate to his learning.
5. A table of contents or index for the student to facilitate review
6. An enumeration of supplementary materials to augment the program.

Providing materials of this type to the student may lessen subsequent orientation and adminis-

trative time. The cautions that follow should be kept in mind:

1. Do not discourage the student from his task; learning is his responsibility.
2. Be honest with the student by indicating difficult spots and potential problem areas.
3. Reduce anxiety by preparing the student for self-study.
4. Keep busywork to a minimum.
5. Do not promote the program as a "quick and easy" way to learn. It may not be!

### *Supervision*

It has been the experience of many educational institutions that the use of programed materials must be supervised. On the whole, students using a program without supervision either rush through it or are poorly motivated and do not finish the program. In either case they may react negatively to the sponsors of the program. The teacher must be ready to diagnose deficiencies, prescribe academic therapy, and arrange administration to facilitate learning. The teacher must maintain the student-program interaction if the program is to work.

Supervision is more than monitoring progress. It consists of calling periodic reviews of material, testing, scheduling small group meetings or seminars, and branching a student in and out of a program as needed. It also involves tutoring both the less able and the more able to prevent difficulties and to capitalize on the learning acquired in the program. Periodic reassessment of goals is essential. Deadlines should be set for completion of units; and peer pressures should not be discounted.

### *Measurement and Evaluation*

Few single elements are able to cope with the total instructional processes. Diagnosis, teaching, and evaluation are parts of a cycle that takes place in education. The test is a part of the process, but only a part; it should sample behavior, not attempt to do the whole instructional job.

Minimum testing should include a prerequisite or diagnostic test, periodic unit tests, and posttests. Some quizzes may be self-scored by students; others should be group- or teacher-scored. In addition, a delayed retention test would be of value. Beware of overtesting. The learning of mathematics is predicated on active student learning without the burden of overemphasis on evaluation.

### *Timing*

How long should a student work on a program? How long should he work on each unit? If a student sets his own schedule, the teacher must provide guidelines to prevent him from spending too much time on any one unit and to prevent his rushing through the more difficult units. Pacing may be set by suggesting a speed of from fifty to seventy-five frames an hour. The teacher must judge whether this seems to need reduction or increase.

Another aspect of timing concerns the optimum length for each session. A rule of thumb is one hour per session—fatigue can easily set in if no time limit is set. It is better for the student to do some work each day of the week rather than to do a whole week's work in one session.

## CONCLUSION

It remains the obligation of the teacher, together with content and learning specialists, to be alert to new and promising trends that emerge as teaching machines and programed instruction assume a proper role as instructional aids in mathematics. Much research remains to be done (14), and many instructional problems remain to be solved. Only by care in selection, evaluation, and use of this medium can its misuse be prevented.

The bibliography that follows will provide a start toward a better education on this subject. It is not exhaustive but has as its goal the inclusion of references for those who would use or develop programed materials.

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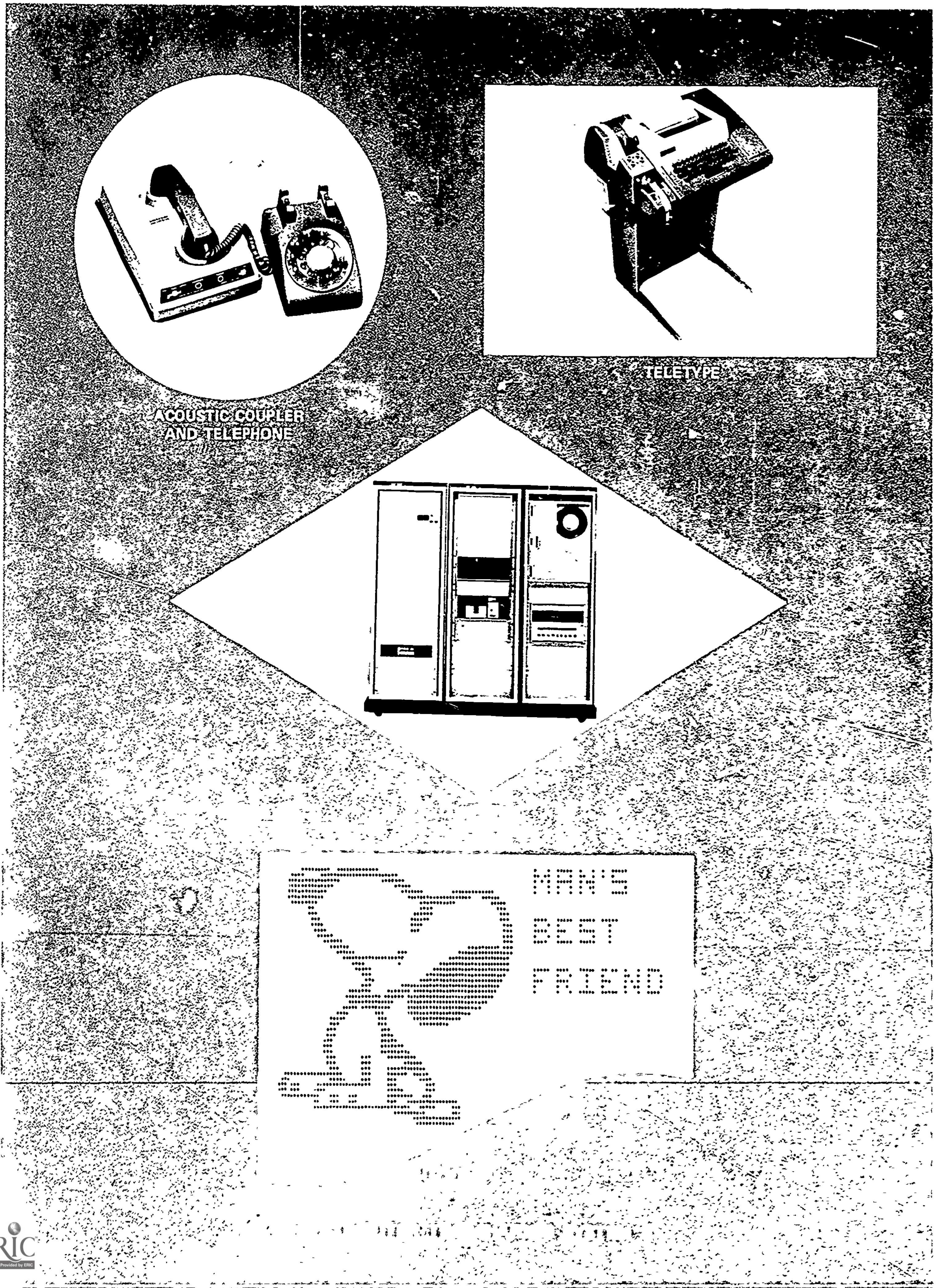
ACOUSTIC COUPLER
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CHAPTER 6

# THE ROLE OF ELECTRONIC COMPUTERS AND CALCULATORS

*by*

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Chapter 6 gives a brief history of electronic computers along with a description of their uses, particularly as they relate to mathematics instruction. Examples of where the computer may be used in mathematics instruction are given, and actual simple computer programs are listed and explained. Comments are also made on the various types of computers available, ranging from the small desk size computer to time-sharing systems. Some discussion of calculators is also included.

# 6. THE ROLE OF ELECTRONIC COMPUTERS AND CALCULATORS

Electronic digital computers embodying the basic concepts of modern computers did not exist thirty years ago. As late as the early 1950s the experts felt that we would never need more than a few computers. Yet the January 1972 issue of *Business Automation* reported that there were then 70,000 computers installed and more on order. Not only are many computers available, but they can now perform in excess of 1,000,000 calculations per second.

It was only a few years ago that the computer was called a "glorified slide rule." The head of a college mathematics department remarked that just as courses in the slide rule were vanishing from the college curriculum, so would courses on how to use the computer. This has not happened. We now have colleges and universities offering the degrees of bachelor of science, master of science, and doctor of philosophy in the new field called "computer science." The number of these schools is increasing rapidly.

Now the use of computers is moving rapidly into the high schools and junior high schools and, in some instances, into the elementary schools. Accordingly, we have this chapter on computing devices and computers.

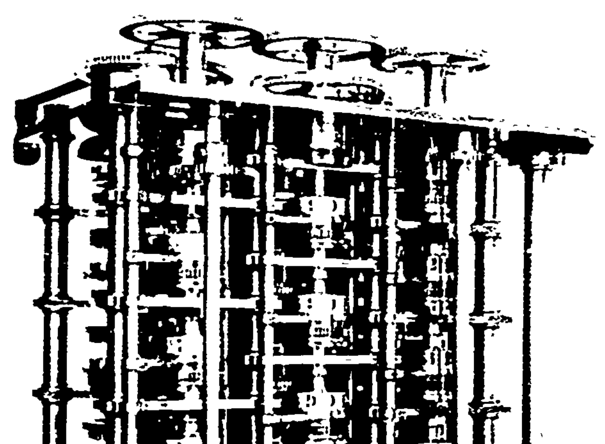
## BRIEF HISTORY

Since the primary concern of this chapter is electronic *digital* computers, the discussions that follow refer only to them. The history of digital computations reaches far back into history to the time when quantification was first conceived by man and he devised schemes for computing things. These schemes undoubtedly could also be related to the beginnings of "modern algebra," as they set up a one-to-one correspondence between sticks, stones, or fingers and animals or whatever else early men may have been trying to keep track of. This led to counters of various kinds, from counters to adders, from adders to calculators and finally to computers.

Among the early names associated with the history of digital computers are Pascal, Leibniz, and Babbage. Pascal, in 1642, designed what appears to have been the first mechanical computing device. It was a simple adder operated mechanically and using cogged wheels. His inspiration came from the fact that he became bored with the job of helping his father with his tax computations. In 1671 Leibniz designed a machine that could multiply as well as add. It was not finished, however, until 1694 and even then did not function reliably.

Babbage, who was born in 1792, built a simple "difference engine" that actually worked but was only a small model (Figure 6.1). When he at-

FIGURE 6.1. *The Babbage "difference engine"*



*Courtesy of the Bettmann Archive*

tempted a large working model, he was unable to make it work because of a lack of drafting and mechanical skill at the time.

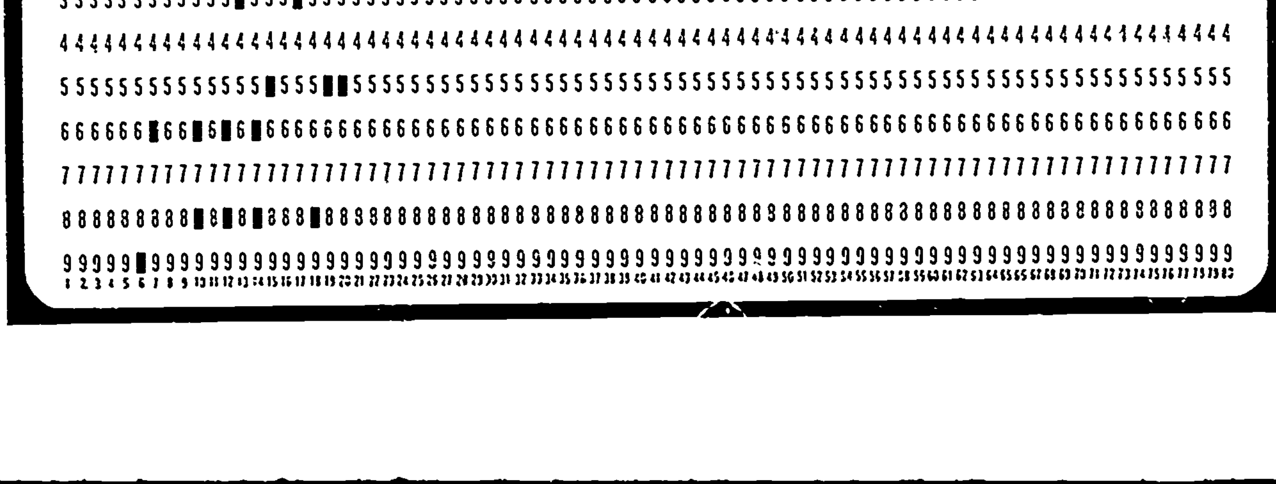
In 1833 Babbage invented a machine he called an "analytical engine." Had it ever worked, it would have done many more things than his difference engine. It included the concept of a memory; that is, it had a section where numbers could be stored. It also included the concept of programing. A program for a computer is a set of instructions directing the computer what to do. His "program" even included the concept of a program's being able to change its own instructions. These are the main reasons why Babbage is thought of as being one of the prime contributors to the ideas of our modern computing machines. Unfortunately, he never completed his machines and died in 1871 after many years of frustration. Clearly, however, he was an outstanding theoretician.

One of the major factors contributing to the early development of the computer field was the use of punched cards. Their use dates back to 1725, even before the Industrial Revolution. An automatic loom was developed by Jacquard in 1801 which was controlled by punched cards, and it really worked! Improvements on these designs for the use of punched cards continued. In 1880 Herman Hollerith invented a machine for sorting punched cards (a present day version of such a card is shown in Figure 6.2), and he built the first set of actual punched-card tabulating machines. These machines were used in work on the census of 1890.

Another stage in the development of computers is illustrated by the IBM 604, shown in Figure 6.3. This equipment had the ability to execute twenty or more operations between the successive reading-in of cards. The wiring of a control panel determined the sequence of operations that were carried out by the machine. It had the capability of looping back to repeat a series of calculations.

The IBM 604 was followed by the card-programed calculator (CPC), which was more advanced. Its operations were determined by reading in a deck of cards containing coded instructions. This represented a significant step forward, since it was considerably easier to work with decks of cards than with wired plugboards or control panels. Note here that the basic ideas of Babbage, dating from 1833, were being employed.

FIGURE 6.2. *The Hollerith card*



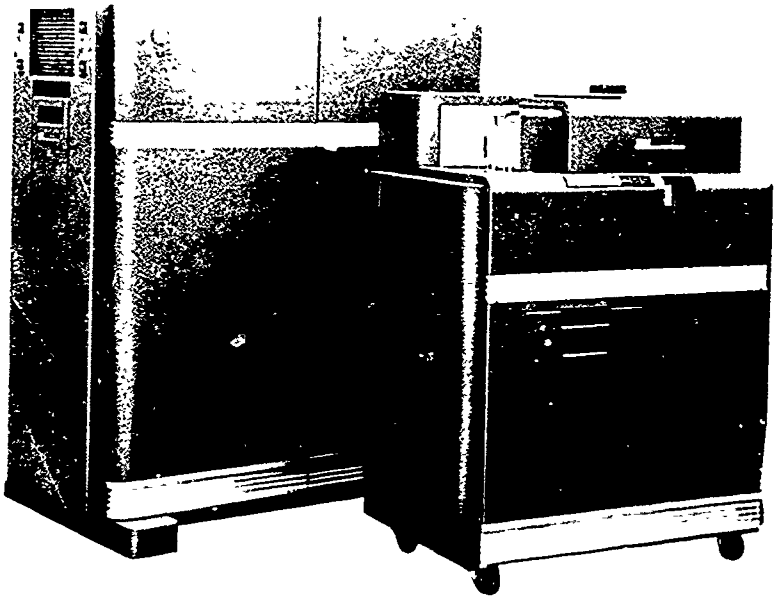


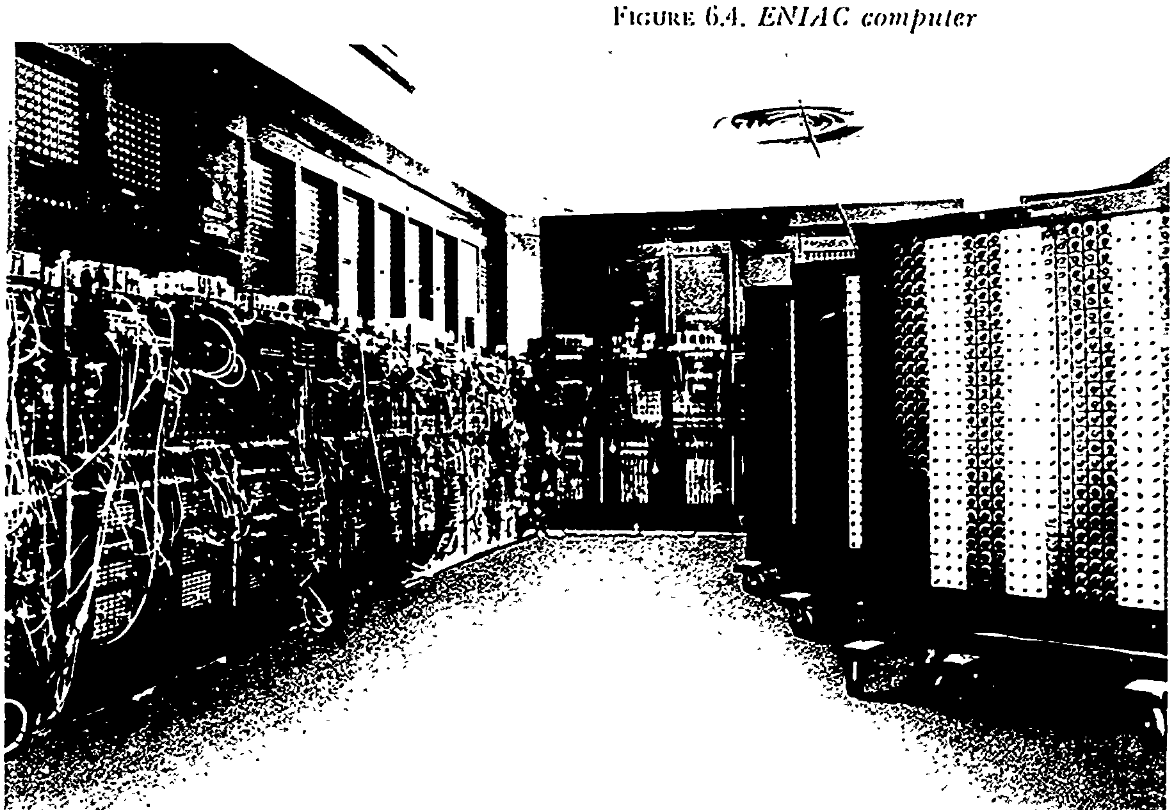
FIGURE 6.3. *IBM 604 and 521*

*Courtesy of International Business Machines Corporation*

Another early computer that used many of Babbage's ideas was the Automatic Sequence Controlled Calculator, or Mark I, built by Professor Howard Aiken of Harvard University and completed in 1944. He worked in conjunction with IBM and used electromechanical devices such as relays, counters, and cams; and reading, punching, and printing units. The sequencing of instructions was mainly by means of paper-tape input. A limited capacity for program branching was available.

The Electronic Numerical Integrator and Computer (ENIAC), designed by J. Presper Eckert and John W. Mauchly at the Moore School of Engineering of the University of Pennsylvania and completed in 1945, was the first electronic computer (Figure 6.4). It was consid-

FIGURE 6.4. *ENIAC computer*



ered a major stride forward, particularly in engineering technology. The ENIAC was a specialized computer built for the U.S. Army to calculate ballistic tables for the Ordnance Department. It occupied a space of about 30 feet by 50 feet and contained about 18,000 vacuum tubes. It would not operate very long, however, before it would break down.

A number of well-known computers quickly followed the ENIAC. This is the point in time where the ideas of John von Neumann and his work on the logical design of computers significantly influenced the field. His design work was apparently influenced by the studies in pure logic by Turing. Von Neumann developed the concept of a *stored program* capable of modifying itself. This idea, and others, helped solve many computer design problems and permitted the development of computers with greatly increased problem solving capability. From these ideas came the first electronic stored-program computer, the Electronic Delayed Storage Automatic Computer (EDSAC), completed in England in 1949. In 1951 the first commercially available electronic stored-program computer became available, namely, the UNIVAC I (*Univ*ersal *A*utomatic Computer), shown in Figure 6.5. The first one of these went to the Census Bureau in 1951. It is now on display in the Smithsonian Institution. The first installation for a private corporation, also a UNIVAC I, went to the General Electric Appliance Park at Louisville, Kentucky, in 1954. In the meantime (1952) the Moore School had completed the Electronic Discrete Variable Automatic Computer (EDVAC).

The electronic computers mentioned above were but the first of a whole series of computers using vacuum tubes. Others were the UNIVAC II, the UNIVAC 1101, 1102, and 1103; the IBM 701, 702, 704, and 709; the Burroughs 205 and 220; and the Bendix G15—to mention only a few. These computers using vacuum tubes are all examples of what are called *first-generation* computers. By 1958–59 the so-called *second-generation* computers began to appear—computers made with transistors instead of tubes. Examples of these are the IBM 1620, 1401, 7090, 7094, 7070, 7074, and 7040; the UNIVAC 1107; the Control Data 160 and 1604; the Burroughs 5500; and the Honeywell 200 and 1800. Maintenance problems with the second-generation computers were considerably less than they were with the first-generation tube machine, since the useful life of transistors is considerably longer than that of vacuum tubes.

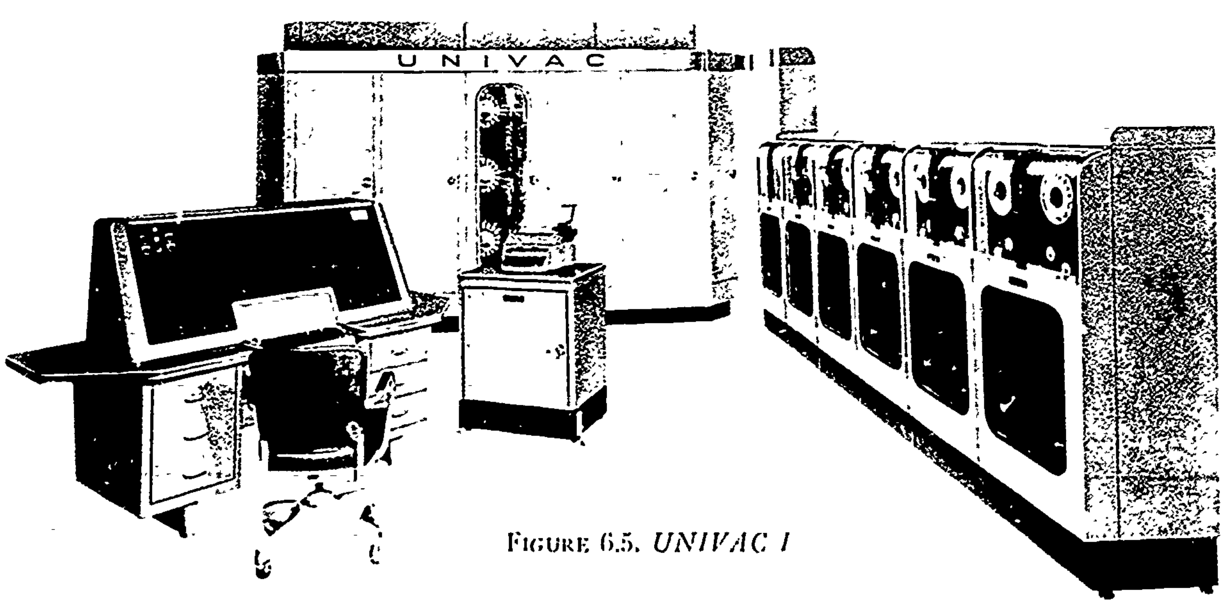



FIGURE 6.5. *UNIVAC I*

*Courtesy of Remington Rand Univac*

The first *third-generation* computers appeared in 1962. These computers employ the new integrated circuits. This means that many different functions can be performed by a very small unit; the integrated circuit card shown at the bottom of Figure 6.6 is capable of doing the work of the 14 printed circuit cards above it. Examples of the third-generation computers are the IBM 360s and the 1130; the Burroughs 6,000 and 8,000 series; and the SDS SIGMA series machines. There now exist portable computers with integrated circuits, such as the Hewlett-Packard and the Wang Laboratories programable calculators, that perform as well as many of the earlier large machines.

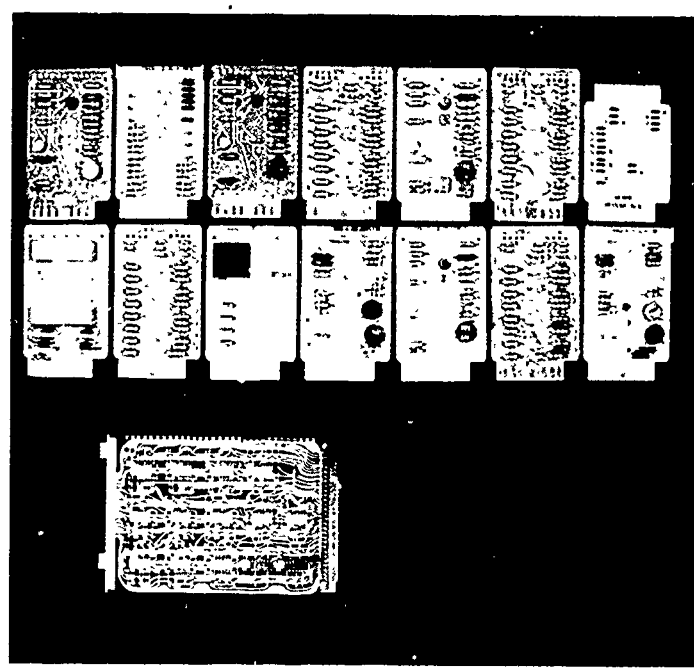


FIGURE 6.6. *The integrated circuit card at the bottom of the picture is capable of doing the work of the 14 printed circuit cards shown above it.*

As each new generation of computers comes along, the problems of maintaining them are lessened. Schools contemplating the purchase or rental of computing equipment should be perceptive about the type of computer they are getting. Obtaining older equipment at "bargain" prices will not be a bargain at all in the long run. Equipment in this rapidly growing field can very quickly become antiquated and expensive in relation to the state of the art. Several computers now advertised as "educational computers" are obsolete machines that actually cost more than new equipment! Maintenance of computers is costly and has to be taken into strong consideration in the purchase of a computer. It is also very important that the right size of computer be obtained.

Another very important aspect of the evolution of the computer field has been the increased efficiency in the use of the machine, that is, in the way of communicating with the machine or, to use the common term, of *programing* the machine. Speaking rather generally, the early first-generation machines required that both their instructions and their data be put in binary, octal, or hexadecimal number form. Each operation, arithmetic or otherwise, required a separate instruction in this form. Several stages in the development of programing followed this. Methods were learned for using decimal instructions and data. Mnemonic letter codes for the instructions came into common use. This was actually a big stride forward in that it reduced human errors.

Current programing methods almost exclusively involve the use of what are frequently called *higher-level* or *problem-oriented* languages. Examples of these are FORTRAN (*For*mula *Trans*lation), COBOL (Common *B*usiness *O*riented *L*anguage), and ALGOL (*Algo*rithm Oriented *L*anguage). These computer languages attempt to have the computer statement of the problem as close to the normal mathematical-formula statement of the problem as possible, as in the case of FORTRAN. COBOL was designed to be very close to the businessman's statement of his problem. ALGOL is a language designed by an international committee in an effort to devise a computer language that would be universally used. It is an excellent piece of wor[illegible] but it did not achieve universal acceptance. [illegible]ally there exist many higher-level languages. [illegible] these ad-


*Courtesy of Chicago Tribune*

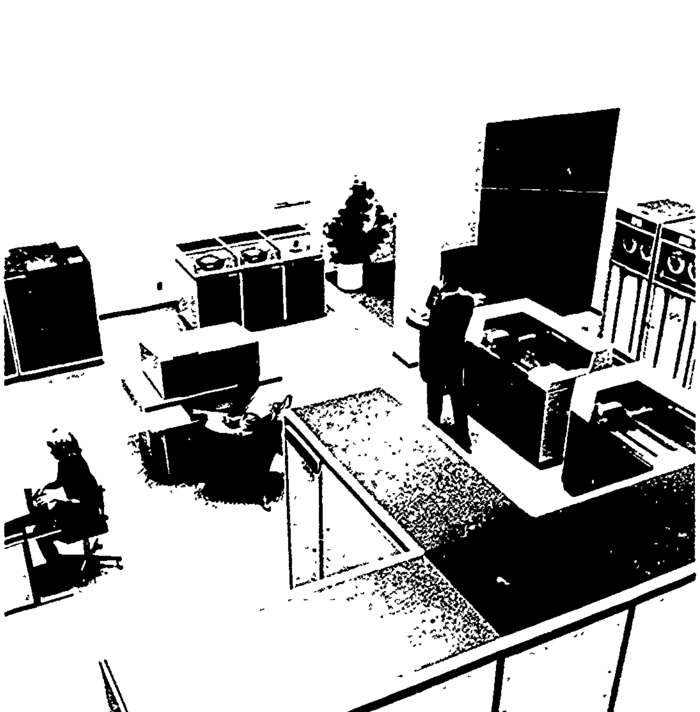
FIGURE 6.7. *"And another thing about computers, they don't goof off behind my back."*

vanced languages a single statement is usually equivalent to many of the earlier, more tedious number codes. In the past few years even simpler languages have been developed for educational (and small business) use. Languages such as BASIC, TELCOMP, IITRAN, QUIKTRAN, CUPL, and APL enable the user to learn programing in a matter of a few hours. These languages are well suited for educational use. Work in the area of programing and the various computer languages is normally called *software* work in contrast with work on the computer itself, which is spoken of as *hardware* work.

The previous paragraphs indicate a very rapid development of both the hardware (the machine and its related equipment) and the software (the programs and computer languages) for the computer field. There is something of a merging of these two aspects as the computer-hardware design goes further and further toward reducing the problems of programing or of man's communication with the machine. One may think of computers as having human characteristics (the cartoonist responsible for Figure 6.7 certainly does!) and may fondly dream of the day when he may just vocalize a problem to the machine and the machine will solve it.

The state of the art is, in fact, quite far from such a reality. Work is going on in this area, and devices have been constructed which do "understand" some spoken words and can thus activate a computer for action; but this is only a small beginning.

Suffice it to say that computers are now very widely used in government, industry, and education and that these uses will clearly expand very rapidly. Society is at a point where if all computers were suddenly removed from the scene, the change would be crippling to our entire economy and progress.



*Courtesy of Control Data Corporation*

FIGURE 6.8. *A computer center*

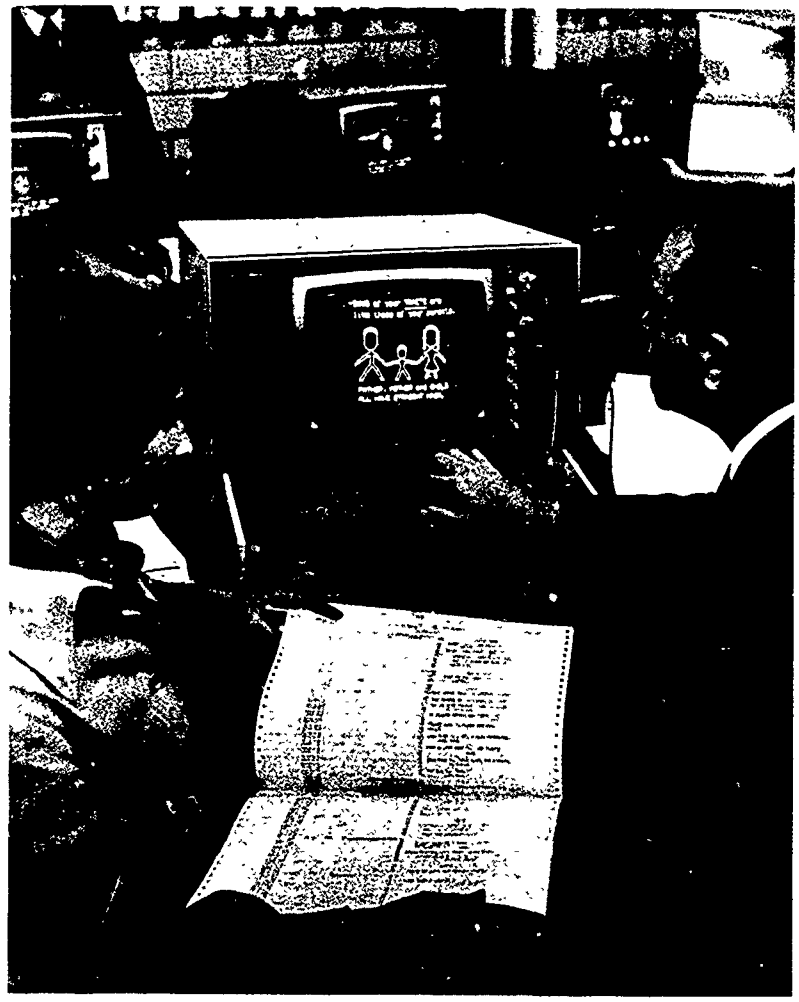
## THE COMPUTER IN MATHEMATICS INSTRUCTION

The preceding section summarized the development of electronic digital computing machines. The impact of these high speed computing and data processing machines on modern society and its activities is increasing at a fantastic rate. Also, the number of occupations in which people must be able to work directly and indirectly with computers is increasing very rapidly. Thus it would be a disservice to students if they did not have ample opportunity to use computers during their precollege or prevocational education. This section includes a summary of the current general uses of computers and a discussion, with illustrative examples, of the applications of computers in conjunction with school mathematics classes.

### *General Computer Uses*

Today's newspapers and journals are full of statements that indeed computers are becoming an integral part of the educational process. In particular, much attention is being given to the use of a computer as a tutor to individualize instruction. Figure 6.9 pictures a center where preparation for such instruction is taking place. This use of a computer is generally referred to as CAI (Computer-*A*ssisted *I*nstruction) or CAL (Computer-*A*ssisted *L*earning) and in a crude sense means the presentation of programed-learning materials by means of computers. For CAI there are usually several terminals and other auxiliary devices connected to a central computer. These terminals may involve many different kinds of equipment, such as visual screens, typewriter keyboards, audio-instruction devices, or even films or slides that are synchronized with the regular terminal display of programed materials.

The computer as a tutor is an infant today, but as more is learned about how to use this development effectively it will become an invaluable aid to the classroom teacher. CAI is considered further in Chapter 5, "Teaching Machines and Programed Instruction."



*Courtesy of Philco-Ford Corporation*

FIGURE 6.9. *At the Philco-Ford plant in Willow Grove, Pa., curriculum for a system of computer-aided instruction is being prepared for the Philadelphia schools. Use of a computer language called INFORM permits teachers to prepare the curriculum without special knowledge of programing skills.*

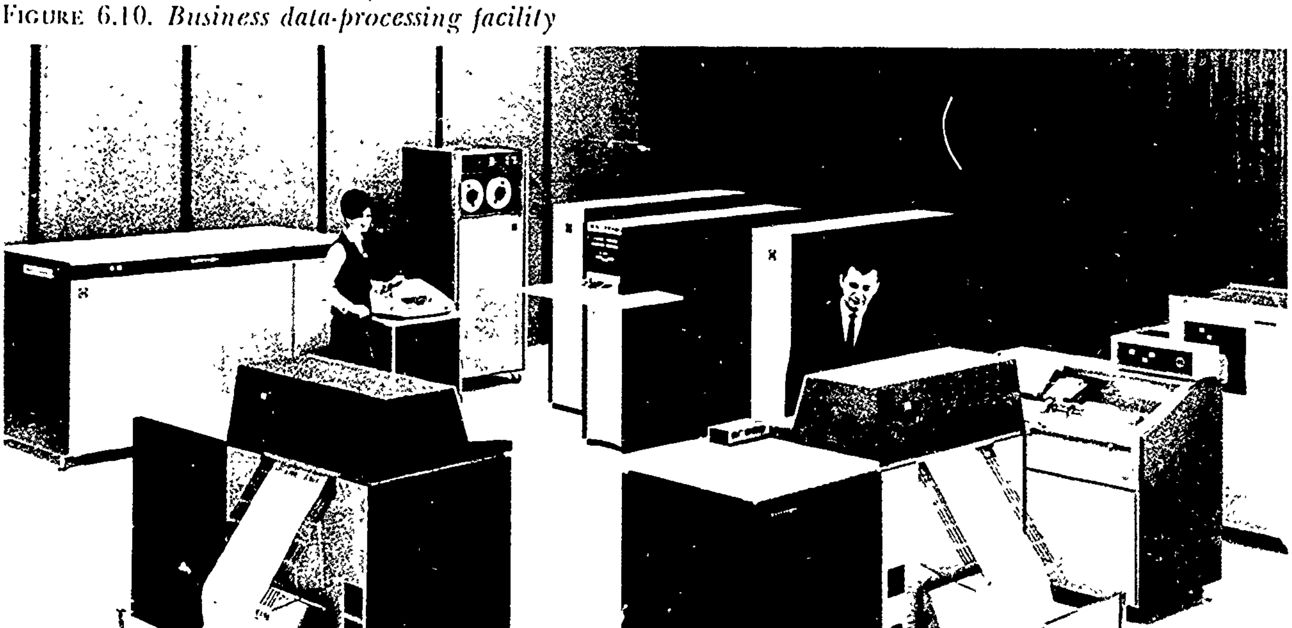
A computer with terminal devices such as those described for CAI may also be used in the following:

1. Preparation of instructional materials
2. Reinforcement of concepts by means of simulation of physical, social, economic, political, and business systems (for example, by playing computer-based economic or political games the student makes decisions based on his knowledge of the factors that influence the situation)
3. Acquisition and storage of student information (responses)
4. Retrieval and presentation of information stored within the system
5. Administrative data processing, record keeping, counseling, and research.

Computers are used in the educational field by many people from many different subject areas. Social science teachers use the computer to study population trends, economic growth, and many other statistical matters. Language teachers have been using computers for the study of linguistics. Music teachers have been working for many years on computer-composed music. Art teachers have investigated the use of computers to mathematically organize certain types of art work. Librarians are using the computer to catalog the books available in a geographical area, to process book orders, and to process interlibrary loans, thus making available more books to more people in the region.

Computer usage is rapidly becoming an integral part of many courses offered in vocational education programs and business education programs. At the college level a variety of programs combining computer science with other disciplines is evolving to prepare students for professional careers in the computer field, particularly in the areas of systems analysis and business data-processing programing (Figure 6.10).

FIGURE 6.10. *Business data-processing facility*



*Courtesy of Burroughs Corporation*

Another area of great potential for computer utilization is that of problem solving. computers and computer programing are being used to solve problems in science. social studies. and mathematics. It is in this area that equipment and materials are available *today* to enable a school to get started. The next section gives a description of computers as a problem solving tool in school mathematics instruction.

In schools where computers are available. one can find almost everyone using the machine:

1. The administrator (for his bookkeeping. scheduling. and student record keeping)
2. The teacher (for demonstration of key ideas in the classroom)
3. The student (for the application of the theories he has learned in the classroom)

   Very often the student will center many of his extracurricular activities about the computer. Models of computers are frequently constructed as projects for science fairs. Dating bureaus. computer-programed by the students. are now common on many campuses.

Several effective uses of computers have already been demonstrated on the educational scene. Education is now at the stage of rapid expansion and development of these uses.

### *The Impact of the Computer in the Mathematics Classroom*

About 1952 the modern methods of teaching mathematics created an upheaval in mathematics classrooms. Textbooks were changed (and are still being changed): teachers started going back to the colleges fo[illegible] more training in modern methods; and colleges started changing their programs for the training of teachers of mathematics. The reverberations are still felt in a majority of school systems as the impact of modern approaches continues to stimulate revisions of elementary and secondary school mathematics curricula.

About 1960 another revolution started with the use of computers in the mathematics classroom. The percentage of schools throughout the country that are involved in the use of computers in the classroom is steadily increasing at an impressive rate. What sorts of facilities are available? Where does the computer aid in teaching the present curriculum? How does the use of a computer change the curriculum? Why has the use of a computer in the classroom been welcomed so enthusiastically? These are some of the questions that are considered in succeeding sections of this chapter.

In the next section sample programs are presented. using several different computer languages. However. before examining specific examples. it is pertinent to discuss the reasons for using computer programing in mathematics classrooms as an integral part of the present curriculum. The use of computers as teaching aids does not necessarily alter the mathematical content of topics being presented but often will suggest modification in the manner of presentation.

A mathematics class that has access to a computer or computer terminal is affected in several ways. One of the often observed differences between this and other classes is the extent of student motivation. Many students of *all ability levels* at all school grade levels become highly motivated when permitted to use a computer. The reasons for this motivation are difficult to specify. but its results are frequently dramatic. In many cases the machine itself is at first the prime target of student interest. However, this interest soon transfers to the magnitude of the problem solving power that is available. The fact that many students will do interesting. challenging. and significant mathematics when allowed to use the computer to pursue their own special interests has been repeatedly demonstrated.

In working with low achievers. many schools have found a desk-top portable computer to be an essential piece of equipment in the mathematics classroom. In using the calculator or computer the student is able to concentrate on process in problem solving. for he has a device that removes the drudgery of computation. Often flowcharting is taught in preparation for prob-

lem solving; then problems are solved by using the flow chart and the computer. The students can study mathematics and mathematical applications rather than do drill on computation, and the interest level is significantly higher because the individual learner is now able to concentrate on meaningful problems.

Teachers of regular classes that have access to a computer have discovered they can now concentrate on the problem solving aspect of their subject and not the arithmetic, which can often obscure the real problem being discussed. For example, consider the second-year algebra problem of finding the solution set of the inequality $ax^2 + bx + c < 0$ for any values of $a$, $b$, and $c$. If students are each asked to write a computer program such that the computer will accept any A, B, and C (in computer language) and then type out the solution set, they must approach the problem in a general manner, since no specific numbers have been given. A valid flow chart for the solution of this problem is given in Figure 6.11.

The student who can develop such a flow chart understands the meaning of the discriminant in the quadratic function, the effect of the sign of A on the graph of the quadratic function, the effect of dividing or multiplying an inequality by a negative number, and other special cases that can occur in this problem. In addition, he is motivated to consider a general mathematical relationship (he uses variables). All the relevant concepts need to be considered without reference to specific numerical problems and thus without the confusion of arithmetic errors. Although the teacher's original assignment to the student was to write a program, the flow chart represents the teacher's purpose in giving the assignment: to promote a complete understanding of the mathematical ideas involved. In addition, the assignment provides the student with the opportunity to design his own algorithm.

To expedite the transfer from the flow chart to a computer program, it is desirable that the student have available a simple computer language that allows him to write the program coding in a very short time. One might ask why a program should be written, since the student has constructed a flow chart. When a computer program is written, the student can have the computer execute (run) his program and receive immediate confirmation as to whether or not his algorithm is correct. If it is incorrect, he can then study the output and refine his program. During this work, the teacher can observe the running of each student's program and can immediately see which students are having trouble with a particular concept. For example: Which students had difficulty when A was negative? Which students could not handle the case with imaginary roots? Which students did not consider the magnitudes of the roots $R_1$ and $R_2$ when they occurred? Which students failed to think of any of the special cases? The answers to these and similar questions permit a teacher to analyze the problems of individual students quickly and in detail.

Modern methods of teaching mathematics require the student to discover the generalization as well as to do the particular problem. Computing methods require the student to understand the generalizations in order to program the particular problem. Because of this relationship the student finds that programing is a natural part and extension of the development of his mathematical thinking. Programing forces the student to be specific and definite.

While a computer can be effectively used in enhancing the teaching of the regular mathematics curriculum, there are also many situations where teachers and students can use a computer facility to extend the study of a particular concept beyond what is traditionally done. Seventh-grade students, for example, can investigate or research a mathematically interesting number-theory problem such as the frequency of primes in an arbitrary interval.

This aspect of computer-assisted problem solving also leads us to a consideration of appropriate teaching methods for using a computer. Teaching that makes full use of a computer's potential requires some changes from the tradi-

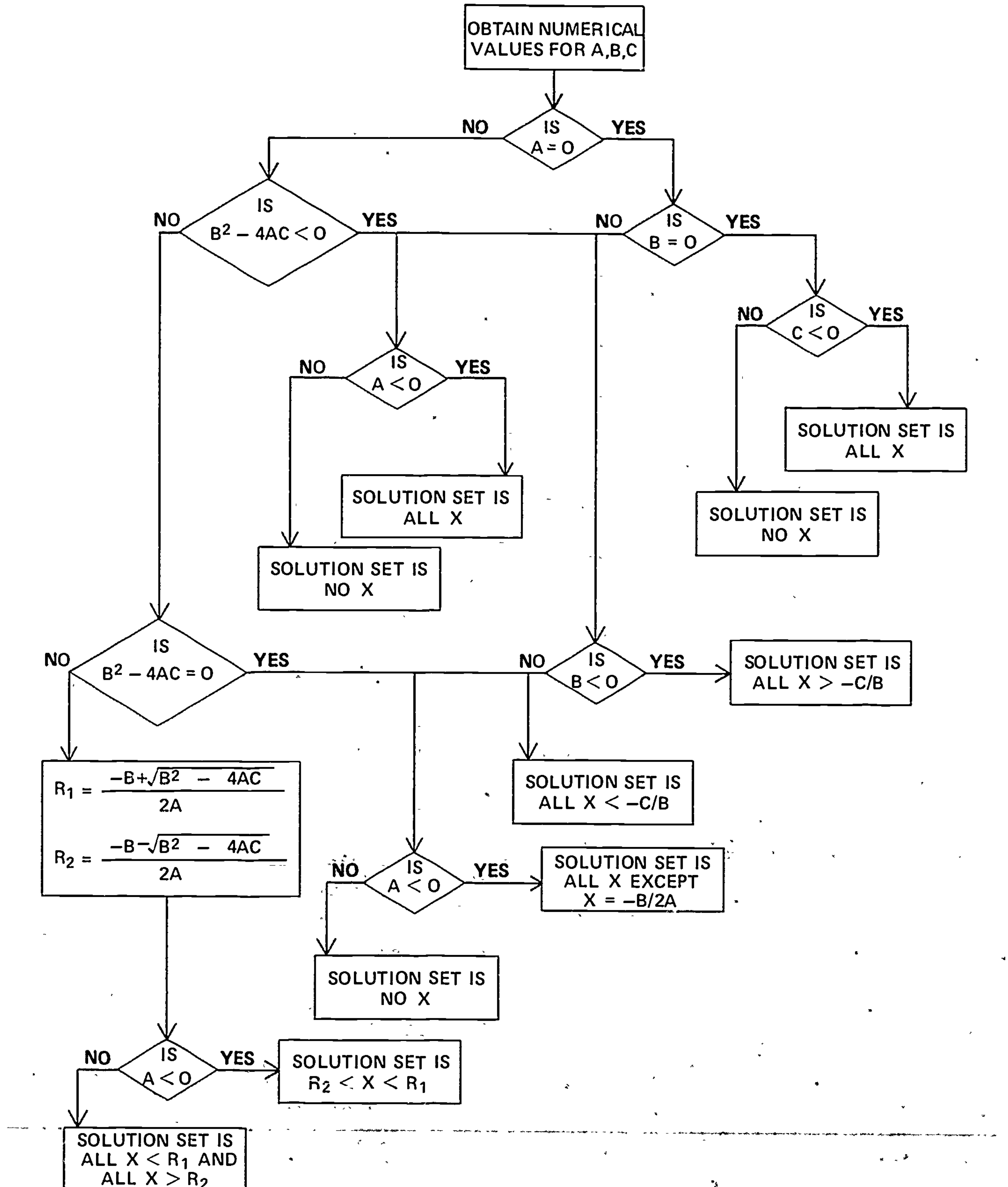



FIGURE 6.11. *Flow chart*

tional presentation-followed-by-drill routine of mathematics instruction. Students should be encouraged to make conjectures and develop (and refine) their own algorithms and then use the computer to test their algorithms. This often means from fifteen to twenty different student programs for a particular problem. The teacher in this situation provides "test" data for the student and may help him in the development of the program; the computer is used to check the program. Often the "giving" of a final answer should be delayed until the student has made a number of trial computer runs.

In addition to enabling actual programing activities, a computer in the mathematics classroom can be an excellent demonstration device. The teacher might write and run a program to form the partial sum of an infinite series or to plot a function that is being discussed. Students may ask questions such as "The approximation may be this for 100 terms, but what is it for 1,000 terms?" or "What happens to the graph if I change this constant or square that term?" The speed of a computer permits almost immediate response to these questions with specific rather than general answers. *A student question "What if . . . ?" rarely goes unanswered when the power of a computer is readily available.*

## EXAMPLES OF PROBLEM SETTINGS AND PROGRAMS

This section contains a set of examples of programs written and executed by students. Each program was written as part of a student's experience in learning mathematics. Each example includes a statement of the problem and a discussion of the purposes that motivated the program. In the first eight examples, the program and its output (a computer printout) are listed, and the name of the programing language and the computer on which the program was run are identified. Examples are presented for various grade levels and for programs run on different computers. Almost any computer, from a small desk-top model to the largest computer, can be used to solve interesting problems. In general, the easier the access to a computer and the more versatile it is, the more useful it will be; but almost any computer can be valuable.

Although some annotation is given to the right of the statements in the programs, it is assumed that the reader has some knowledge of one or more computer languages. Even if he does not, this section should still be of interest because it includes a discussion of the purposes of the different problems and shows the computer solutions. If the teacher has had some contact with a computer language, he will find that with only a little effort he can read many of the programs, since languages for most computer systems have many similarities and there is extensive carryover in knowledge from one programing language to another. Text materials designed to teach computer languages are available from commercial publishers, computer companies, and educational institutions. An annotated bibliography including such texts is provided at the end of the chapter.

EXAMPLE 1

*Grade:* 6

*Problem:* Demonstration of scientific notation

*Purpose:* In the intermediate grades students are usually introduced to scientific notation. The following two one-statement programs were used in a sixth-grade class to augment the discovery approach to scientific notation. A computer prints the small numbers in integer form but at some point changes to the exponential form for larger numbers.

In the first program only nonnegative integral powers of 10 are used. For relatively small numbers the computer types the number in decimal form, but at 10,000 the computer changes to $1 \cdot 10^4$. Since the first program is limited to powers of 10—a circumstance which tends to suggest that one need only add zeros to multiply by 10—it is followed by more general numbers in the second program to illustrate the general use of scientific notation.

*Language:* TELCOMP

*Computer:* PDP 7 TIME-SHARE

*Program:* TYPE 1, 10, 100, 1000, 10000, 100000, 1000000, 10000000, 100000000, 1000000000

*Output:*

```
         1 =    1
        10 =   10
       100 =  100
      1000 = 1000
     10000 =     1*10↑4
    100000 =     1*10↑5
   1000000 =     1*10↑6
  10000000 =     1*10↑7
 100000000 =     1*10↑8
1000000000 =     1*10↑9
```

*Output:* TYPE 1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 1234567891

*Program:*

```
         1 =    1
        12 =   12
       123 =  123
      1234 = 1234
     12345 =     1.2345*10↑4
    123456 =     1.23456*10↑5
   1234567 =     1.234567*10↑6
  12345678 =     1.2345678*10↑7
 123456789 =     1.23456789*10↑8
1234567891 =     1.234567891*10↑9
```

## EXAMPLE 2

*Grade:* 6, 7

*Problem:* To find the intersection of two finite sets specifically, to find the intersection of

$$A = \{1, 3, 5, 7, \ldots, 29\} \text{ and } B = \{2, 5, 8, 11, 14, \ldots, 41\}$$

*Purpose:* Primarily algorithm design, a consideration of what one does in finding the intersection of two sets

This problem serves to reinforce the concept of set intersection and to demonstrate that searching for a pattern is a desirable way to approach the task of generating a set.

*Language:* BASIC

*Computer:* HONEYWELL 235 TIME-SHARE

*Program:*

```
10 FOR J = 1 TO 29 STEP 2
20 FOR K = 2 TO 41 STEP 3
30 IF J = K THEN 50
40 GO TO 60
50 PRINT J
60 NEXT K
70 NEXT J
80 END
```

(The variable J takes on all values of the elements of set A and the variable K takes on all values of the elements of set B.)

*Output:*

```
5
11
17
23
29
```

## EXAMPLE 3

*Grade:* 9, 10, 11

*Problem:* To demonstrate computer printing of graphs of trigonometric functions using standard library programs

*Purpose.* In the ninth, tenth, or eleventh grade, students normally begin the study of trigonometric functions. In the following one-step program, the graphs of several trigonometric functions are plotted on one set of axes to show the effect of $a$ on the plot of the function $y = \sin ax$. The plot routine comes as a standard library program for most large computers. The availability of a quick and a fairly accurate graph of almost any function encountered in high school mathematics is a very useful feature of a classroom computer or terminal.

*Language:* TELCOMP

*Computer:* PDP 7 TIME-SHARE

*Program:*

```
PLOT SIN(X/2), SIN(X), SIN(2*X) ON IP(100*X)/100 FOR X = -SPI:SPI/25:SPI
```

*Output:* (See graphs on next page.)

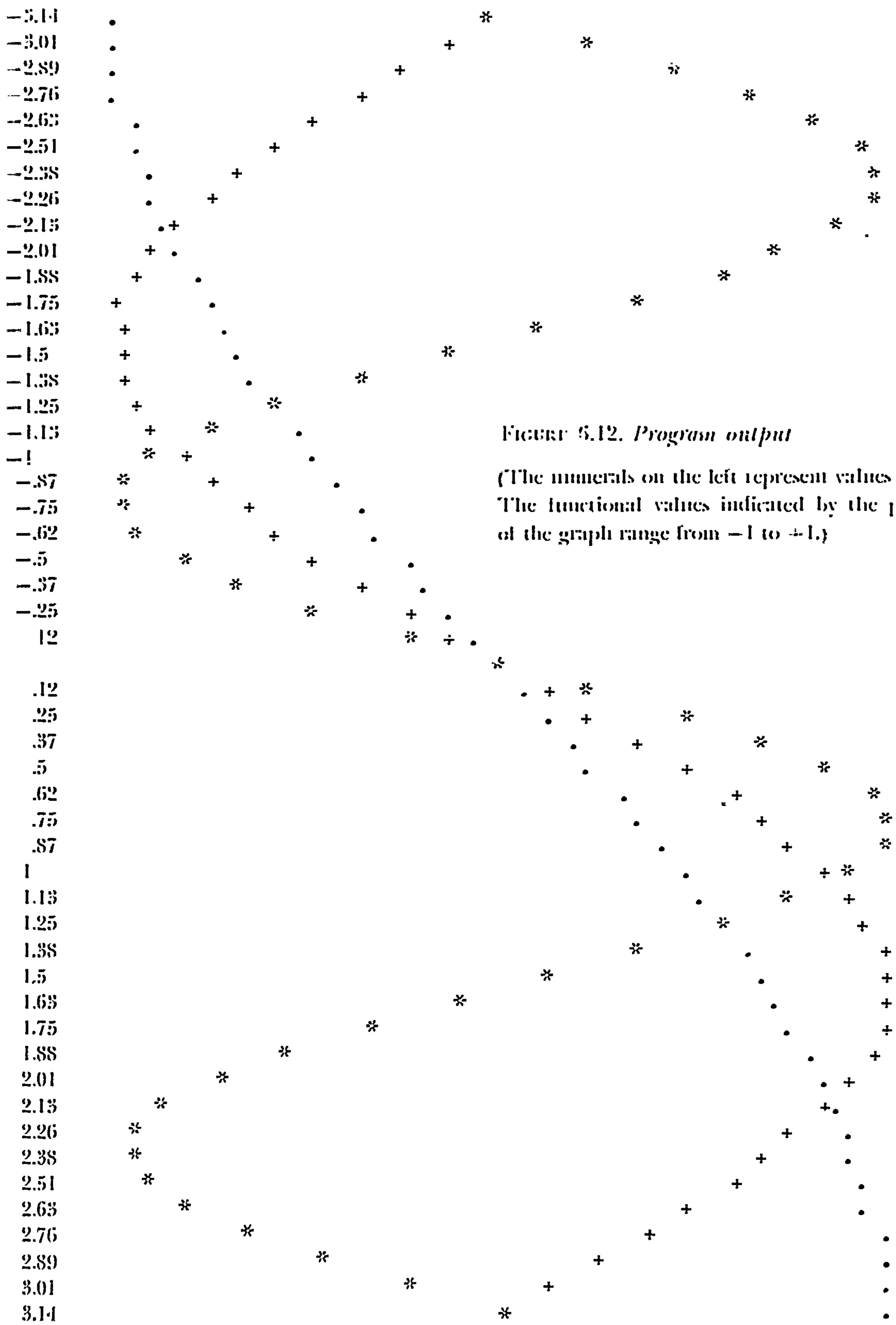



FIGURE 5.12. *Program output*

(The numerals on the left represent values of X. The functional values indicated by the points of the graph range from −1 to +1.)

## EXAMPLE 4

*Grade:* 9, 11

*Problem:* To solve a system of *n* linear equations in *n* unknowns (Gauss-Jordan reduction)

*Purpose:* The student is required to know an important algorithm and to use it to solve the problem. The output will be the solution set of the system of equations or the reduced form of the equations in case the system of equations has no solution or has an infinite set of solutions.

*Language:* TELCOMP

*Computer:* PDP 7 TIME-SHARE

*Program:*

```
1.005 TYPE #. "YOU HAVE N EQUATIONS IN N UNKNOWNS. WHERE"
1.01  DEMAND N
1.015 TYPE "NOW GIVE ME THE COEFFICIENTS"
1.02  DEMAND A[I.J] FOR J = 1:1:N+1 FOR I = 1:1:N
1.025 TYPE #. #                                         (Spaces the paper up)
1.03  K = 0. W = 0. T = N
1.04  K = K + 1
1.05  TO STEP 1.09 IF A[K.K] = 0
1.06  A[K.J] = A[K.J]/DIV FOR J = K:1:N+1 FOR DIV = A[K.K]
1.07  DO PART 2 FOR J = K:1:N+1 FOR MUL = A[I.K] IF I >< K FOR I = 1:1:N
1.08  TO STEP 1.04 IF K < N                              (Note: >< means ≠.)
1.09  TO STEP 1.14 IF K = N
1.10  C = I IF A[I.K] >< 0 FOR I = K+1:1:N FOR C = 0
1.11  S = A[K.J]. A[K.J] = A[C.J]. A[C.J] = S FOR J = 1:1:N+1 IF C >< 0
1.12  T = K. K = K+1. W = 1 IF C = 0
1.13  TO STEP 1.06
1.14  W = 1 IF A[N.N] = 0
1.15  TYPE "NO SOLUTIONS" IF W = 1 AND A[T.N+1] >< 0
1.16  TYPE "DEPENDENT" IF W = 1 AND A[T.N+1] = 0
1.17  TYPE "REDUCED MATRIX CONTAINS" IF W = 1
1.18  TYPE A[I.J] FOR J = 1:1:N+1 FOR I = 1:1:N IF W = 1
1.19  TYPE "SOLUTIONS ARE" IF W = 0
1.20  TYPE A[I.N+1] FOR I = 1:1:N IF W = 0
1.21  TYPE #. #
2.1   A[I.J] = A[I.J] - A[K.J]*MUL
2.2   A[I.J] = 0 IF |A[I.J]| < 10↑-8
```

*Output:*

```
DO PART 1
YOU HAVE N EQUATIONS IN N UNKNOWNS. WHERE
     N = 2
NOW GIVE ME THE COEFFICIENTS
  A[1.1] = 2
  A[1.2] = -3
  A[1.3] = 1
```

(The underlined symbols are typed in during execution time.)

```
  A[2.1] = 5
  A[2.2] = -7
  A[2.3] = -2
SOLUTIONS ARE
  A[1.3] = -34
  A[2.3] = -24
DO PART 1
YOU HAVE N EQUATIONS IN N UNKNOWNS, WHERE
     N = 3
NOW GIVE ME THE COEFFICIENTS
  A[1.1] = 2
  A[1.2] = 4
  A[1.3] = -1
  A[1.4] = -3
  A[2.1] = 3
  A[2.2] = -5
  A[2.3] = 4
  A[2.4] = 5
  A[3.1] = 5
  A[3.2] = -1
  A[3.3] = 3
  A[3.4] = -4
NO SOLUTIONS
REDUCED MATRIX CONTAINS
  A[1.1] = 1
  A[1.2] = 0
  A[1.3] = .5
  A[1.4] = .227272727
  A[2.1] = 0
  A[2.2] = 1
  A[2.3] = -.5
  A[2.4] = -.863636364
  A[3.1] = 0
  A[3.2] = 0
  A[3.3] = 0
  A[3.4] = -6
```

(When there is no solution, this program gives the coefficients of the reduced matrix.)

## EXAMPLE 5

*Grade:* 11, 12

*Problem:* To find the zeros of a polynomial up to degree 5, using Newton's method

*Purpose:* The student is to construct an algorithm to solve the problem. The final program utilizes the following techniques: The computer is given the coefficients of a polynomial function, the upper and lower bounds of a search area, and an increment of search. The program then has the computer search for changes in sign of $f(x)$ on each given increment between the given bounds. When a sign change is detected, the program branches to a Newton's method routine that prints $x$ and $f(x)$ for each of eight passes through the routine. More sophisticated programs of this type can set their own bounds, check for looping in the algorithm routine, use the zeros of the derivatives of $f(x)$, or employ many other techniques. However, this program was introduced in a more elementary setting—thus the more complicated input but simpler program.

*Language:* FORTRAN

*Computer:* IBM 1620

*Program:*

```
 10 READ 105, A, B, C, D, E, F, S, T, R          (A, B, C, D, E, and F are the poly-
    PRINT 110, A, B, C, D, E, F, S, T, R          nomial coefficients. S and T are
    CT = 0.                                       the lower and upper bounds of
    FS = ((((A*S + B)*S + C)*S + D)*S + E)*S + F  the search area, and R is the in-
    V = S + R                                     crement of search.)
 20 FV = ((((A*V + B)*V + C)*V + D)*V + E)*V + F  (Steps 20 to 40 scan between the
    IF (FS*FV) 70, 40, 30                         bounds looking for sign changes.)
 30 FS = FV
    V = V + R
    IF (T - V) 10, 20, 20
 40 IF (FV) 30, 50, 30
 50 PRINT 120, V, FV
 60 CT = CT + 1
    IF (CT - 5.) 30, 10, 10
 70 V1 = V
    DO 100 I = 1, 8
    FV1 = ((((A*V1 + B)*V1 + C)*V1 + D)*V1 + E)*V1 + F
    V1 = V1 - FV1/((((5.*A*V1 + 4.*B)*V1 + 3.*C)*V1 + 2.*D)*V1 + E)
100 PRINT 120, V1, FV1             (Newton's algorithm, x = x1 - f(x1)/f'(x1).)
    GO TO 60                       (The computer prints on each loop.)
105 FORMAT (9F5.2)
110 FORMAT (1H0, 9(F7.2, 2X))
120 FORMAT (1H0, 4HX = ,F14.8, 4X, 8H F(X) = ,F14.8)
    END
```

*Output:* 0.00 0.00 1.00 −12.00 0.00 2.00 −6.00 6.00 .10

```
X = -.40158730 F(X) =  .01600000
X = -.40158403 F(X) = -.00003310
X = -.40158403 F(X) = -.00000010
X = -.40158403 F(X) = -.00000010
X = -.40158403 F(X) = -.00000010
X = -.40158403 F(X) = -.00000010
X = -.40158403 F(X) = -.00000010
X = -.40158403 F(X) = -.00000010
X =  .42222223 F(X) = -.87500000
X =  .41555568 F(X) = -.06398910
X =  .41550518 F(X) = -.00017750
X =  .41550519 F(X) =  .00000010
X =  .41550519 F(X) = 0.00000000
X =  .41550519 F(X) = 0.00000000
X =  .41550519 F(X) = 0.00000000
X =  .41550519 F(X) = 0.00000000
```

(The first line lists the coefficients of the polynomial (0, 0, 1, −12, 0, 2), the bounds of the search (−6 to +6), and the increment of search (.1). The output gives the values of $x$ and $f(x)$ each time a root is detected.)

## EXAMPLE 6

*Grade:* 8, 9

*Problem:* To find the slope of a line, given the coordinates of two points on the line

*Purpose:* In elementary algebra the student is introduced to the idea of the slope of a line. It becomes important to know how to find the equation of a line, given a point and a slope, given two points, and so on. This simple program requires the student to use the general notation for the coordinates of two points to find a slope. The definition of the slope of a line is used.

It is an easy step from this program to the generation of the equation of a line through the two points. Or, if algorithm design is the objective, a more complete algorithm would also test the $x$ and $y$ values to make sure that the points are distinct and to learn whether the slope is undefined (infinite-vertical).

*Language:* BASIC

*Computer:* HONEYWELL 235 TIME-SHARE

*Program:*

```
10 READ X1, Y1, X2, Y2
20 PRINT (Y2-Y1)/(X2-X1)
30 GO TO 10
40 DATA 0, 0, 1, 2, 3, 5, 7, 7, 3, 12, -7, 72
50 END
```

*Output:*

```
2
.5
-6
```

## EXAMPLE 7

*Grade:* 11, 12

*Problem:* To find an approximation of the number $e$

*Purpose:* The number $e$ is introduced to junior or senior students in order to discuss the function $\log_e x$. Many texts mention that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

but of course no proof is offered. The simple program that follows prints $\left(1 + \frac{1}{n}\right)^n$ for $n = 1$, $n = 1001$, $n = 2001$, $n = 3001$. Although this demonstration is no proof of the convergence to $e$, it is convincing and simple.

The function $\left(1 + \frac{1}{n}\right)^n$ approaches $e$ very slowly, and round-off error becomes very important, thus a 28-digit mantissa was used in this run. (The approximation is still only accurate to about 5 places for $n = 10{,}001$.)

*Note:* This same problem can also be solved effectively on a desk-top machine—particularly if the machine has an $a^x$ routine or key. However, the output is often restricted to a fixed number of digits. The program below illustrates the use of a FORTRAN program that provides flexibility in the number of digits desired.

*Language:* FORTRAN II

*Computer:* IBM 1620

*Program:*

```
   *LIST PRINTER
   *FANDK2801
   DO 10 N = 1, 3001, 1000
   E = 1.
   A = N
   P = 1. + 1./A
   DO 20 I = 1, N
20 E = E*P
10 PRINT 30, N, E
   CALL EXIT
30 FORMAT (1H0, I4, F32.28)
   END
```

(This control card changes the mantissa length to 28 digits.)

(In this program $(1 + 1/N)$ is multiplied by itself N times, thus avoiding the exponential function using $e$.)

(The 1H0 format spaces the printer in FORTRAN II.)

*Output:*

```
   1 2.0000000000000000000000000000
1001 2.[illegible]52875345538925243607420
2001 2.7176029086275465383455130720
3001 2.7178290707477605883673329440
```

### EXAMPLE 8

*Grade:* 12

*Problem:* To approximate the value of an integral, specifically, to find the area bounded by

$$y = \sin x + e^x, x = 1, x = 2, y = 0$$

*Purpose:* SMSG's *Elementary Functions* and any one of a number of new senior texts introduce a scheme of approximating the area of a region bounded by a parabola $y = ax^2 + bx + c$, two vertical lines $x = m$ and $x = n$, and the $x$-axis. This method can be expanded to Simpson's rule for approximate integration of any continuous function.

In the following program a student used Simpson's rule to solve the problem given above.

*Language:* FORTRAN II

*Computer:* IBM 1620

*Program:*

```
   Y = SIN(1.) + EXP(1.)
   DO 20 I = 1, 99, 2
   P = I
   X1 = 1. + P/100.
   X2 = 1. + (P + 1.)/100.
20 Y = Y + 4.*(SIN(X1) + EXP(X1)) + 2.*(SIN(X2) + EXP(X2))
   Y = (Y - SIN(X2) - EXP(X2))*.01/3.
   PRINT 30, Y                 (Calculates the last or right-end value and mul-
   CALL EXIT                   tiplies by ΔX*1/3)
30 FORMAT (1H0, F13.8)
   END
```

*Output:* 5.62721760 (The correct value is close to 5.627223.)

## *Selected Problems*

Several additional examples of computer-oriented problems are now offered, without their programs. Each of these problems *has been programed by students.*

### EXAMPLE 9

*Grade:* 7

*Problem:* Generate a table of factorials.

*Purpose:* To demonstrate the idea of factorials and to provide the student with a chance to see how rapidly factorials increase in size

The limitation of the word length (number of significant digits) of a computer can become an overriding factor. The program will approximate for a large number (20! has 19 digits) and will fail for a very large number.

## EXAMPLE 10

*Grade:* 11, 12

*Problem:* Given the coordinates of $n$ points on the Cartesian plane, find the equation of the best linear approximation for these points.

*Purpose:* In science courses there is occasion to collect data, to plot points on a graph, and to attempt to extrapolate or interpolate from the data. If in studying the graph the relationship seems linear, the student then obtains the following information:

1. The equation of the best linear approximation for the points (method of least squares)

   The equation of the line is $y = mx + b$, where the parameters $m$ and $b$ are defined as follows:

$$m = \frac{n\Sigma xy - \Sigma x \cdot \Sigma y}{n \Sigma x^2 - (\Sigma x)^2},$$

$$b = \frac{m \Sigma x - \Sigma y}{n}.$$

2. Some indication of how near these points are to the line

   Error can be indicated by the maximum distance of any point from the line or the average of the distances of all points from the line.

## EXAMPLE 11

*Grade:* 7–12

*Problem:* Generate an arithmetic, geometric, or some other sequence of numbers.

*Purpose:* Many different goals can be emphasized:

1. In the junior year, infinite series for arctan $x$, sin $x$, log $(x + 1)$, $\pi$, etc. are often introduced but their usefulness is seldom demonstrated. The sequence of partial sums of a series can be printed by a computer to any desired number of terms to demonstrate that the use of a series to calculate the value of a function or number is feasible. The series shown below is typical:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots .$$

2. Geometric and arithmetic sequences and series can be studied with a computer. For example, before discussing underlying mathematical proofs many terms of the series $5 + \left(\frac{2}{5}\right) \cdot 5 + \left(\frac{2}{5}\right)^2 \cdot 5 + \ldots$ can be calculated and their sum compared to $\dfrac{5}{1 - \frac{2}{5}}$, which is the algebraically determined sum of the infinite series.

3. In the seventh and eighth grades, students are often given part of a sequence and asked to guess the general rule governing generation of the terms of the sequence. The computer can be programed to generate a particular term and stop, then generate another term and stop, giving students a chance to check their hypotheses without any practical limit on the number of terms generated.

EXAMPLE 12

*Grade:* 9-12

*Problem:* Calculate the number of possible combinations of $n$ things taken $r$ at a time (where $n$ and $r$ are positive integers) and the number of permutations of $n$ things taken $r$ at a time.

*Purpose:* To enable the student to obtain quick answers to these calculations when working in probability and statistics:

$$nCr = \frac{n!}{r!(n-r)!},$$

$$nPr = \frac{n!}{(n-r)!}.$$

A program is easily written for any machine from a small desk-top calculator to a large time-sharing computer and will save time in doing the calculations.

EXAMPLE 13

*Grade:* 7-12

*Problem:* Find the cube root of any number $n$.

*Purpose:* This problem can be programed in several ways to meet various purposes:

1. The teacher may want to demonstrate how $\sqrt[3]{n}$ varies. Newton's method will generate $\sqrt[3]{n}$ about as fast as any algorithm.
2. The teacher may want to demonstrate the general idea of iterative processes. The following formula might be used for this purpose:

$$x_{i+1} = \frac{\sqrt{\frac{n}{x_i}} + x_i}{2}.$$

3. The teacher may wish to demonstrate the use of Newton's method of finding the roots of $x^3 - n = 0$. In this case one of the following equations can be used:

$$x_{i+1} = x_i - f(x_i)/f'(x_i),$$
$$x_{i+1} = x_i - (x_i^3 - n)/3x_i^2.$$

EXAMPLE 14

*Grade:* 7, 8

*Problem:* Test associative, commutative, and distributive properties of addition and multiplication mod $m$.

*Purpose:* To reinforce the concepts of associativity, commutativity, and distributivity and to apply an important technique in proof, the search for a counterexample

The general idea of mod $m$ arithmetic is reviewed in the writing of the program.

EXAMPLE 15

*Grade:* 9, 11

*Problem:* Find the roots of the general equation $ax^2 + bx + c = 0$.

*Purpose:* To review the general solution of the quadratic equation and to encourage the student to consider special cases of the solution, such as $a = 0$, $b^2 - 4ac$ is negative, and so on.

EXAMPLE 16

*Grade:* 7

*Problem:* Find the prime factors of a positive integer.

*Purpose:* To encourage the student to review the following ideas as he writes the program:

1. The meaning of "*a* is a divisor of *b*"
   The student may have to devise a test to decide whether *a* is a divisor of *b* when the remainder of the division is not available.
2. The meaning of "*a* is a prime number"
   The student must choose some algorithm to test for primes.

EXAMPLE 17

*Grade:* 7–9

*Problem:* Write a program to print the absolute value of any integer.

*Purpose:* To encourage the student to use the mathematical definition of absolute value when writing the program.

Although the sub-routine "ABS(X)" is available on most machines, it should not be used here. The student should test to determine if *x* is positive, zero, or negative, and take appropriate action in each case in order to get the absolute value of *x*. Doing this will motivate the use of the definition of absolute value and reinforce the student's comprehension of the concept.

EXAMPLE 18

*Grade:* 9–12

*Problem:* Write a program that reads the numerator and denominator of a rational number and prints the nonrepeating digits and the repetend of the infinite decimal expansion.

*Purpose:*

1. To review with the student the general idea that rational numbers can be represented by repeating infinite decimals and to "discover" some algorithm for finding the nonrepeating and repeating parts.
2. To provide examples that promote the "discovery" of theorems which state that the decimal form of every rational number repeats and that the number of digits in the repetend is a function of the denominator.

### EXAMPLE 19

*Grade:* 11

*Problem:* Calculate the area above the $x$-axis and under the curve $y = 1/x$, for $x$ between $a$ and $b$ (where $a$ and $b$ are positive).

*Purpose:* In many modern eleventh-grade texts, the logarithm function is introduced as the area described above—with $a = 1$ and $b = x$. Using this definition of the logarithm function the fundamental rule of logarithms can be demonstrated. If a computer program is written before formal proofs are offered, the following ideas can be introduced:

1. The area under a curve can be interpreted as the limit of a sequence of areas of rectangles.
2. Several tests can be run and, if desired, the rule "discovered" that

$$\log a + \log b = \log ab.$$

## COMPUTING EQUIPMENT

Although some computer topics can be taught without actual use of a computer, the real benefits of a school program involving computers can be obtained only if teachers and students actually use computing equipment. Many devices are used to teach computer-related concepts, including computer simulators, "toy" computers, logic boards, and digital trainers. Some of these devices are useful in teaching about computers—that is, in instructional programs or units in which the computer itself is the primary object of instruction. In general, however, these devices have very limited usefulness in mathematics instruction. Therefore this section will describe the types of computing equipment that are useful as problem solving tools and instructional aids in the regular mathematics curriculum.

A school that has decided to include the use of computers in its mathematics curriculum can do this in many ways. These can be categorized as follows:

1. *Direct access.* Students and teachers *operate* the computing equipment. This is sometimes called "hands-on" use.
2. *Indirect access.* Programs are sent to a computer center for processing, and the results are returned.

Direct access is provided by transporting students to a computer center, by installing a computer in the school, by bringing a small portable computer into the classroom, or by installing a terminal in the school to communicate directly with a time-shared computer system.

The advantages and disadvantages of actually having a computer in the school must be related to the costs of existing computers. With older, obsolete computers, the cost per student in the program is quite high. However, new equipment is being introduced that is much more useful and considerably less expensive than the computers available during the past few years. Also, many of the new computers are small enough that they can be brought directly into the classroom when needed.

If indirect access is used, student programs are sent to a computer center by mail or by courier. In this type of access, an important consideration is the *turn-around time,* that is, the elapsed time between sending the program and receiving the results. The means of using a computer in an individual school is determined by the type of problems to be programed. If students are working on individual projects and do not need immediate results, then indirect access by courier or mail service to a computer

facility is often adequate. If, however, results are necessary for immediate use, as for a particular class period, fast turn-around is essential and a time-sharing teletype or on-site computer is desirable.

A courier service provides an efficient and low-cost method of transmitting student programs. Programs are sent by messenger to a computer center for execution, and the results are returned in the same way. Programs may be sent in handwritten or typed form. As an alternative, equipment such as a key punch or converter (for mark-sense cards which can be converted—punched), for preparation of programs in computer-readable form, can be installed in the school. Turn-around time may range from a few hours to a few days, depending on the proximity of the computer center to the school, the means of transporting information, and the efficiency of the computer center. Some school districts already have computers for administrative data processing that could also serve the needs of students. Another possibility is buying computer time from commercial service bureaus or from computer users in the vicinity of the school. A good choice is a college or university center; in fact, several universities now provide computer services to schools in their geographical areas. In some parts of the country, educational data centers are being established to provide a variety of computer services to participating schools. In many cases, these services include processing of student programs.

If a school district is geographically remote from a computer center, indirect access can be achieved by sending and receiving programs by mail. The main disadvantage of this method is the longer turn-around time.

The availability of low-cost communication equipment permits direct telephonic transmission of student programs to a computer center for fast turn-around processing. The school installs a paper-tape, punched-card, or mark-card device that is directly attached to a telephone line for the communication of information to the computer center. When the device is not being used for actual transmission, it may be used by students for the preparation and verification of tapes or cards for later transmission. The program is reproduced on tapes, cards, or magnetic tapes at the computer center; it is processed, and the results are transmitted back to the student. Student programs might be sent in the evening, with the results available at the beginning of the next school day.

Many schools are now using computers in a mode that is referred to as time sharing. A school district can install a terminal by paying a flat monthly charge plus additional charges that depend on how much the computer is used. In this way, the cost can be geared directly to the number of students participating in the educational program and the manner in which the program is conducted.

Terminals may be installed in individual schools or in a central location (e.g., the administration building). Students may have direct access or indirect access to the computer, or a combination of both. For example, if a terminal is installed in a school, students may be given primary access during the school day; then at night administrative programs can be run so that results will be available the next morning. In many cases the terminal itself can be used as an off-line program-preparation device. Programs can be prepared "off line" (that is, with no connection to the computer) and transmitted at a later time. This results in reduced operating costs.

Projects are under way to evaluate the several ways of providing student access to computers at a reasonable per-student cost. It may happen that some blend of two or more approaches will be the best way.

The following paragraphs describe several types of computing equipment in current use in mathematics instruction and give at least one specific example of each type. The list is far from complete! It includes examples of the following types of equipment:

Calculators
Programable calculators

Digital trainers
Small general-purpose computers
Time-sharing systems.

## *Calculators*

Conventional, electrically driven calculators have been around a long time. They are used in hundreds of school districts, particularly in instructional programs for low achievers. Figure 6.13 is a picture of a printing calculator in common use.

A more recent development is the electronic calculator. Machines of this type are far more powerful problem solving tools than the conventional machines. Of course, they are also more expensive! Figure 6.14 is a picture of an electronic calculator system that can be used simultaneously by four people working on four different problems.

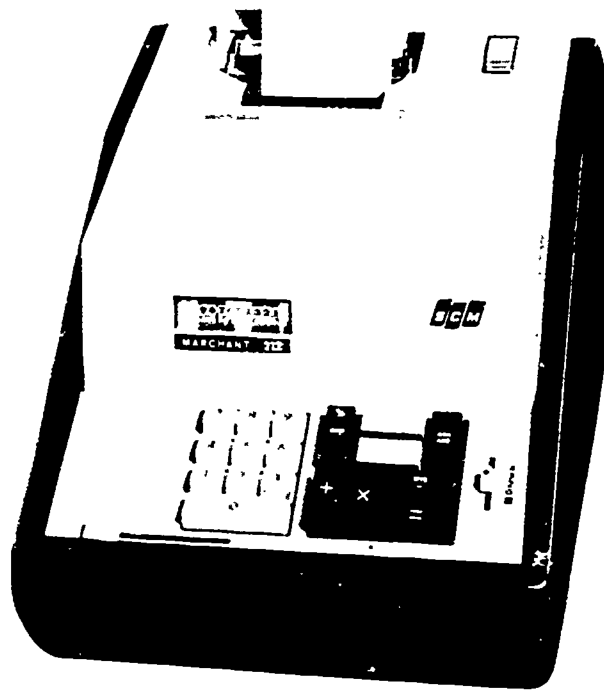
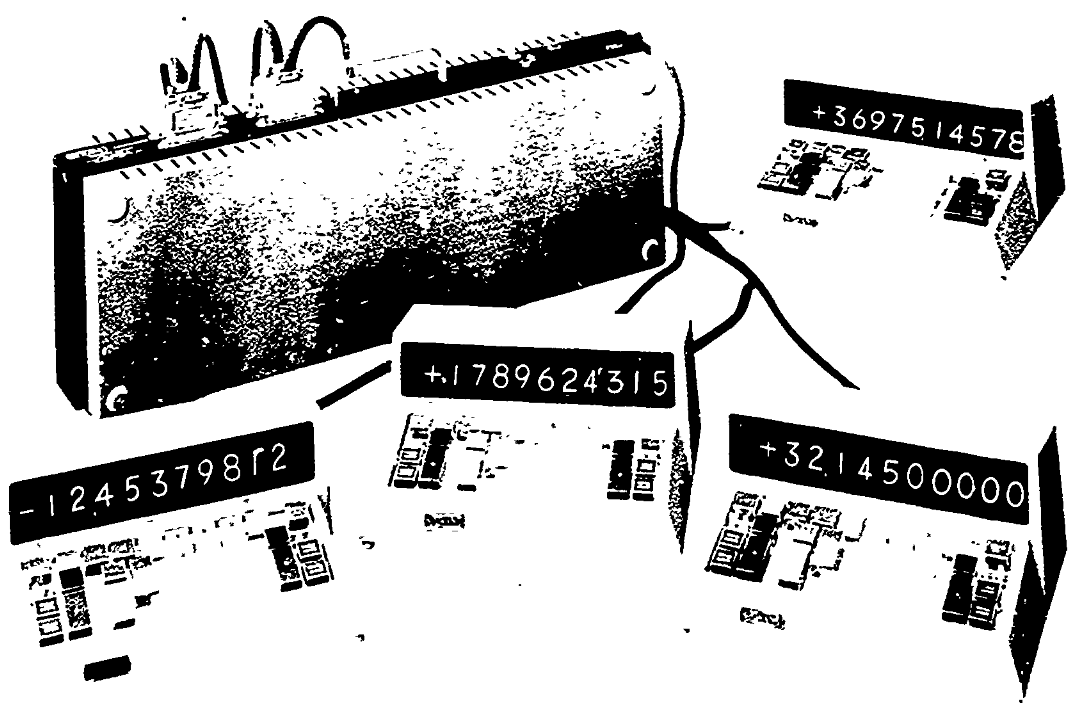



FIGURE 6.13. *Printing calculator*

*Courtesy of Smith-Corona Marchant*

FIGURE 6.14. *Electronic calculator system*




*Courtesy of Wang Laboratories*

A diagram of one of the keyboards of the Wang Laboratories system is shown in Figure 6.15.

In addition to the usual operations of addition, subtraction, multiplication, and division, this calculator also provides direct evaluation (one keystroke) of $x^2$, $\sqrt{x}$, ln $x$, and $e^x$.

Figure 6.16 shows the sequence of keystrokes required for each of several applications of the calculator. In each case, the result can be read in the display following the last keystroke.

In order to solve a problem with the calculator, one must first know what happens when each key is pressed. Then he must design a sequence of

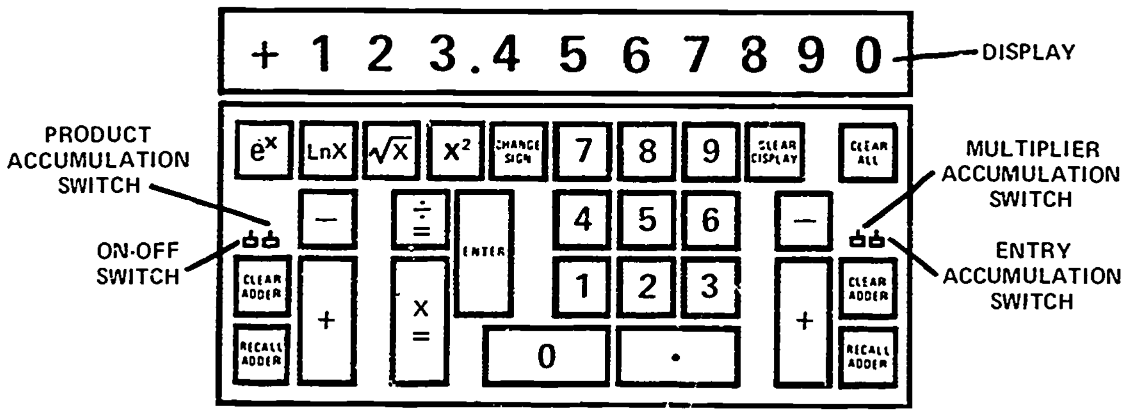



FIGURE 6.15. *Wang Laboratories calculator keyboard*

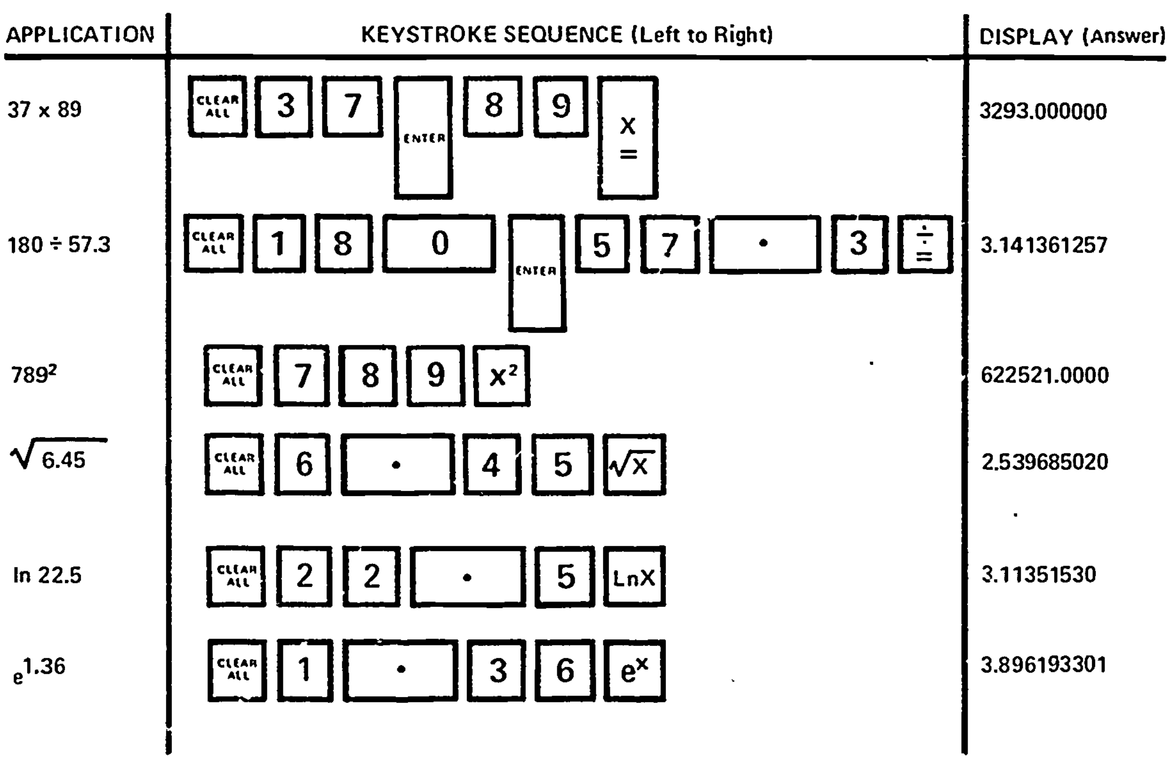



FIGURE 6.16. *Keystroke sequence*

keystrokes to solve the problem—that is, design a *program*. If the program (sequence of keystrokes) is long, it is written down for future reference. A program to evaluate $\sqrt{a^2 + b^2}$ follows.

| STEP | INSTRUCTION |
|---|---|
| 1 | Press the "CLEAR ALL" key. |
| 2 | Input the value of $a$ on the keyboard. |
| 3 | Press the "$X^2$" key. |
| 4 | Press the left-hand "+" key. |
| 5 | Input the value of $b$ on the keyboard. |
| 6 | Press the "$X^2$" key. |
| 7 | Press the left-hand "+" key. |
| 8 | Press the "$\sqrt{X}$" key. |

Time can be saved by writing the instructions in abbreviated form. The program now looks like this:

| STEP | INSTRUCTION |
|---|---|
| 1 | CLEAR ALL |
| 2 | KBD $a$ |
| 3 | $X^2$ |
| 4 | + $A_L$ |
| 5 | KBD $b$ |
| 6 | $X^2$ |
| 7 | + $A_L$ |
| 8 | $\sqrt{X}$ |

## *Programable Calculators*

A program for a conventional calculator is executed (carried out) manually. That is, the operator manually presses the instruction keys, enters data, and records the results.

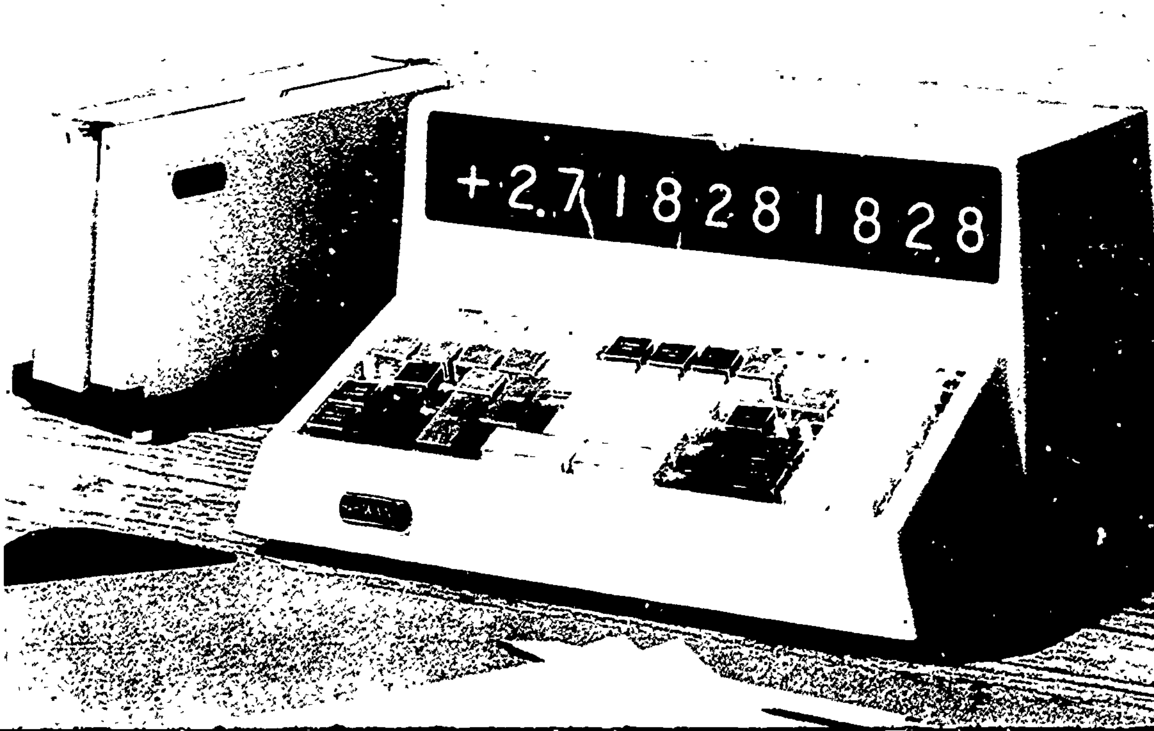
Some electronic calculators can execute a program of several steps *automatically*. These are referred to as *programable calculators*. The companies that build these machines frequently call them *desk-top computers*.

The programable calculator bridges the gap between the manually operated calculator and the general-purpose computer. Programs that would take hours to execute manually can be carried out in minutes or even seconds when executed automatically.

Two methods of programing are in common use. In one type, used by the Wang calculator, programs are punched into a *program card*. To use the program, the program card is inserted into a *card programer* and the start button pressed. The system executes the program automatically at high speed, pausing whenever input of data is required. The card programer is shown in Figure 6.17.

*Courtesy of Wang Laboratories*

FIGURE 6.17
*Wang keyboard and card programer*



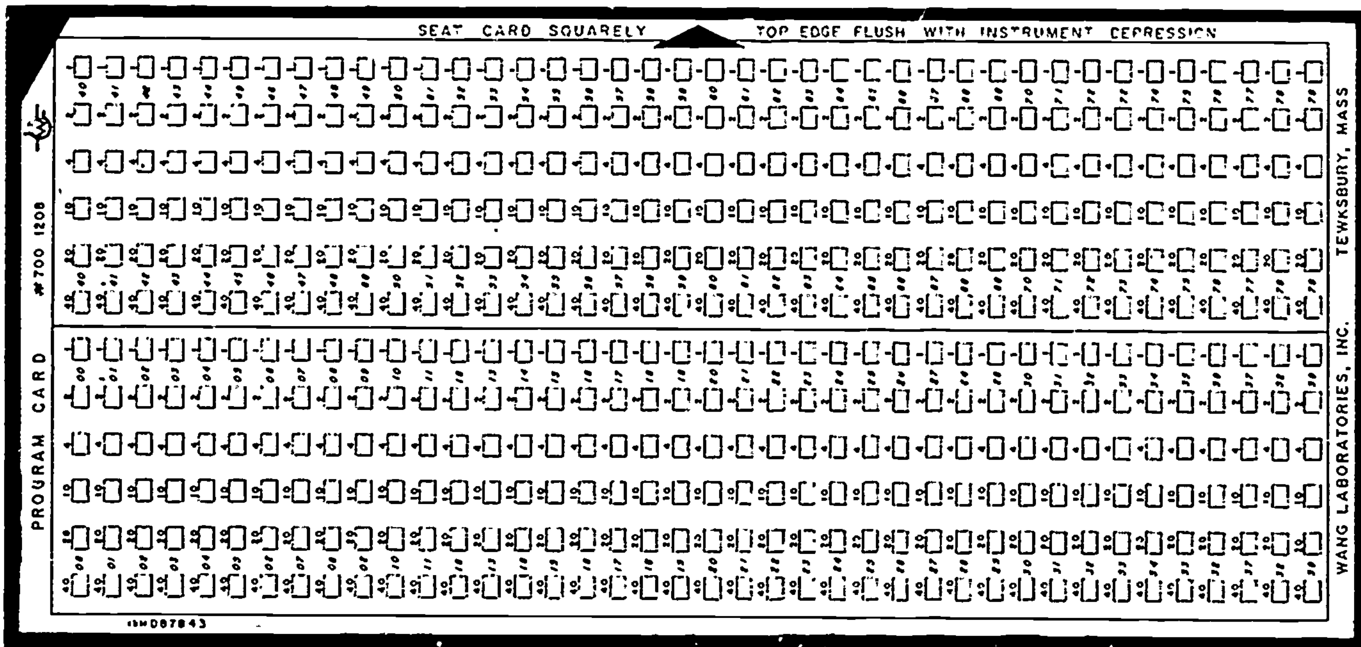



FIGURE 6.18. *Wang Laboratories program card*

*Courtesy of Hewlett-Packard*

FIGURE 6.19
*Hewlett-Packard Model 9100A programable calculator*

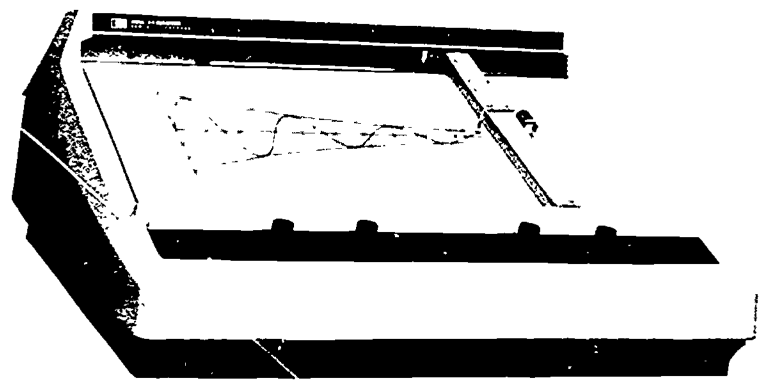
The program card is simply a prescored (punched) Hollerith card, shown in Figure 6.18. The card pictured can hold up to 80 program steps, each represented by a pattern of holes punched into the card. The holes may be punched into the card manually by using a paper clip, pencil, or stylus.

A different type of programable calculator is shown in Figure 6.19. In this calculator instructions and data are stored internally in a magnetic-core memory. The memory consists of three "working" registers and sixteen storage registers, each capable of storing a 14-digit decimal numeral. Three of the registers are displayed on the television-like screen. They are involved in the actual work of computing and data manipulation. The other sixteen registers are used to store up to 196 program steps, input data, and intermediate results. This machine features direct evaluation (1–3 keystrokes) of these functions: ln $x$, log $x$, $e^x$, sin $x$, cos $x$, tan $x$, arcsin $x$, arccos $x$, arctan $x$, as well as the hyperbolic functions and their inverses. It also has a special arithmetic unit that performs computations with complex numbers or vectors, including single-keystroke conversion from rectangular to polar coordinates and the reverse.

The machine can be used manually as a calculator or automatically as a stored-program computer. In using the machine manually, keys are pressed in the sequence required to solve the problem, just as with the Wang calculator described previously. To store a program, a switch is set to "PROGRAM" mode; then the program is entered by pressing keys in the required sequence. As each key is pressed a corresponding code is stored in the memory. The following is a program to compute $n!$, where

$$n! = n(n-1)\cdots(3)(2)(1).$$

(Here $n$ is entered manually and then the program is executed; therefore $n$ is in the X register before step 00, which copies it into the Y register.)



Courtesy of Hewlett-Packard

FIGURE 6.20. *Hewlett-Packard plotter*

| STEP | KEY | COMMENTS | CONTENTS OF REGISTERS X | Y | Z |
|---|---|---|---|---|---|
| 00 | ↑ | | $n$ | $n$ | |
| 01 | ↑ | | $n$ | $n$ | $n$ |
| 02 | 1 | | 1 | $n$ | $n$ |
| 03 | − | (Subtract) | 1 | $n-k$ | $n(n-1)\cdots(n-k+1) = P_{k-1}$ |
| 04 | IF X > Y | (Conditional | | | |
| 05 | 0 | branch to | | | |
| 06 | d | address 0d) | | | |
| 07 | ROLL ↓ | | $n-k$ | $P_{k-1}$ | 1 |
| 08 | X | (Multiply) | $n-k$ | $P_k$ | 1 |
| 09 | ROLL ↑ | | 1 | $n-k$ | $P_k$ |
| 0a | GO TO | (Unconditional | | | |
| 0b | 0 | branch to | | | |
| 0c | 3 | address 03) | | | |
| 0d | END | | 1 | 0 | $P_n = n!$ |

After entering the program, the switch is set to the "RUN" mode, $n$ entered, and "CONTINUE" button pressed to tell the machine to run the program starting with step 00. The computer computes the value of $n!$ and stops with the value displayed on the screen. The program can be repeated as often as desired for different values of $n$.

The program in the memory can be copied onto a magnetic card. Then, when one wishes to use the program again, he merely drops the card into a slot on the calculator and records the program back into the memory.

The usefulness and efficiency of the programable calculator is increased by the addition of *peripheral units*. For example, a printer and a plotter can be plugged into the Hewlett-Packard machine, providing the means for automatically printing or graphing results (Figure 6.20).

In an educational setting, the programable calculator has the following advantages:

*Low cost—and a minimum of maintenance*

*Portability*

As portable as a typewriter, it can be brought into the classroom when desired.

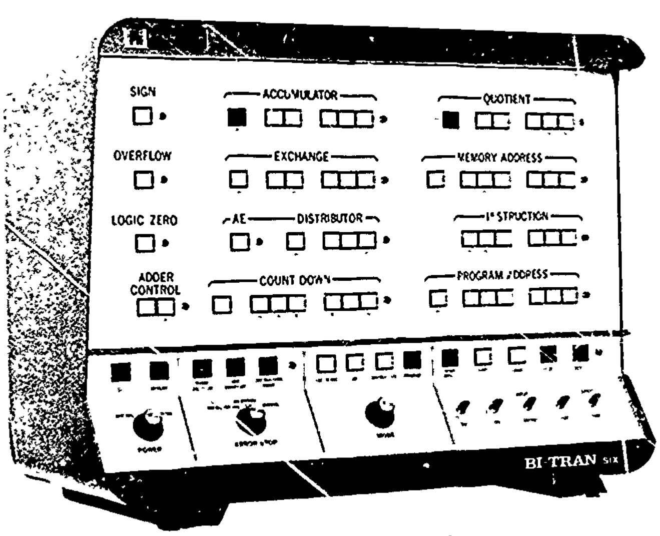
*Capability*
It has sufficient computing power to solve most of the problems in the standard mathematics curriculum. For example, these machines can be used to solve almost every problem in the previous section; the only limitation is that in solving simultaneous equations the number of equations cannot exceed three.

*Teachability*
Average and even below-average students can quickly and easily learn to use a programable calculator, yet it provides plenty of challenge to bright students.

*Transferability*
Methods learned on the programable calculator are readily transferred to more powerful equipment.

There are also, of course, disadvantages. The programable calculator has limitations in the following areas:

*Data storage*
Data storage is usually limited to a few locations; therefore, problems that require internal storage of large amounts of data (as for large matrix problems and most business data-processing applications) cannot be handled.




*Courtesy of Fabri-Tek, Incorporated*
FIGURE 6.21. *Bi-Tran 6 digital trainer*

*Program storage*
The usual limit in program storage is from 100 to 200 program steps.

*Numeric only*
The programable calculator is designed primarily for numerical calculations. It is not designed to handle variables.

*Program language*
Programing is generally in a machine language or an octal type of coding rather than a simple compiler language. This will present considerable difficulty when the mathematical problem involves a large number of algebraic manipulations and decisions.

*Program library*
These small computers may not have immediately available a library of programs—a set of programs (business, statistics, record keeping, scheduling, etc.) stored in the computer and made available to the user when needed. The program library must be stored externally on punched cards or magnetic cards. On larger computers the program library is available "on line" as part of the system. Such programs are often useful in science, social studies, and mathematics.

In brief, the programable calculator probably provides an economical and effective way to introduce computers into the mathematics curriculum. It is generally desirable, however, to supplement its use with access to more powerful equipment.

## Digital Trainers

Digital trainers are small computers that are designed specifically to help teach fundamentals in particular areas of computer-science education. They are particularly useful in teaching binary arithmetic, computer logic, computer circuitry, computer operation, and troubleshooting and repair of computers. These are subject areas in which the computer itself is the primary object of instruction. A digital trainer in widespread use is shown in Figure 6.21.

The digital trainer is not designed to be used as a problem solving tool or as a practical, "real life" computing instrument. Hence its usefulness in the mathematics curriculum is quite limited. If a trainer is available, however, it may be used occasionally to demonstrate computer concepts to students who are using other equipment for mathematical problem solving, experimentation, and discovery.

### *Small General-Purpose Computers*

One way to extend and expand the use of computers in the school is to acquire and use a small general-purpose computer, that is, a computer that includes each of the following:

*Several thousand storage locations,* each capable of storing one alphanumeric character
The memory is magnetic core or equivalent.

*Expandability*
Additional storage can often be added.

*Ability to use storage locations interchangeably* for data or instructions

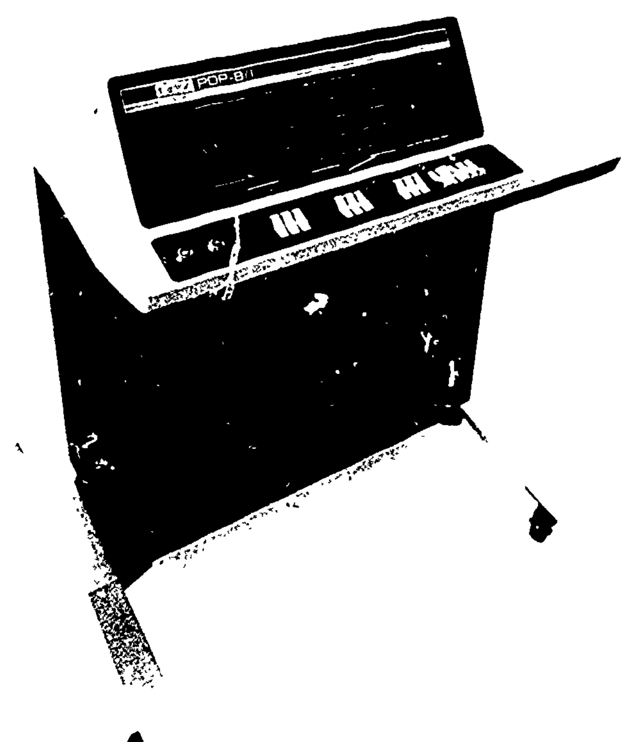
*Ability to use most of the standard peripheral devices* such as teletypewriters, paper-tape readers and punches, card readers and punches, printer, plotters, cathode-ray-tube displays, magnetic tape units, and disc storage units
These should be available as optional equipment.

*Software for processing student programs written in an algorithmic language* such as ALGOL, FORTRAN, BASIC, TELCOMP, APL, or TEACH
Languages such as BASIC or TELCOMP are particularly desirable.

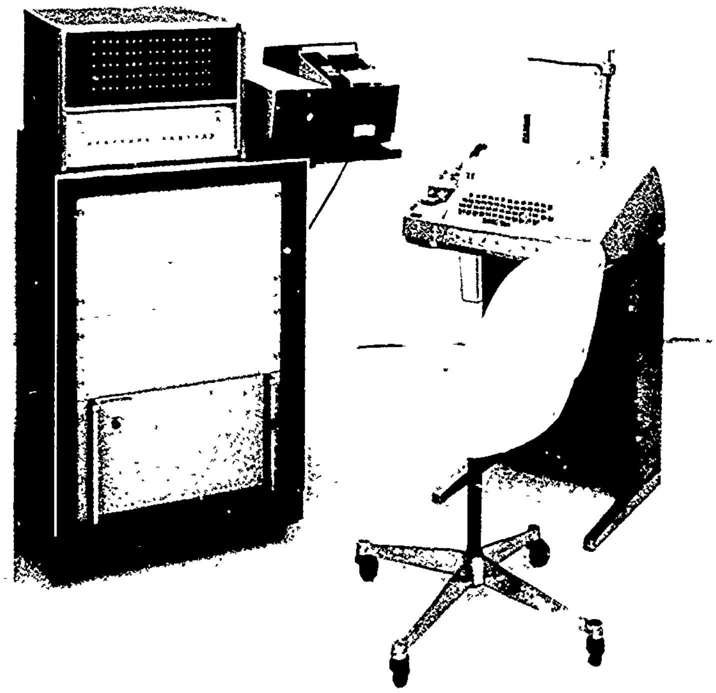
Great care must be exercised in the selection of this type of equipment. Some school districts are still purchasing old, obsolete computers that actually *cost more* than modern, third-generation machines. The older machines are slow and are terribly inefficient in processing algorithmic-language programs. Furthermore, they are generally too large (e.g., the size of a standard office desk) and too heavy to be easily portable: the newer machines are small, compact, and portable. The following briefly describes two small computers that meet these requirements.

Figure 6.22 pictures the Digital Equipment Corporation PDP-8/I computer. It is a third-generation machine using integrated circuits. The unit shown includes the computing unit, core memory, connections for input/output equipment, and an operator's console. Also included (not shown) is a teletypewriter that is connected to the computer by a cable. A complete array of peripheral devices is available as optional equipment. Software is available for running programs written in BASIC, FOCAL (similar to TELCOMP), and FORTRAN.

FIGURE 6.22. *PDP-8/I computer*




*Courtesy of Digital Equipment Corporation*



Courtesy of Hewlett-Packard

FIGURE 6.23. *Hewlett-Packard 2115A computer*

Figure 6.23 pictures the Hewlett-Packard 2115A computer, along with a teletypewriter and an optical card-reader. The optical card-reader reads information from Hollerith cards that have been prepared by marking preprinted boxes with an ordinary soft pencil.

The unit on which the computer is sitting contains the power supply and empty compartments for additional core storage and a disc storage unit. Everything is on wheels so that the system can be easily moved to where it is needed.

The small computers shown here have sufficient capability for almost every instructional use in secondary school mathematics. They may be used as problem solving tools and instructional aids in the regular mathematics curriculum. In addition, they may be used in elective courses in computer science and in vocational programs involving computers.

### *Time Sharing Systems*

A powerful new type of computer system has been developed—one that permits many users to share the simultaneous use of a large, fast system in a convenient and practical manner. This type of system is called a time-sharing system.

Each user communicates with the computer system by means of a terminal such as a teletype, an electric typewriter, or a keyboard for input and a printing device or small TV screen for output. Terminals are connected directly to the system by coaxial cable or indirectly by means of ordinary telephone lines.

If the connection is by means of telephone lines, a terminal can be geographically remote from the computer system. It can be in a different city or even in a different state.

Typical time-sharing systems service from 2 to 200 terminals each. Many of these systems include features designed specifically for educational use, and terminals are in use at secondary schools, colleges, and universities throughout the United States. Some of these systems operate nearly twenty-four hours a day, every day.

If the number of users is large, the time-sharing system will require a very fast computer with a large drum or disc auxiliary storage unit. In addition, it must have hardware devices that permit many terminals to be attached to the system. Figure 6.24 diagrams a time-sharing system.

The time-sharing system operates under control of a very powerful set of software. The functions normally performed by computation-center personnel are controlled by the time-sharing software. These administrative functions include scheduling the use of the computer (determining which user gets to run his problem and when), recording automatically (for charging purposes) the computer time used, and managing the library of programs—the set of programs (business, statistics, record keeping, etc.) stored in the computer and made available to subscribers for their use.

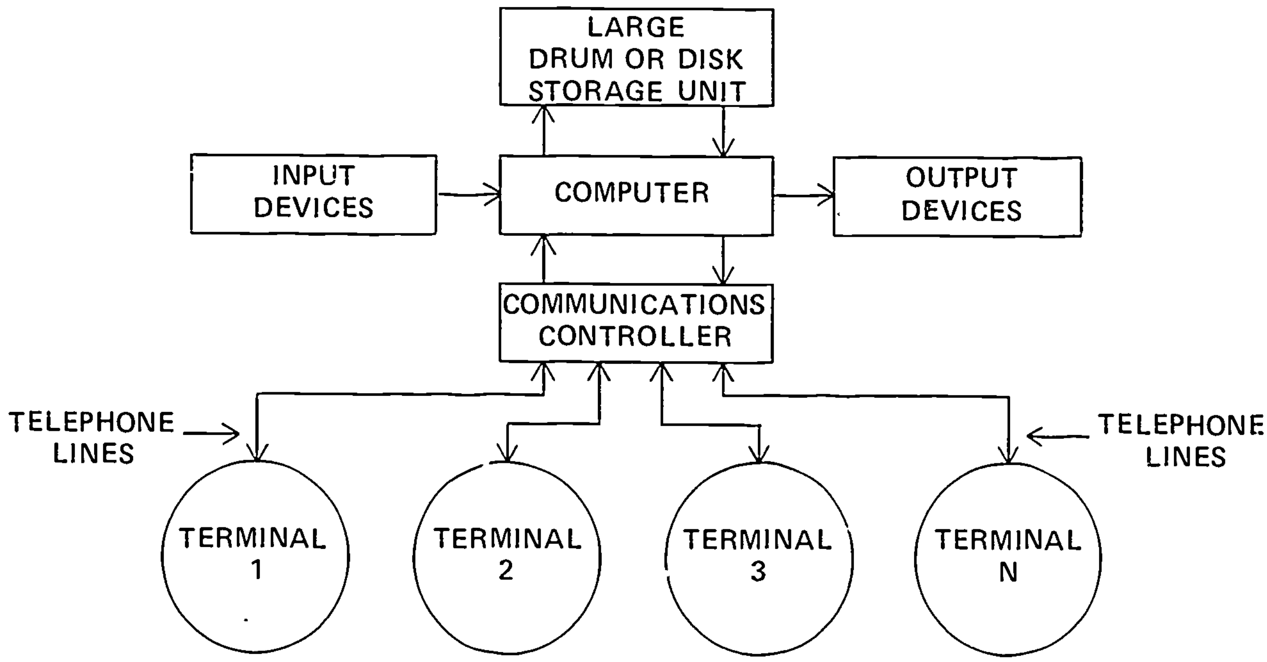



FIGURE 6.21. *Diagram of time-sharing system*

Note should be made that smaller time-sharing systems are currently available (for 2–8 users) that require far less equipment and, of course, are much less expensive. Such systems may be used effectively in a single school or shared by two or three schools.

An understanding of the use of a time-sharing system can perhaps best be gained by examining the procedure for solving a problem on the system, using one of the terminals. Assume that the user (a teacher or student) has written a program in an algorithmic language and now wishes to process his program on the system. Also assume that he is using a terminal connected to the system through an ordinary dial telephone. (See illustration opposite title page of this chapter.)

First, he gets the attention of the system by dialing. He is then able to communicate with the system by means of the terminal.

He enters the program by typing it on the keyboard of the terminal, running a punched tape that has been prepared previously on the terminal, or submitting cards (punch or mark-sense cards that have been prepared previously) through a card reader. As the program is entered, it is stored on the drum or disc storage unit.

Now suppose that the user, having entered his program, requests the computer to run his program (which exists now on the disc unit). If the computer is not being used at the moment by any other user, other than for routine input and output, the program will be running in a matter of a fraction of a second. If his problem requires only a second or so to complete (the case with many student exercises), it will be run to completion and then the output printed on the terminal. This output may consist of the desired results or may be one or more error messages pointing out typing or other errors in his program.

If the computer *is* being used when the run request is made, the request is held in a list with a certain priority. The time-sharing software contains a scheduling portion that determines, among all those programs requesting computer time, which program is to be run next and how much time is to be allocated. Eventually (in a

few seconds or less) the user is scheduled to be next. The software system causes the computer to cease running the program then being worked on (even if it is not finished), causes it to be written off onto the disc unit, brings in the new program from the disc, and starts running it.

If the new program requires longer than a second or so, the problem may in turn be interrupted, written off on the disc, and held in abeyance for a short time while other users are serviced. Later (again, in a matter of seconds) the program will be brought back from the disc and continued from the exact place where it was interrupted. This process is repeated until the program is completed and all the results have been printed.

The time-sharing system is especially well suited to situations in which many short programs are to be processed. This, of course, is the kind of situation typically encountered when a computer is used as an instructional tool in teaching elementary and secondary mathematics.

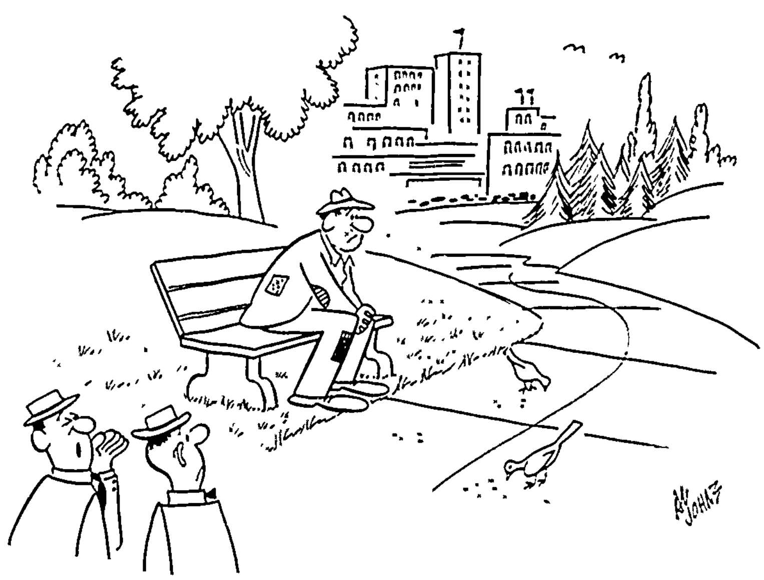
It should be noted also that the type of computer used in a time-sharing system is usually 100 to 1,000 times as fast as small drum- or disc-memory computers that might be acquired by a school for classroom use. Hence, a program that might run for several minutes on a small drum-memory computer would be executed in a few seconds on the time-sharing system.

In addition, the cost of a time-sharing system is determined by the amount of use. Typically, the user is charged only for the time he is "on line" to the computer and for the rental of the terminal device.

## CONCLUSION

In view of the rapid development of computing equipment, many of the facilities described in this chapter will probably be improved on or even replaced within a very few years, if not months. Although changes are not quite so rapid as Figure 6.25 depicts, the exaggeration has some basis in truth.

However, regardless of the equipment available, the mathematical experiences considered in this chapter will be appropriate as long as these are included in the school's curriculum objectives.



"Sad case there . . . brilliant computer man—took a six weeks vacation and fell too far behind in his field."

FIGURE 6.25 *Courtesy of F. D. Thompson Publications*

## SELECTED BIBLIOGRAPHY
*Annotated*

This bibliography is divided into four sections. Part A contains sources of classroom materials (student or teacher) and suggestions for classroom use of the computer as a problem solving tool in mathematics. Part B lists references in the areas of computers in education and computer science. Part C contains other information, including articles of interest to teachers and students, journals, and bibliographical material. Part D lists professional societies.

### A. CLASSROOM MATERIALS


1. Albrecht, Robert L. *Teach Yourself BASIC*. 2 vols. San Carlos, Calif.: Tecnica Education Corp., 1970. Paperback.
(Student text)
An introduction for junior high school or slower high school students. The pace is very slow with continued efforts to increase student motivation.

2. Albrecht, Robert L., et al. *Computer Methods in Mathematics*. Palo Alto, Calif.: Addison-Wesley Publishing Co., 1968.
(Student text)
For high school juniors or seniors or for college freshmen; designed for use as a supplementary text in mathematics courses in which computers are used for demonstration or to solve problems. It may be used as a primary text for an elective course in computer-oriented mathematics.

3. Andree, Richard V. *Computer Programming and Related Mathematics*. New York: John Wiley & Sons, 1967.
(Teacher material or college student text)
A book on programing with a good chapter on computer-related mathematics. A college-level text, but a number of the examples are appropriate for secondary school use.

4. Braun, Ludwig, and Marian Visich. *The Uses of Computers in High Schools*. 8 vols. Brooklyn, N.Y.: Huntington Project, Polytechnic Institute of Brooklyn, 1969, 1970. Paperbacks.
(Teacher reference)
A guide covering many topics in mathematics, science, and social science which gives useful programs and teaching strategies.

5. Carnahan, Brice, et al. *Applied Numerical Methods*. New York: John Wiley & Sons, 1969.
(Teacher reference)
An intermediate treatment of the theory and applications of numerical methods. The methods discussed are illustrated by fully documented programs.

6. Claxman, David. *Mathematics Laboratory Work Book*. Tewksbury, Mass.: Wang Laboratories, 1967. Paperback.
(Teacher resource book)
Thirty-seven mathematical topics that can be taught using the Wang calculator.

7. Committee on the Undergraduate Program in Mathematics. *CUPM Newsletter*, August 1969.
(Teacher reference)
This issue, with the theme "Calculus with Computers," contains information on the use of computers in first-year calculus courses in colleges and universities in the United States. Brief outlines are presented so that individuals can contact appropriate institutions for further information.

8. Computing and Mathematics Curriculum Project. *Complex Numbers, Functions, Limits,* and *Natural Bases for Logarithms*. Denver: University of Denver, 1970. Paperbacks.
(Student supplements)
These four computer-extended instruction units use the BASIC language and assume the availability of computer facilities. They were written especially for secondary school seniors or college freshmen taking a calculus or precalculus course.

9. Conte, Samuel D. *Elementary Numerical Analysis: An Algorithmic Approach*. New York: McGraw-Hill Book Co., 1965.
(Teacher reference)
An undergraduate text that is particularly clear. FORTRAN IV is used in programing examples.

10. Corlett, Peter Norman, and J. D. Tinsley. *Practical Programming*. New York: Cambridge University Press, 1968.
(Teacher reference)
This is an introductory text employing ALGOL as a language. It has both numerical

and nonnumerical applications, but the emphasis is strongly on the former.

11. CRICISAM. *Proceedings of an Invitational Conference on Calculus and the Computer*. Tallahassee: Florida State University, 1970.
(Teacher reference)
Transcript and summary of a three-day conference to discuss all aspects of using the Stenberg and Walker CRICISAM calculus text. The suggestions reflected during the conference, although they are made specifically for this text, are valuable to anyone teaching calculus.

12. Dalton, J. *Orbital Mechanics*. Hanover, N.H.: Secondary School Publications, Dartmouth College Kiewit Computation Center, 1970. Paperback.
(Student text)
An excellent exploration of the physical forces involved in orbital flight. Better students of second-year algebra or more advanced courses can understand this rewarding material. Good application of trigonometry and analytic geometry.

13. Damaskos, N. J., and M. P. Smyth. *You and Technology*. Chester, Pa.: PMC Colleges, 1969. Paperback.
(Student text)
The purpose of this text is to teach about engineering and technology using the case study approach. Several problems are enhanced by computer availability, and the problem solving approach is very useful even without available computer facilities. Filled with ideas for teachers interested in applications.

14. Dartmouth College Kiewit Computation Center. *Secondary School Project* (mid-year report) and *Demonstration and Experimentation in Computer Training and Use in Secondary Schools*. Hanover, N.H.: Dartmouth College, 1968.
(Teacher reference and source listing of student materials)
The booklets give a description of the Dartmouth–Secondary School Computer Project. In addition to listing many mathematics curriculum areas with computer potential, they provide information regarding the topic outlines and other instructional materials of the project.

15. Dodes, Irving A., and S. L. Greitzer. *Numerical Analysis with Scientific Applications*. New York: Hayden Book Co., 1964.
(Student text)
College freshmen and better high school math students will find this introduction to numerical analysis a fascinating challenge. The text assumes access to some type of computing facility and cannot be completed using only pencil and paper. No programming is taught, and the text is compatible with any computing facility.

16. Dorn, William S., and Judith B. Edwards. "Finding the Best Solution via Computer." *Journal of Educational Data Processing* 6, no. 2:90–107.
(Teacher reference)
This article describes how the computer can be used in solving the problem(s) of maximizing an area. It is an excellent example of how the computer can be used to extend instruction to solve interesting problems in mathematics.

17. Dorn, William S., and Herbert J. Greenberg. *Mathematics and Computing: With FORTRAN Programming*. New York: John Wiley & Sons, 1967.
(Student text)
An excellent text for grade 12 students or college freshmen which takes account of the role of the computer in mathematics instruction. (See review in the MAA's *American Mathematical Monthly*, January 1968, p. 103.) Also a good source of ideas for teachers.

18. Engineering Concepts Curriculum Project (ECCP). *Man Made World*. 3 pts. New York: McGraw-Hill Book Co., 1969.
(Student text, lab manual, and teacher materials)
ECCP has developed a science program for high school students. Selected sections deal with the teaching of computer concepts and using the computer in solving problems.

19. Galler, Bernard A. *The Language of Computers*. New York: McGraw-Hill Book Co., 1962.
(Teacher reference material)
Some very good "individual" problem settings for senior high school. A teacher would find this a good source of interesting problems. The book is quite readable and would be an appropriate addition to the school library.

20. General Electric Co. Mark I Time Sharing Service Teaching Guides. 4 vols. Bethesda, Md.: General Electric Information Service Dept., 1969.
(Teacher reference)
Each of these four paperback books (*Algebra I*, *Algebra II*, *Trigonometry*, and *Physics*) contains a number of programs for problems in mathematics or science, along with some exercises. The material was developed by teachers in the Altoona (Pa.) Area School District.

21. Gruenberger, Fred, and George Jaffray. *Prob*

*lems for Computer Solution*. New York: John Wiley & Sons, 1965. Paperback.

(Teacher resource manual)

A very good series of problems of widely varying difficulty which can be used to complement the regular mathematics curriculum or form the nucleus of an elective course on computer applications.

22. Gualtieri, Domenic N. *Basic Mathematics: An Electronic Approach; General Mathematics: An Electronic Approach;* and *The Real Numbers: An Electronic Approach*. Tewksbury, Mass.: Wang Laboratories, 1968, 1969. Paperbacks.

(Student texts and teacher's editions)

These books are part of a five-volume series prepared for use with Wang desk-top computers. The two books on basic and general mathematics emphasize business applications of mathematics and corresponding use of calculating or computing equipment. *Real Numbers* includes discussions and problem settings appropriate for students in grade 7 through junior college. The material on sequences (including Farey sequence) is particularly appropriate for computer application.

23. Haag, James N. *Comprehensive Standard FORTRAN Programming*. New York: Hayden Book Co., 1969. Paperback.

(Student text)

One of the lengthiest introductions to FORTRAN programing. The book is intended for secondary school students or college underclassmen who have completed two years of algebra. Many interesting applications are illustrated in the example problems.

24. Hamming, Richard W. *Calculus and the Computer Revolution*. Boston: Houghton Mifflin Co., 1968.

(Teacher material or material for selected students)

This is an expansion of an earlier monograph by the same title published by the CUPM. The booklet discusses some aspects of computing as they are related to the beginning calculus course. The material is well written and easy to follow.

25. Hansen, V. P. Discovering Mathematics through Computers series. 6 vols. Hartsdale, N.Y.: Olcott Forward, 1968. Paperbacks.

(Student texts)

Titles in this series are as follows: *Rational Numbers* (pt. 1 and pt. 2), *Algebra, Geometry, Advanced Algebra and Trigonometry, Probability and Statistics*. The material was originally written for (and published by) Olivetti Underwood Corp. for use with its desk-top computers; thus the programing settings are tied closely to the procedures of the Programma 101. The mathematical settings are standard to the regular 7–12 curriculum.

26. Hemmerle, William J. *Statistical Computations on a Digital Computer*. Waltham, Mass.: Blaisdell Publishing Co., 1967.

(Teacher reference)

An advanced text that might be used for particular topics appropriate to advanced students in secondary schools.

27. Hickey, Albert E. "The Use of the Computer in Mathematics Instruction." *Two-Year College Mathematics Journal*, Spring 1970, pp. 44–51.

(Teacher reference)

The article gives a general discussion of the role of the computer in the classroom and indicates a number of specific settings where the computer can be used as a problem solving tool.

28. Higgins, G. A. *The Elementary Functions: An Algorithmic Approach*. Hanover, N.H.: Secondary School Publications, Dartmouth College Kiewit Computation Center, 1970.

(Student text)

A course in elementary functions in which use of a computer plays an intimate part. Knowledge of BASIC and availability of computing facilities are assumed. This text, written for secondary school or first-year college students, is a good source of ideas for teachers of students in all secondary school mathematics courses.

29. Holden, Herbert L. *Introduction to FORTRAN IV*. New York: Macmillan Co., 1969. Paperback.

(Student text)

Text designed to teach FORTRAN IV for use as a scientific tool. Good for the beginner, but not appropriate as a reference work. The lack of flow charts is sometimes confusing.

30. Hull, T. E. *The Numerical Integration of Ordinary Differential Equations*. Berkeley, Calif.: Committee on the Undergraduate Program in Mathematics, 1966.

(Teacher material or college student text)

College-level mathematics and applications of the computer.

31. Hull, T. E., and David D. Day. *Computers and Problem Solving*. Reading, Mass.: Addison-Wesley Publishing Co., 1969.

(Student text)

The purpose of the book is to explain the function of a computer and show how the com-

puter can be used to solve a wide variety of interesting proble: While the book is primarily oriented to computer science, many of the exercises have implications for the mathematics classroom.

32. International Business Machines Corp. *Computer Mathematics in Secondary Schools*. Publication no. C20-1687-0.
(Teacher material)
Course administration and curriculum guide are described for a computer mathematics course. The book contains units on history of the computer, flow charts, machine language, derived language, FORTRAN, and advanced mathematical topics (number theory, probability, matrices, etc.).

33. ———. *An Introduction to Engineering Analysis for Computers*. White Plains, N.Y., IBM Technical Publication Dept.
(Teacher material)
A college-level text showing computer applications in engineering problems.

34. Iverson, Kenneth E. *Elementary Functions: An Algorithmic Treatment*. Chicago: Science Research Associates, 1966.
(Student text)
A textbook for precalculus students at the high school or college level. Although the programing language, APL, is difficult and formalism very rigorous, the text is a good source of ideas for teachers.

35. Johnson, David C., Larry L. Hatfield, Pamela W. Katzman, Thomas E. Kieren, Dale E. LaFrenz, and John W. Walther. Computer Assisted Mathematics Program (CAMP) series, 6 vols. Glenview, Ill.: Scott, Foresman & Co., 1968–1970. Paperbacks.
(Student texts with teacher commentaries)
Supplementary student booklets for each of the grades 7 through 12. The materials contain many excellent exercises for use with the standard mathematics curriculum; students write computer programs in BASIC to study concepts and solve problems.

36. Johnson, Donovan A., and Gerald R. Rising. *Guidelines for Teaching Mathematics*. Belmont, Calif.: Wadsworth Publishing Co., 1967.
(Teacher reference)
The section "The Role of Computers" in this secondary school mathematics methods text shows some interesting settings at the junior high school and senior high school levels.

37. Katzan, Harry, Jr. *APL Programming and Computer Techniques*. New York: Van Nostrand Reinhold Co., 1970.
(Student text)
A comprehensive sourcebook with examples included from mathematics, science, engineering, and business.

38. Kemeny, John G., and Thomas E. Kurtz. *BASIC Programming*. New York: John Wiley & Sons, 1967. Paperback.
(Teacher material or supplementary student text)
An introduction to techniques of computer programing using the language BASIC. There is an applications section that contains a variety of problem settings.

39. Killen, Michael. *Introduction to Programming: A Mathematical Approach*. Tewksbury, Mass.: Wang Laboratories, 1969.
(Student text and teacher's edition)
This book is one of a series prepared for use with Wang desk-top computers. It emphasizes computer fundamentals with examples taken from mathematics.

40. Knuth, Donald E. *Fundamental Algorithms* and *Semi-Numerical Algorithms*. Reading, Mass.: Addison-Wesley Publishing Co., 1968, 1969.
(Teacher reference)
These are the first two volumes in a projected seven-volume series, The Art of Computer Programming, designed to explain and illustrate most of what is known about basic computer-programing techniques (exclusive of numerical analysis). The first volume contains many excellent settings for illustrating algorithm design.

41. Koetke, Walter J. *Computers in the Classroom*. Maynard, Mass.: Digital Equipment Corp., 1968. Paperback.
(Teacher resource manual)
An indexed reference work containing specific suggestions on how and when to use the computer within a school's present mathematics curriculum.

42. Kovach, Ladis D. *Computer-Oriented Mathematics: An Introduction to Numerical Methods*. San Francisco, Calif.: Holden-Day, 1964. Paperback.
(Teacher reference or supplementary student material)
This booklet deals with "numerical mathematics" and presents a number of interesting problems. While the material is written to be done without a computer facility, the content has implications for computer use.

43. McCracken, Daniel D., and William S. Dorn. *Numerical Methods and FORTRAN Programming*. New York: John Wiley & Sons, 1964.
(Teacher reference)

The book is designed for undergraduate students in engineering or science. Students should have an analysis or calculus background before attempting many of the problems. The book is recommended as a teacher reference in the areas of evaluation of functions and solving equations and systems of equations.

44. Marcovitz, Alan B., and Earl J. Schweppe. *An Introduction to Algorithmic Methods Using the MAD Language*. New York: Macmillan Co., 1966. Paperback.
(Teacher reference or college student text)

A lower-division college text to introduce students to those methods that are applicable to the implementations of algorithms on computing machines.

45. *Mathematics Teacher.* "Computer-Oriented Mathematics" department. Edited by Walter J. Koetke.
(Teacher reference and student material)

Articles describing how teachers and students used the computer as a problem-solving tool in the mathematics classroom are included in the following list of contributions to the department.

"Quadratic Equations—Computer Style." Thomas E. Kieren. April 1969, pp. 305–9.

"Generating 'Random' Numbers Using Modular Arithmetic." Brother Arthur Indelicato, F.S.C. May 1969, pp. 385–90.

"Prime Triplets." Joseph Hirsch. October 1969, pp. 467–71.

"Computers in Mathematics Education." Charles J. Zoet. November 1969, pp. 563–67.

"A Case Study in Mathematics—the Cone Problem." Nikander J. Damaskos. December 1969, pp. 642–49.

"Patterns in Algorithms for Determining Whether Large Numbers Are Prime." Aaron L. Buchman. January 1970, pp. 30–41.

"Computer-extended Instruction: An Example." William S. Dorn. February 1970, pp. 147–58.

"Topics in Numerical Analysis for High School Mathematics." Stephen D. Schery. April 1970, pp. 313–17.

"Polynomial Synthetic Division." Irwin Hoffman and Larry Kauvar. May 1970, pp. 429–31.

"Predicting the Outcome of the World Series." Richard Brown. October 1970, pp. 494–500.

"An Eulerian Development for Pi: A Research Project for High School Students." Thomas W. Smithson. November 1970, pp. 597–608.

"A Student Computer That Really Works." William A. Leonard. December 1970, pp. 681–84.

"Introducing Matrix Algebra with Computer Programming." Margariete Montague. January 1971, pp. 65–72.

"Gauss, Computer-assisted." Helen S. Hughes. February 1971, pp. 155–66.

46. Mullish, Henry. *Modern Programming: FORTRAN IV*. Waltham, Mass.: Blaisdell Publishing Co., 1968. Paperback.
(Student text)

An introduction to programing for the beginner which can be used by students who have completed second-year algebra (although some examples would have to be skipped). Good problem development in later chapters.

47. National Council of Teachers of Mathematics. *Introduction to an Algorithmic Language (BASIC)*. Washington, D.C.: The Council, 1968.
(Teacher reference)

The pamphlet contains an easy-to-read introduction to the programing language BASIC, with illustrations of computer applications in school mathematics. It is intended for teacher use, but students in grade 9 or above could use it to learn how to program in BASIC.

48. Pennington, Ralph H. *Introductory Computer Methods and Numerical Analysis*. New York: Macmillan Co., 1965.
(Teacher reference or student text)

A post-integral calculus text for college; however, there are settings with implications for secondary school mathematics.

49. Price, Wilson T., and Merlin Miller. *Elements of Data Processing Mathematics*. New York: Holt, Rinehart & Winston, 1967.
(Student text)

For students who have completed first- or second-year algebra with a vocational interest in the fields of data processing and computer programing. The major emphasis is on problem solving rather than proof. No specific programing language is taught or assumed.

50. Ralston, Anthony, and H. S. Wilf, eds. *Mathematical Methods for Digital Computers*. 2 vols. New York: John Wiley & Sons, 1960, 1967.
(Teacher reference)

An advanced reference text too difficult for students. Format consists of a mathematical dis-

cussion of a topic followed by a series of related topics of particular interest to a computer user.

51. Rich, Barnett. *Modern School Mathematics—an Electronic Approach: Algebra I.* Tewksbury, Mass.: Wang Laboratories, 1970. Paperback.
(Student text)
A complete first-year algebra course built around the Wang calculator. This is a good idea resource book for teachers using any type of electronic calculator.

52. Rosenthal, Myron R. *Numerical Methods in Computer Programming.* Homewood, Ill.: Richard D. Irwin, 1966.
(Teacher reference)
The presentation assumes a good high school mathematics background. Included in the book are sections on recurrence relations, relaxation and Monte Carlo methods, random-number generators, determinants and matrices, and simultaneous equations.

53. Rule, Wilfred P. *FORTRAN IV Programming.* Boston: Prindle, Weber & Schmidt, 1970. Paperback.
(Student text)
Contains many sample engineering applications that can be understood by secondary school students. No previous computer experience is assumed, and almost all mathematical material is suitable for students with a solid background in second-year algebra.

54. Sage, Edwin R. *Problem Solving with a Computer.* Newburyport, Mass.: Entelek, 1969. Paperback.
(Teacher reference or student text)
The book identifies problems for computer solution through the mathematics curriculum, grades 8–12.

55. School Mathematics Study Group. *Algorithms, Computation, and Mathematics* (with student supplements for FORTRAN and ALGOL). Pasadena, Calif.: A. C. Vroman, 1966. Paperback.
(Student texts with teacher commentaries)
For a grade 12 mathematics and computing course. Spends excessive time on how a computer works; however, the later sections utilize the computer in studying appropriate mathematics. Good source of ideas. (See also Forsythe reference in part B).

56. Science Research Associates, Inc. *Computing Concepts in Mathematics.* 2 vols. Chicago: Science Research Associates, 1968.
(Student texts and teaching guides)
A good history of computers is presented in the first volume. The second surveys several computational algorithms and presents some topics in mathematics which are suitable for advanced students.

57. Serisky, Melvin. *Computer Math Experiences.* Hartsdale, N.Y.: Olcott Forward, 1970. Paperback.
(Teacher reference)
The book is written for use with the Olivetti Programma 101 desk-top computer. It is a collection of programs from different mathematical levels, from algebra to calculus, including some business and recreational mathematics.

58. Sharpe, William F. *BASIC: An Introduction to Computer Programming Using the BASIC Language.* New York: Free Press, 1967. Paperback.
(Teacher reference)
This textbook is an elementary introduction to programing in the time-sharing language BASIC, and it includes examples taken from mathematics, science, and business. The book is easy to read and understand. The emphasis is not on hardware but on getting the reader programing as quickly as possible.

59. Smith, Robert E. *The Bases of FORTRAN.* Minneapolis: Control Data Corp., 1967. Paperback.
(Supplementary student text)
An excellent source of problems (mainly number theory) for computer solution—junior high school and senior high school. The author is very clever in the way he motivates the problems in a story setting.

60. ———. *BASIC Ideas.* Minneapolis: International Time Sharing Corp., 1969. Paperback.
(Teacher reference or student text)
The text includes forty-one lessons and fifty review problems attempting to teach the BASIC language and various applications. Problem settings are not as interesting as those in *The Bases of FORTRAN,* listed above.

61. Steinbach, Robert C. *Programming Exercises for Problem-Oriented Languages.* Beverly Hills, Calif.: Glencoe Press, 1969. Distributed by Macmillan Co., New York.
(Teacher reference and source of problems)
The book provides a series of laboratory exercises designed to "develop the student's ability to communicate freely with a digital computer using a problem-oriented language." The statement of exercises is independent of any specific language. Included are a number of business-type applications.

62. Stenberg, Warren B., and Robert J. Walker. *Calculus: A Computer Oriented Presentation.* 2 vols. Tallahassee: Florida State University, CRICISAM, 1968.
(Student text)
The ideas of calculus are introduced using algorithmic concepts. Although the order of topics differs from that in other first-year calculus texts, rigor and content are not reduced but enhanced with the help of the computer. The text is independent of a specific programing language and thus can be used with any algebraic language.

63. Thompson, Bruce L. *Statistics; Biology, Physiology; Mathematics of Chemistry;* and *Secondary Level Physics.* Minneapolis: Control Data Corp. Paperbacks.
(Source of problem settings for teachers or student texts)
These four student booklets illustrate a wide variety of applications of the computer in mathematics and science.

64. Weeg, G. P., and G. B. Reed. *Introduction to Numerical Analysis.* Waltham, Mass.: Blaisdell Publishing Co., 1966.
(Teacher reference)
An advanced text that can be used for certain topics that are appropriate on a more elementary level, such as errors in computation, real roots of an equation, and the Gauss reduction.

65. Weissman, K. *School BASIC.* Hanover, N.H.: Secondary School Publications, Dartmouth College Kiewit Computation Center, 1970. Paperback.
(Student text)
A language manual, but one written for the student who has *not* completed even one semester of algebra. For users of the Dartmouth system, but easily adapted to any BASIC system.

## B. SURVEY OF COMPUTER-SCIENCE REFERENCE MATERIALS

66. Adams, J. Mack, and Robert Moon. *An Introduction to Computer Science.* Glenview, Ill.: Scott, Foresman & Co., 1970.
An elementary computer-science text with primary emphasis on FORTRAN programing.

67. Allen, Paul, III. *Exploring the Computer.* Reading, Mass.: Addison-Wesley Publishing Co., 1967.
A programed introduction to computer hardware. Very elementary, with little emphasis on use of the facility. Appropriate for students.

68. Atkinson, Richard C., and H. A. Wilson, eds. *Computer-Assisted Instruction: A Book of Readings.* New York: Academic Press, 1969. Paperback.
An excellent collection of papers on a variety of topics in computer-assisted instruction, from general survey papers to specific application in instruction, including discussions of hardware, languages, and economics.

69. Barker, P. J., and W. F. Beveridge. *Basic Computer Studies.* Edinburgh: Oliver & Boyd, 1970.
This is a general introduction to the construction, operation, and applications of computers. ALGOL and FORTRAN are discussed briefly.

70. Bolt, Albert B., and M. E. Wardle. *Communicating with a Computer.* New York: Cambridge University Press, 1970. Paperback.
The book is an elementary introduction to what a computer is and how it works. Although it was written for teachers, it is well within the grasp of junior high school students.

71. Bowles, Edmund A., ed. *Computers in Humanistic Research.* Englewood Cliffs, N.J.: Prentice-Hall, 1967.
This collection of papers presents short surveys on the use of computers in anthropology and archaeology, language and literature, and musicology.

72. *Bulletin of the National Association of Secondary School Principals.* February 1970.
This issue contains a number of articles on the theme "The Computer in Education." One by Helen Hughes discusses an experimental program in computer-assisted mathematics in which the students programed the computer to solve problems in mathematics, grades 11 and 12.

73. Bushnell, Donald D., and Dwight W. Allen. *The Computer in American Education.* New York: John Wiley & Sons, 1967. Paperback.
Should be of interest to teachers. Readings on the many roles of the computer in education—individualizing instruction, assisting in instruction and research, teaching the computer sciences (secondary school and college), and information processing.

74. Calingaert, Peter. *Principles of Computation.* Reading, Mass.: Addison-Wesley Publishing Co., 1965.
A reference book or general-interest library book for students and teachers. It includes an introduction to the field of computing and a discussion of number bases as well as some material on numerical approximation.

75. Carroll, John M. *Careers and Opportunities in Computer Science.* New York: E. P. Dutton & Co., 1967.

A survey of opportunities in computer technology applied to solving problems in manufacturing, transportation, medicine, engineering, and many other fields. A fairly comprehensive overview of the computing community.

76. Control Data Corp. *The Teacher-Student Approach to Computer Programming Concepts.* 2 vols. Minneapolis: Control Data Corp., 1963. Paperback.

Good reference material. In addition, there are a number of settings for possible classroom use.

77. Crawford, Rudd A., and David H. Copp. *Introduction to Computer Programming.* Boston: Houghton Mifflin Co., 1969. Paperback. (Student text)

An interesting introduction to machine-language programing which teaches students to write programs for an imaginary computer. No computer facilities are needed, and first-year algebra is the only prerequisite.

78. Crowley, Thomas H. *Understanding Computers.* New York: McGraw-Hill Book Co., 1967. Paperback.

An introduction to how a computer works. Readable by students.

79. Darnowski, Vincent S. *Computers—Theory and Uses.* Washington, D.C.: National Science Teachers Association, 1964.

A pamphlet written for student use (with a teacher commentary) in learning about computers. The short unit includes materials dealing with what computers are, the development of computers, how computers solve problems, and how computers are used.

80. Digital Equipment Corp. *Introduction to Programming.* Maynard, Mass.: Digital Equipment Corp., 1970. Paperback.

An excellent introduction to machine-language programing which is well suited for better secondary school students and teachers.

81. ———. *Programming Languages.* Maynard, Mass.: Digital Equipment Corp., 1971. Paperback.

A compact reference for the programing languages FOCAL, BASIC, and FORTRAN and for several assembly languages. The text does not teach programing techniques, and assembly languages are unique to the DEC family of computers.

82. Farina, Mario V. *Computers: A Self-Teaching Introduction.* Englewood Cliffs, N.J.: Prentice-Hall, 1969. Paperback.

This book is an introduction to the field of digital-computer usage. It assumes no previous computer experience and is appropriate for use by high school students.

83. Forsythe, Alexandra I., Thomas A. Keenan, Elliott I. Organick, and Warren Stenberg. *Computer Science: A First Course.* New York: John Wiley & Sons, 1969.

An excellent introduction to computer science. The first chapters deal with algorithms and flow charts. Mathematical examples are used throughout. The text is organized around flowcharting, with a supplementary book on the more widely used languages, and it is appropriate for high school or undergraduate use. (Based on SMSG reference in part A)

84. Gerard, R. W. *Computers and Education.* New York: McGraw-Hill Book Co., 1967.

A general reference book for educators which includes a survey of developments in computer-assisted instruction and learning, library utilization, and administrative record keeping and procedures. The book is a report of a conference held at the University of California to discuss the future of computers and education.

85. Gruenberger, Fred. *Computers and Communications: Toward a Computer Utility.* Englewood Cliffs, N.J.: Prentice-Hall, 1968.

Publication of the Computer Communications Symposium, 1967. This is a series of articles on new directions in computer organization—notably a computer utility—in government, industry, or the community.

86. Haga, Enoch. *Understanding Automation.* Elmhurst, Ill.: Business Press, 1965. Distributed by Taplinger Publishing Co., New York.

A reference book designed specifically for teachers who are teaching or planning to teach courses about data processing.

87. Hassitt, Anthony. *Computer Programming and Computer Systems.* New York: Academic Press, 1967.

A computer-science textbook for advanced college students. It assumes knowledge of programing and deals with further refinements of machine-language programing, monitor systems, computer hardware, and advanced programing.

88. Lohberg, Rolf, and Theo Lutz. *Computers at Work.* New York: Sterling Publishing Co., 1969.

An easy-to-read, accurate introduction to computers at work. It relates the different computer languages and their uses; it traces the growth in complexity of computers and illustrates the differences in their potentials.

89. Lovis, F. B. *Computers 1* and *Computers 2*. Contemporary School Mathematics Series. Boston: Houghton Mifflin Co., 1964. Paperback.

Booklets about computers for junior high or senior high school students.

90. Moursund, David G. *How Computers Do It*. Belmont, Calif.: Wadsworth Publishing Co., 1969. Paperback.

The book emphasizes algorithm design and flow charts to illustrate how the computer is used to solve problems. It was written to be used by high school students with a background through second-year algebra or by undergraduate college students.

91. Murphy, John S. *Basics of Digital Computers*. 3 vols. 2d ed., rev. New York: Hayden Book Co., 1970.

A good elementary introduction to basic computer hardware elements and how they work, with a slight emphasis on electronics. Usable by high school students.

92. National Council of Teachers of Mathematics. *Computer-Assisted Instruction and the Teaching of Mathematics*. Washington, D.C.: The Council, 1969. Paperback.

This booklet reports the proceedings of a National Conference on Computer-Assisted Instruction conducted at Penn State University. The report discusses hardware development, software development, and the developing role of CAI in education.

93. ———. *Computer Facilities for Mathematics Instruction*. Washington, D.C.: The Council, 1967. Paperback.

A survey of computer facilities for possible classroom use. Some sample problems are given, and the results (times and outputs) for running on different pieces of hardware (computers) are compared. Should answer many questions for schools just getting started.

94. Nolan, Richard L. *Introduction to Computing through the BASIC Language*. New York: Holt, Rinehart & Winston, 1969.

This is a good lower-level undergraduate text.

95. Post, Dudley L. *The Use of Computers in Secondary School Mathematics*. Newburyport, Mass.: Entelek, 1970.

A very good general introduction to computers for teachers and administrators with no computer-related experience.

96. *Scientific American* Editors. *Information*. San Francisco: W. H. Freeman & Co., 1966. Paperback.

A reprint of the September 1966 *Scientific American*. A comprehensive review of computer technology in many areas.

97. Smith, Eugene, and Joseph Hirsch. *Computer Mathematics 1*. Birmingham, Mich.: Midwest Publications Co., 1966. Paperback.

The student text is a two-week unit, for grades 8 through 12, on how a computer works. The material deals with such topics as the binary system, coding, floating-point arithmetic, and computer programing and history. There is a teacher commentary.

98. Sterling, Theodor D., and Seymour V. Pollack. *Computing and Computer Sciences: A First Course with PL 1*. New York: Macmillan Co., 1970.

This is an undergraduate text giving an introduction to computer science and employing PL 1 as a language.

99. Stolurow, Lawrence M. *Computer Assisted Instruction*. Detroit: American Data Processing, 1968. Paperback.

An excellent overview of CAI.

100. Walker, Terry M., and William W. Cotterman. *An Introduction to Computer Science and Algorithmic Processes*. Boston: Allyn & Bacon, 1970.

The book is intended as a text for an introductory course in computer science for students of all disciplines. The emphasis is on the analysis of classes of problems and the design of algorithms.

## C. MISCELLANEOUS ARTICLES AND PUBLICATIONS

101. *Arithmetic Teacher*, March 1969.

This issue is devoted to applications of the computer in elementary school instruction. Sample articles: "Computers and Art," by John Mott-Smith; "A Teaching Program for Experimentation with Computer-Assisted Instruction," by James L. Fejfar; and "A Workshop on Computer-Assisted Instruction in Elementary Mathematics," by Max Jerman and Patrick Suppes.

102. Association for Computing Machinery. "Curriculum 68, Recommendations for Academic Programs in Computer Science." *Communications*, March 1968, pp. 151–97.

An outline and description of programs and courses in computer science at the college undergraduate and graduate levels. Students may be interested in this for college planning.

103. *Business Automation* (journal), Hitchcock Publishing Co., 288 Park Ave. West, Elmhurst, Ill. 60126

104. *Computer Decisions* (journal). Hayden Publishing Co., 850 Third Ave., New York, N.Y. 10022.

This monthly publication includes articles on all aspects of computers and computer use, well written and readable by the non-computer scientist. It is also an excellent source of up-to-date information on hardware and software developments.

105. *Computers and Automation* (journal). Berkeley Enterprises. 815 Washington St., Newtonville, Mass. 02160.

Almost every issue includes good material for keeping abreast of developments and issues in the computing field in education, and once a year an entire issue is devoted to computers in education. Many articles on sociological, cultural, and moral issues involving computers and automation. Highly recommended.

106. *Computerworld* (weekly newspaper). Computerworld, 797 Washington St., Newton, Mass. 02160.

Devoted exclusively to the current world of computers. Hardware, software, and all applications are included. This paper is a must for all school libraries: it is motivating material for both mathematics and science departments.

107. *Computing Surveys* (journal). Association for Computing Machinery, 1133 Avenue of the Americas, New York, N.Y. 10036.

This is the survey and tutorial journal of the ACM. It contains many interesting and relatively easy-to-read articles on current activities in the computer field.

108. *Computopics* (journal). Association for Computing Machinery, Washington, D.C. Chapter, Reiss Science Bldg., Rm. 234, Georgetown Univ., Washington, D.C. 20007.

Two special issues (February 1965 and March 1967) contain bibliographies on career information and films.

109. *Datamation* (journal). F. D. Thompson Publications, 35 Mason St., Greenwich, Conn. 06830.

A good source of up-to-date information on hardware and software developments—the state of the art. Frequent articles of interest to teachers and students.

110. *Data Processing Magazine*. North American Publishing Co., 134 N. 13th St., Philadelphia, Pa. 19107.

Monthly collection of three to seven main articles on data processing in general. Also contains a section on new products, some short news items, and occasional articles pertaining to educational applications.

111. Dorn, William S. "Computers in the High School." *Datamation*, February 1967, pp. 34–35.

An article describing the role of the computer in problem solving or (in the words of the author) computer-extended instruction, CEI.

112. *Educational Media*. Educational Media, Inc., 1015 Florence St., Ft. Worth, Tex. 76102.

Monthly publication that contains occasional articles on school computer activities.

113. *Educational Technology* (journal). Educational Technology Publications, Englewood Cliffs, N.J. 07632.

This monthly publication deals with technological developments, and its coverage includes "all lines of inquiry and professional practice which offer methods of making teaching more rational."

114. Hesse, Allen R. "Iterative Methods in High School Algebra." *Mathematics Teacher*, January 1964, pp. 16–19.

The article contains some suggestions for teachers.

115. Hughes, J. L., and K. J. Engvold. "Hexapawn: A Learning Demonstration." *Datamation*, March 1968, pp. 67–73.

A simple demonstration program for explaining to the uninitiated how a computer can "learn" through experience. An excellent article for students and teachers.

116. *Interface*. Association for Computing Machinery, 1133 Avenue of the Americas, New York, N.Y. 10036.

This is the bulletin of the ACM Special Interest Group on [Computer] Uses in Education (SIGUE). Always contains a variety of interesting papers.

117. *Journal of Data Education*. Society of Data Educators, 247 Edythe St., Livermore, Calif. 94550.

This monthly publication includes a number of articles of interest to teachers from research reports to discussions of programing languages.

118. *Journal of Educational Data Processing*. Educational Systems Corp., 25 Churchill Ave., Palo Alto, Calif. 94306.

The journal is a quarterly publication "devoted primarily to the publication of technical information, original research and descriptions of the organization and operation of data processing systems." A special issue, Spring 1968, included three articles relevant to computer use in the mathematics classroom: "Starting Computer Instruction? The Third Phase is Most

Difficult," by John O. Parker; "Computer Instruction—a Three Dimensional Approach," by Judith B. Edwards, and "The Computer, A Mathematics Laboratory," by William S. Dorn.

119. Knuth, Donald E. "What Is an Algorithm?" *Datamation*, October 1967, pp. 30–32.

This excellent article discusses a portion of the first chapter of *Fundamental Algorithms*, listed in part A under "Knuth". The author introduces the concept of an algorithm, using the Euclidean algorithm for GCF as an example.

120. LaFrenz, Dale E., and Thomas E. Kieren, "Computers for All Students." *School Science and Mathematics*, January 1969, pp. 39–41.

An overview that proposes the point of view that computers should be used more extensively in the secondary schools; it suggests some sample settings.

121. "The New Computerized Age." *Saturday Review*, 23 July 1966, pp. 15–42.

An excellent set of articles dealing with many aspects of the computer revolution—social implications, educational uses, business applications, etc. Appropriate for high school students and teachers.

122. Schaaf, William L. *The High School Mathematics Library*. 4th ed., rev. Washington, D.C.: National Council of Teachers of Mathematics, 1970.

This pamphlet contains a section with a bibliography of many general or survey books and pamphlets on computers and automation and on linear programing (pp. 47–51).

123. Spencer, Donald D. "Computers: Their Past, Present, and Future." *Mathematics Teacher*, January 1968, pp. 65–75.

History of computers; useful for background information for computer-oriented teaching.

## D. PROFESSIONAL SOCIETIES

124. American Educational Research Association, 1126 Sixteenth St., NW, Washington, D.C. 20036. This group of educational researchers includes many persons actively engaged in computer-oriented problems. Excellent sources of information concerning recent work include the organization's annual meeting and its publications. Of special help is the annual abstract of papers read at the year's meeting, available through the office in Washington. This organization also has a special interest group for researchers in mathematics education.

Periodicals: *American Educational Research Journal*, *Review of Educational Research*, and *Educational Researcher* (newsletter).

125. American Federation of Information Processing Societies, 210 Summit Ave., Montvale, N.J. 07645. This is a federation of societies to serve as a national voice of the computing field. The fall and spring joint computer conferences are noteworthy. Dates of these conferences are given in the ACM publications (see below).

126. Association for Computing Machinery, 1133 Avenue of the Americas, New York, N.Y. 10036. A professional group seeking to "advance the sciences and arts of information processing" and "promote the free interchange of information about the sciences and arts of information processing both among specialists and among the public." It has several special interest groups and special interest committees dealing with selected areas.

Periodicals: *The Journal*, *Communications*, *Computing Reviews*, and *Computing Surveys*.

127. Association for Educational Data Systems, 1201 Sixteenth St., NW, Washington, D.C. 20036. Seeks to "promote a greater flow of information among educators regarding data and information processing ideas, techniques, materials, and applications."

Periodicals: *AEDS Monitor* and *AEDS Journal* (quarterly).

128. Association for Machine Translation and Computational Linguistics, 1755 Massachusetts Ave., NW, Washington, D.C. 20036. Concerns itself with significant application of computers to verbal data processing, including methods of textual analysis, abstracting, editing, psychological and sociological analysis of interview transcripts, group discussions, mass communication, and so forth.

Periodicals: *The Finite String* (newsletter) and *Mechanical Translation*.

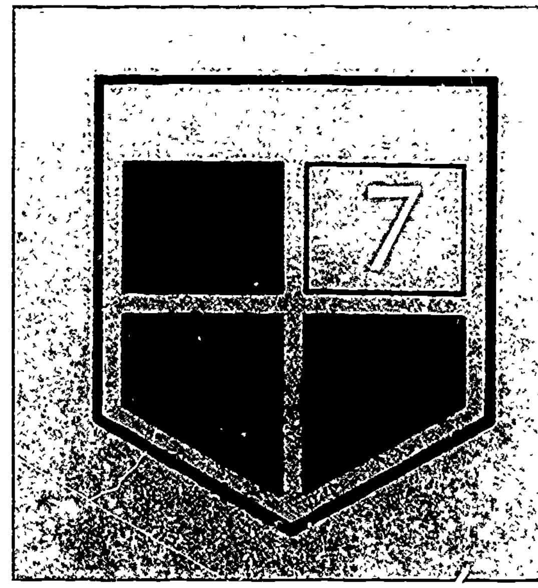
129. International Federation of Information Processing Societies, IFIP Secretariat, P.O. Box 311, 1211 Geneva 11, Switzerland. A federation of societies to serve as an international voice of the computing field. It has a technical committee on computer science education (TC3) and a working group on computer education at the secondary level (WG 3.1). Every three years the federation sponsors an international computer meeting; the most recent one was held in the summer of 1971.

130. National Council of Teachers of Mathematics, 1201 Sixteenth St., NW, Washington, D.C. 20036.


$a^2+b^2=c^2$

CHAPTER 7

# PROJECTION DEVICES



*by*
DONOVAN R. LICHTENBERG
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Chapter 7 treats such instructional aids as motion pictures, television, filmstrips, slides, and overhead and opaque projectors. The overview of the kinds of devices and accompanying materials that are available, particularly recently developed ones, should help teachers make wise selections. Also included are suggestions for making effective use of projection devices, hints on the preparation of transparencies, and advice on keeping informed about films, filmstrips, and other media.

◄ A VIDEOTAPE RECORDING SYSTEM includes a television camera and monitor, a videotape recorder, and a receiver.

# 7 PROJECTION DEVICES

Of all the contributions that modern technology has made to education, the materials discussed in this chapter are among the most used and also the most abused. In some respects they seem to hold the greatest promise for the future.

This chapter deals with the use of devices that project a light image on a screen. Strictly speaking, television does not fall within this classification, but it has much in common with projection devices and is therefore included here.

Projection devices are usually looked on as instruments for bringing common experiences to large groups of students who often vary greatly in backgrounds and abilities. With our burgeoning school populations, this function of these aids becomes increasingly significant. However, in recent years more and more attention has been paid to individualizing instruction; and teachers are finding that projection devices can play a unique role in this aspect of education, also.

The purposes of this chapter are to give the reader an overview of the kinds of materials available, to discuss some ways that they can be and are being used, and to offer some suggestions for effective use. We begin our discussion with motion pictures and then move on to television and to the several means of projecting still pictures.

## MOTION PICTURES

Although motion pictures intended for classroom use have been available since the twenties, it is not possible to assert that they have met with a great degree of acceptance on the part of mathematics teachers. Indeed, there are probably thousands of teachers who have never utilized a film in the mathematics classroom. The reasons for this are varied. For some teachers the task of ordering a film, previewing it, and arranging for its showing seems too time-consuming and too complicated to make it worthwhile.

Another reason for the lack of acceptance is undoubtedly the inconsistent quality, in terms of mathematical content, of many of the films on the market. With respect to this there has fortunately been much improvement in recent years. Film producers have begun to pay heed to the fact that a film on mathematics is not of much use if it is filled with inaccuracies.

Many mathematics teachers reject motion pictures because they question their appropriateness. They contend with a great deal of conviction that motion pictures are not as appropriate in the teaching of this subject as they are in the teaching of the social studies or science. But many of the reasons given by the proponents of the medium for using motion pictures are as relevant for mathematics as for any other subject.

As any experienced teacher knows, the problem of motivation is critical in the teaching of mathematics. There are many films available which can do much to arouse and sustain interest. Good examples of such films are the ones in the Adventures in Number and Space series.[1] These nine films feature Bil Baird and his puppets, and they deal with topics ranging from arithmetic to topology. From a mathematician's point of view the films may leave something to be desired, but any junior high teacher will find that they generate considerable interest in the subject matter. And, like so many other films, they add variety to our teaching and provide a dramatic impact not easily obtained in another way.

The motion picture also affords a way of bringing to students in the classroom experiences that would not otherwise be theirs. In the film *The Mathematician and the River*,[2] viewers are shown how mathematics is used in analyzing the flow of the Ohio and Mississippi rivers for the

1. Published by Association Films, 16 mm, b & w, 30 min. each.
2. Published by Educational Testing Service, 16 mm, color, 20 min.

purpose of flood control. It is unlikely that a teacher could present the ideas contained in this film as well through another medium.

The most obvious characteristic of motion pictures is, of course, the creation of the illusion of motion. Although certain mathematical concepts such as locus are now taught from a static point of view, there are still many aspects of mathematics in which movement contributes to clarification. This is particularly true in discussions of the applications of mathematics. In many good mathematics films, motion is produced through the technique of animation. This allows the portrayal of phenomena that are not readily observable in real life. In the film *Time*,[4] for example, a rocket circling the earth is shown to illustrate the necessity for the international date line.

Animation is cleverly used in the *Seven Bridges of Königsberg*[3] to show how Euler analyzed the problem of crossing the seven bridges in a unicursal walk. In *Possibly So, Pythagoras*,[5] the Pythagorean theorem is demonstrated by a transformation of areas which is effected through animation. In the five-film Discovering Solids series,[6] animation is used repeatedly to develop the formulas for volume and surface area of prisms, cylinders, pyramids, cones, and spheres. Other outstanding examples of the use of animation to communicate mathematical concepts can be found in the student films of the NCTM series Elementary Mathematics for Teachers and Students.[7]

Aspects of motion-picture technology such as change-speed photography and microphotography can also be used advantageously in mathematics films. Students will be fascinated by the close-up shots of bees manipulating wax in *Mathematics of the Honeycomb*,[8] a film that shows how bees minimize the amount of wax needed by using hexagonal prisms.

## *Using Motion Pictures Effectively*

In order to select a film that will help achieve a desired instructional goal, a teacher naturally has to know what is available. This means that he must be familiar with the catalogs put out by commercial distributors and film libraries. The principal distributors of films in the field of mathematics are listed preceding the references at the end of this chapter.

The teacher should also be familiar with some of the comprehensive sources of film information. For many years *The Educational Film Guide*, published by the H. W. Wilson Company, was the leader in its field. Unfortunately, the publication of this guide was discontinued after the annual supplement for 1962 was issued. Copies of the guide are still available in libraries, however, and it is still useful.

Many films are listed in *The Educational Media Index* (5). This fourteen-volume index is a compilation of available instructional resources other than printed materials. Videotapes, films, kinescopes, filmstrips, slides, and transparencies are included in the listings. Volume 10 is devoted entirely to resources for mathematics teaching from the junior high school level upward. Volume 1 *(Preschool and Primary)* and volume 2 (*Intermediate*) contain listings of materials useful in the teaching of elementary school mathematics. Supplements to the *Educational Media Index* were planned, but none has yet been published.

In 1967, the National Information Center for Educational Media (NICEM) was established at the University of Southern California. NICEM plans to produce a series of reference volumes on audiovisual materials. One volume presently available is its *Index to 16-mm Educational Films* (16). Approximately five-hundred

3. Published by Indiana University, 16 mm, color, 15 min.
4. Published by International Film Bureau, 16 mm, color, 4 min.
5. Published by International Film Bureau, 16 mm, color, 11 min.
6. Published by Cenco Educational Aids, 16 mm, color, 15 min. each.
7. Developed by the National Council of Teachers of Mathematics in cooperation with General Learning Corp. and distributed by Silver Burdett Co., 16 mm, color, 7 to 13 min.
8. Published by Moody Institute of Science, 16 mm, color, 13 min.

mathematics films are listed in this volume.

The most recent comprehensive index is the *Learning Directory* (13). This is a seven-volume listing of films, slides, filmstrips, transparencies, and other instructional materials.

Another publication with which the teacher should be familiar is *Educators Guide to Free Films* (6). This guide is issued annually and lists the many films that are available from business, industry, and governmental agencies.

When a teacher finds a listing of a film that seems appropriate for his purposes, it is a good idea to look for an evaluative review of the film. Reviews of approximately 150 films were published in the December 1963 issue of the *Mathematics Teacher*. Other reviews have been published in more recent issues of the *Mathematics Teacher*. The *Film Evaluation Guide* published by the Educational Film Library Association contains evaluations of about 50 mathematics films. Film reviews can also be found in the *Booklist*, published twice a month by the American Library Association.

If a teacher decides to use a particular film and it is not available locally, he should order the film from a distributor considerably in advance of the date for showing. To fit the film presentation in at the proper point in a course obviously requires careful planning.

When the film arrives, arrangements should be made to preview it. No teacher should be so imp ..dent as to neglect this step, for the preview serves many purposes. Even if a review of the film has been favorable, a teacher might decide upon previewing it that the film does not do what he thought it would do. Certainly there is no way for a teacher to make his final decision regarding the use of a film except by familiarizing himself with it. The preview also provides an opportunity to plan the class activities that will precede the showing of the film and those that will follow the showing.

When a film is to be shown, the room and the projection equipment should be readied ahead of time if at all possible. The film should be threaded into the projector, the machine should be focused, and the sound adjusted. When it is time for the class to view the film, a flick of the switch is all that should be necessary.

It is, of course, the responsibility of the teacher to make sure that every child in the class is seated so that he can see and hear. A rule of thumb is that no child should be seated further from the screen than a distance six times the width of the screen. But the best way to assess seeing and hearing conditions is simply to walk from place to place in the room.

A motion picture should not be regarded as a lesson in itself. Research has demonstrated the importance of the beforehand preparation and the follow-up activities. Maximum benefit will be gained from a film only if the teacher prepares the class properly. Students should be given suggestions about what to expect in the film. Some teachers experienced at using films find that presenting a list of questions about the key concepts to be dealt with is an effective way of setting the mood. It may be necessary to discuss some of the vocabulary used in the film. Students quite naturally tend to lose interest if they hear too many unfamiliar words. The teacher must consequently learn to anticipate difficulties of this nature.

If a list of questions has been given to the class before the showing of a film, the follow-up activities would obviously include a discussion of these questions. But a good teacher will go beyond this. He will use the film as a springboard to further learning. Suppose, for example, that *Mathematics of the Honeycomb* is shown to a class. This film should lead to an examination of the relationship between volume and surface area of three-dimensional figures and between area and perimeter of plane figures. The possibilities for extending the concepts developed in such a film are almost unlimited.

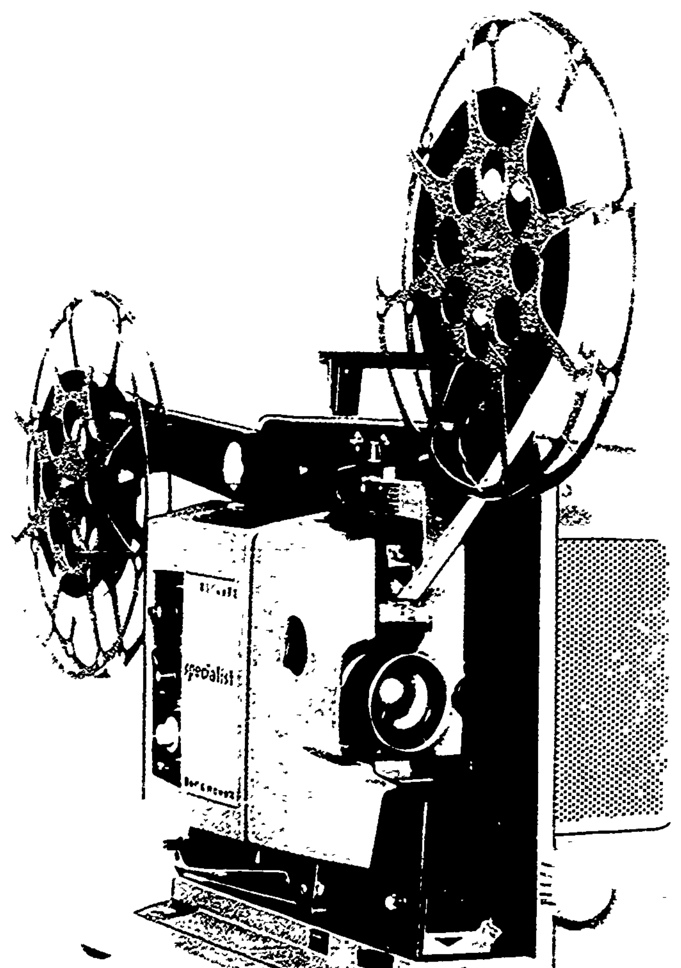
One of the immediate objectives of the follow-up session is to help the teacher ascertain whether the film was used at the right place and whether it is worth using again. Some teachers

keep a record on index cards of the films they use and their effectiveness. In this way it is possible to build a valuable card file in a few years. If a teacher finds that a certain film is especially worthwhile and thinks he will be using it over and over, he should consider recommending its purchase by the local school system.

## *Developments in Projectors*

The standard classroom motion-picture projector for a number of years has been the 16-mm sound projector. Many of the newer models are self-threading (Figure 7.1). This feature certainly eliminates one of the main obstacles to frequent use of motion pictures. The better projectors are also capable of stopping on a single frame without burn damage to the film, and they can be reversed so that a sequence in a film can be repeated for emphasis and clarification.

Despite the great improvements in 16-mm projectors, recent developments in 8-mm equipment will probably have an even greater impact. We are referring to the development of cartridge projectors and the improvements in 8-mm film.



*Courtesy of Bell and Howell*

FIGURE 7.1
*Self-threading movie projector*

Loading a cartridge projector is simply a matter of pushing a film cartridge into a slot (Figure 7.2). The film is ready for immediate showing. As it is viewed it is rewound in the cartridge. The problems of threading, adjusting, and rewinding are nonexistent. The cartridges range in capacity from four minutes of film up to thirty minutes.

The prospective purchaser of a cartridge projector has a choice of several types. Both silent and sound projectors are being marketed. Some projectors are of the normal front-screen type (Figure 7.3), while others make use of rear projection (Figure 7.4). In the rear-projection system the projected beam is reflected from a mirror onto the rear of a screen made of a translucent material. The main advantage of this system is that it requires little darkening of the viewing room. Rear projection devices are available as attachments for units designed for front projection (Figure 7.5).

Cartridge projectors use 8-mm film. When the first ones were introduced, the 8-mm film that was available was of the type designed for home movie making. This film left much to be desired in the way of picture quality, although it was satisfactory for individual or small-group viewing. In 1965, super-8 film was introduced by Eastman Kodak. This film has many advantages over the standard-8 film, the chief one being an approximately 50-percent-larger picture area. The larger area is obtained by decreasing the

FIGURE 7.2. *Cartridge film projector*

*Courtesy of Ealing Corporation*

space for sprocket holes and the space between frames. Because of the increase in picture size, the projected image is much brighter and sharper. Super-8 film is consequently suitable for viewing by groups of as many as a hundred people.

Another advantageous feature of super-8 film is the 4:3 ratio of picture width to picture height—the same as for 16-mm film. This means that super-8 prints can be made from 16-mm film without cutting off part of each frame. The teacher can expect to see the printing of 8-mm versions of many educational films previously available only in 16-mm format. As a matter of

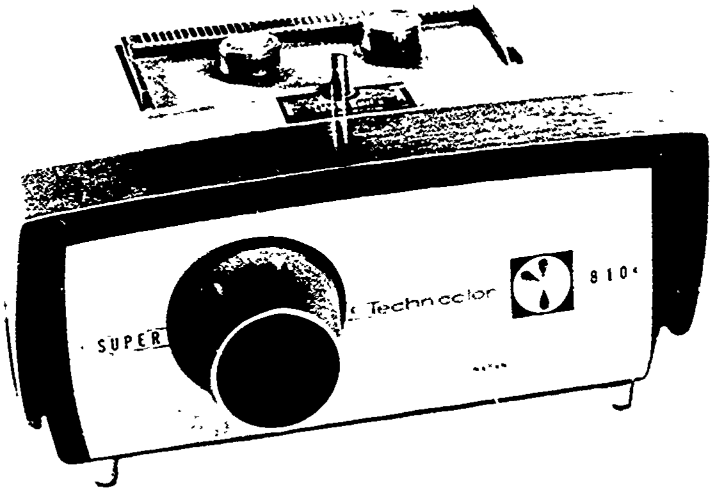



FIGURE 7.3. *Front-screen type*

*Courtesy of Technicolor Corporation*

*Courtesy of Fairchild Camera and Instrument Company*

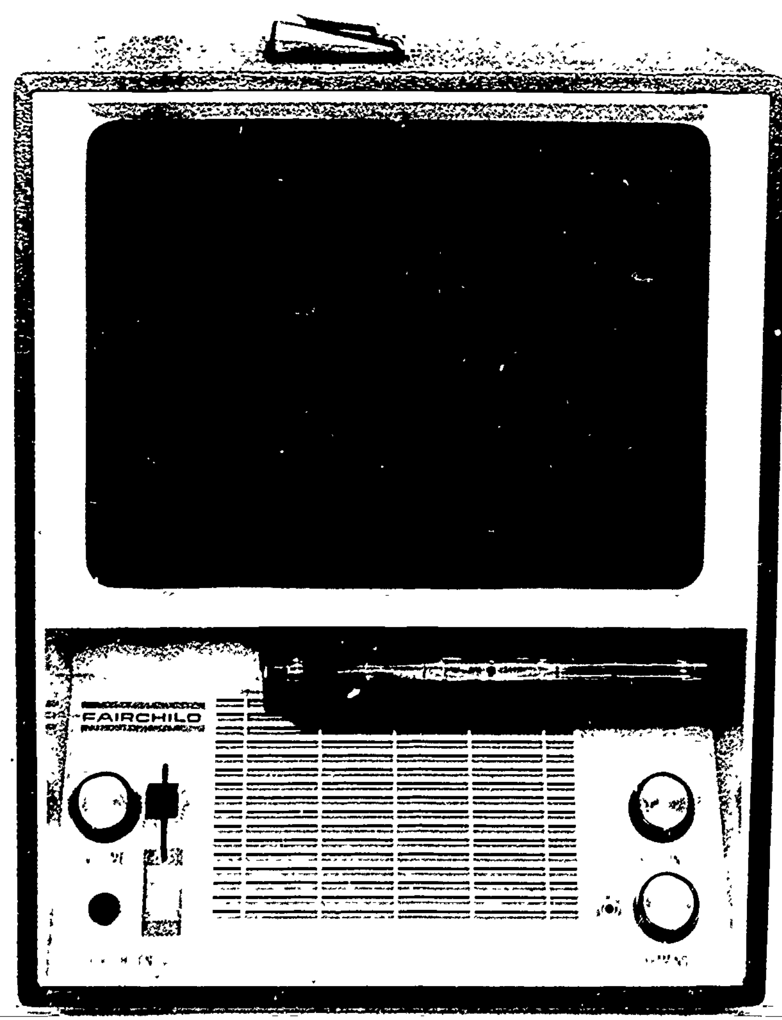
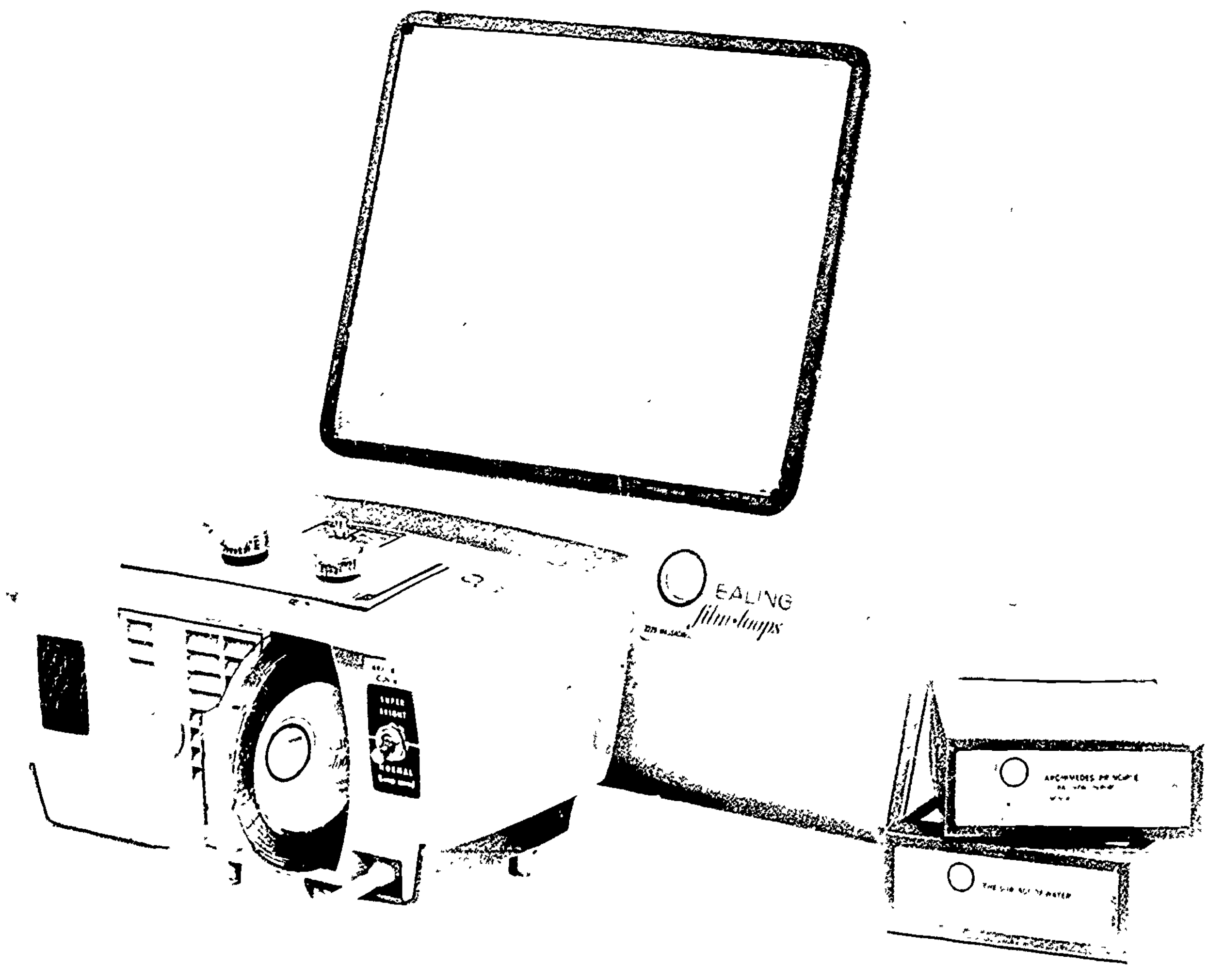



FIGURE 7.4. *Rear projection*

fact, at least one leading distributor advertises that it will make 8-mm prints of any title in its library as soon as it has orders for at least five prints of that film. This means that the motion-picture budget will stretch much further. Not only is the original cost for 8-mm films considerably less than that for 16-mm films, but because hands never have to touch a cartridged film it should stay in good shape for many more showings.

The majority of the cartridged films that are currently available are short, single-concept "loops." These are ideal for individual or small-group instruction. The simplicity of operation of the cartridge projector means that almost any child of school age can operate one by himself. And since the film rewinds itself in the cartridge, an individual student can view it as many times as he wishes. The motion-picture film has finally become as accessible to a child as a library book.

FIGURE 7.5. *Rear-projection device as attachment*




*Courtesy of Ealing Corporation*

## TELEVISION

The possibility of utilizing television for educational purposes was recognized almost from the beginning of experimentation with the medium. At the height of the depression in the early 1930s, the University of Iowa, Kansas State, and Purdue University produced trial educational programs. New York University began to experiment with such programs in 1936.

After the interruption caused by World War II the pace of development of educational television picked up rather slowly at first, but it accelerated in the early 1950s. In February 1950, WOI-TV began operations on the campus of Iowa State at Ames. WOI-TV was licensed as a commercial station, but from the outset it devoted much time to educational programing. In 1952 the Federal Communications Commission reserved 242 channels for exclusive educational use. The number of reserved channels has since been increased. KUHT in Houston began broadcasting in June 1953, thus becoming the first noncommercial educational television station. In the next few years dozens of other stations were established.

Concurrently with the development of broadcasting stations, experiments were being conducted with closed-circuit installations. The Pennsylvania State University experimental program beginning in 1954 and the Hagerstown (Washington County) Maryland project initiated in 1956 were particularly noteworthy in the early years. Both attracted widespread attention and amply demonstrated the feasibility of direct teaching by television.

Commercial networks have also played a role in educational television. In October 1958 the National Broadcasting Company initiated its "Continental Classroom" with a course in physics. A year later a chemistry course was telecast, and in 1960-61 "Continental Classroom" presented a course in modern algebra during the fall semester and a course in probability and statistics in the spring.

At present there are more than four hundred educational television stations on the air in the United States. There are more than eight hundred closed-circuit systems in educational institutions, and more of these are being put into operation every year. Several states have statewide educational networks. It is difficult to estimate how many students are receiving at least some of their instruction by means of television, but the number unquestionably exceeds 10 million.

Despite these figures, television has not had the profound influence on education that many enthusiasts had predicted for it in the 1950s. Few will dispute the statement that television is the most potent, versatile, and comprehensive medium of communication yet devised by man, but its impact on education has nevertheless been negligible. This is particularly true at levels beyond the elementary school.

It is not that students cannot learn by means of television. In the years since the advent of educational television, hundreds of studies have compared the effectiveness of instruction using this medium with that of more conventional means of instruction. Most of the research indicates that there is no significant difference—students learn at least as well from televised instruction as they do from ordinary classroom instruction.

Why, then, has the potential of educational television not been realized? Certainly one of the factors is the inertia in the American educational system. Most change takes place slowly. Many teachers, particularly at the higher levels, seem perfectly willing to conduct classes in the usual manner and stubbornly refuse to even consider the possibilities of television.

Also hindering greater acceptance of television is the fact that the vast majority of the programs produced so far have not taken advantage of the medium. The programs are adequate—as stated above, children learn from them—but most of them do not generate great amounts of enthusiasm among the participants. In February 1967 a conference on television in mathematics education was held at the National Center for School and College Television in Bloomington, In-

diana. The conferees were mathematics educators and television personnel. For two days these people viewed videotapes and kinescopes of televised mathematics lessons from all parts of the country. They were not impressed with what they saw (15). The general conclusion of the group at the end of the conference was that only a few of the lessons had exhibited any real creativity. In far too many cases a telelesson is produced by simply transferring a classroom setting to a television screen. It is not surprising that the response to such programs has been indifferent.

The inflexible schedule of broadcast television has been another deterrent to greater utilization. This problem may eventually be conquered with the aid of some recent developments. One is the production of low-priced videotape recorders. A school of moderate size can now afford this piece of equipment. The tape machine (Figure 7.6) enables a school to record, for later playback, programs that are broadcast at an inconvenient time. It also allows repeat showings for classes that need the extra exposure.

Another significant development is the opening up for educational purposes of some frequencies in the upper regions of the electromagnetic spectrum. Only one program at a time can be broadcast over a given television channel, and in the VHF and UHF portions of the spectrum only one channel can be assigned to a station. In 1963, the Federal Communications Commission opened up 31 channels in the 2500-2690–megacycle frequency range. A licensee can be assigned as many as four of these channels and can hence broadcast as many as four pro-

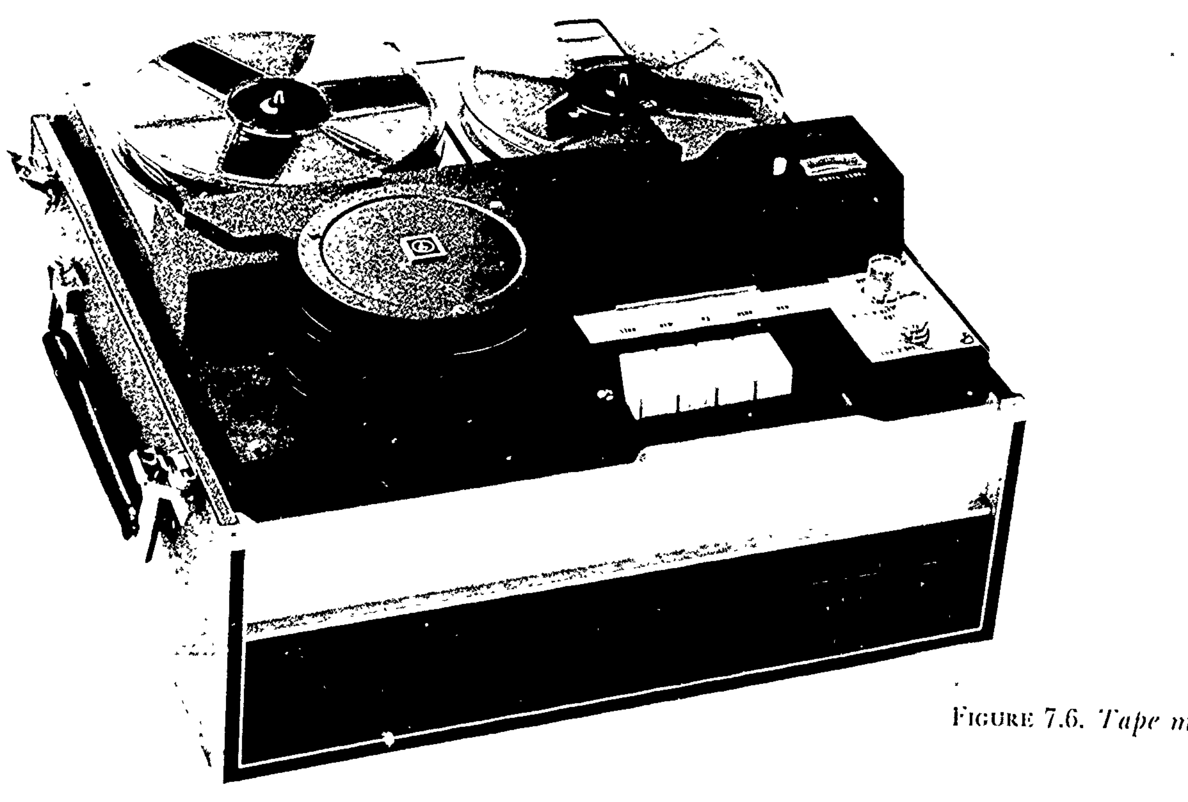


FIGURE 7.6. *Tape machine*

grams simultaneously. The Instructional Television Fixed Service (this is the term the FCC uses for the 2500-megacycle system, also known as the 2500-megahertz system) is still in its experimental stage (9), but it looks promising.

Often raised as an objection to instructional television is the contention that the prospective learner plays a passive role. The student has no opportunity to ask a question or participate in a discussion with the instructor. Students experienced at learning by television have, however, discovered that this difficulty can be overcome by taking careful notes and by jotting down questions as they come to mind. The questions can then be discussed in a follow-up session. As a matter of fact, some experimental studies have noted that a positive result of televised instruction is that students feel responsible for their own learning to a greater degree than otherwise. The skills of listening, concentrating, and taking notes are immeasurably sharpened.

### *Effective Utilization of ITV*

There is a distinction between *instructional television* (ITV) and *educational television* (ETV). The former refers to formal school or college instruction using television, whereas the latter term applies to all cultural educational programing. The discussion here is limited to ITV.

Although there still is certainly much to learn about ITV, it is possible to give some definite suggestions to the classroom teacher who wishes to utilize this medium.

The most important thing for the classroom teacher to keep in mind is that he is part of an instructional team for a television lesson. His duties as part of the team differ from those of the studio teacher, but they are just as important.

The classroom teacher must prepare himself and his students for the lesson. If there is a written guide that accompanies the program, he should familiarize himself with it and make use of the hints therein.

He should do everything possible to promote interest in the lesson. Studies have shown that an enthusiastic attitude on the part of the classroom teacher is extremely important. A negative or indifferent attitude will be quickly communicated to the students, and the televised lesson will not be effective.

If the lesson guide lists certain materials that students will need during the telecast, the teacher should make certain that the materials are available and that students have them organized before the lesson begins.

The classroom teacher should emphasize the importance of careful listening and viewing. He may want to give suggestions to the students regarding note-taking.

It is up to the classroom teacher to determine the effectiveness of the televised lesson for his particular class. During the telecast he should be using his intimate knowledge of his students to assess the degree to which they are comprehending various portions of the lesson. This will give him some idea as to which areas need reinforcing during the follow-up. Some carefully formulated questions after the telecast is over will shed still more light on the degree of comprehension. The classroom teacher then does what he can to reinforce, clarify, and expand the ideas presented by the television teacher. The follow-up session should be deemed an integral and important part of the total lesson.

One of the criticisms sometimes voiced about televised instruction is that it doesn't take individual differences into account. It is true that every student is presented the same material at the same rate of speed. But here, once again, is where the classroom teacher makes a contribution. It is primarily his responsibility to provide for the individual differences. The methods for doing so are not essentially different from the methods used in the conventional classroom situation. Some students, for example, are going to require a great deal of individual attention. It is conceivable that after a telecast the majority of students will be able to carry out assigned tasks independently while the teacher works with a few students on an individual basis.

### *Classroom Arrangement for ITV*

In arranging the classroom for a televised lesson it is important to keep in mind (as obvious as it seems) that television is an audiovisual tool. As with motion pictures, it is not possible to achieve maximum effectiveness unless every student can see and hear what is intended for him. Placement of the receiving set, seating arrangement, lighting, and acoustics must all be considered.

The maximum viewing distance is naturally determined by the size of the picture tube. According to a definitive study done for Educational Facilities Laboratories (3), students with normal vision can be placed at distances from the screen that range up to 12 times the actual width of the televised image (not the diagonal measure, which is the nominal size of a tube). This means that for a 23-inch set the maximum viewing distance is about 20 feet.

The minimum viewing distance is dependent upon the height at which the set is mounted, and this height depends to a certain extent on the seating arrangement to be used. The set should definitely be placed above eye level, but if viewers are to be placed one behind the other the set will have to be considerably higher in order for each student to have an unobstructed line of sight. If a staggered seating arrangement is used, it is not necessary to mount the set so high. Once the height of the screen is established, the minimum viewing distance should be determined by keeping in mind that the angle of elevation from the eye to the top edge of the screen should be no greater than about 30° (Figure 7.7).

There is also a horizontal viewing angle to consider (Figure 7.8). If the angle formed by the line of sight and the center axis of the picture tube is greater than 45°, the distortion of the image usually becomes objectionable. For the viewing of some kinds of materials the distortion becomes objectionable if this angle is much over 30°.

It should not be necessary to darken a room

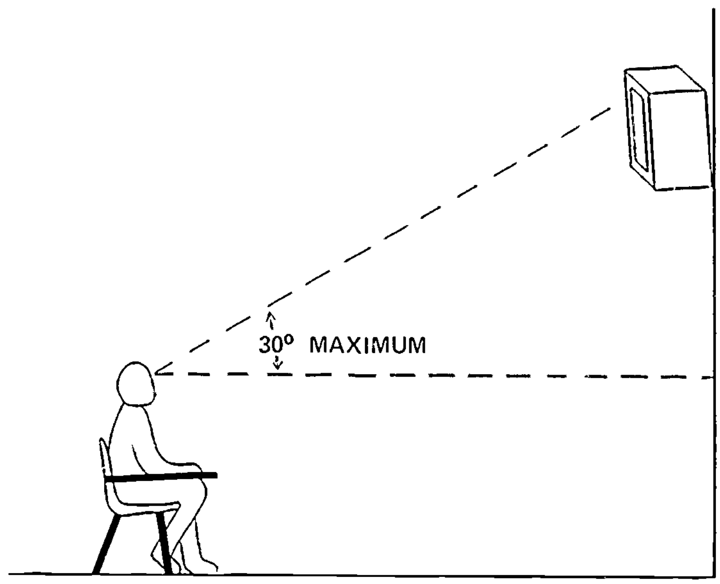



FIGURE 7.7. *Vertical viewing angle*

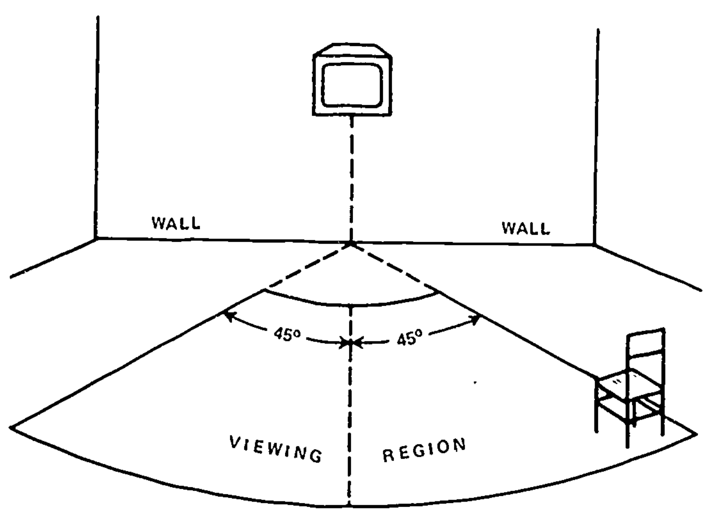


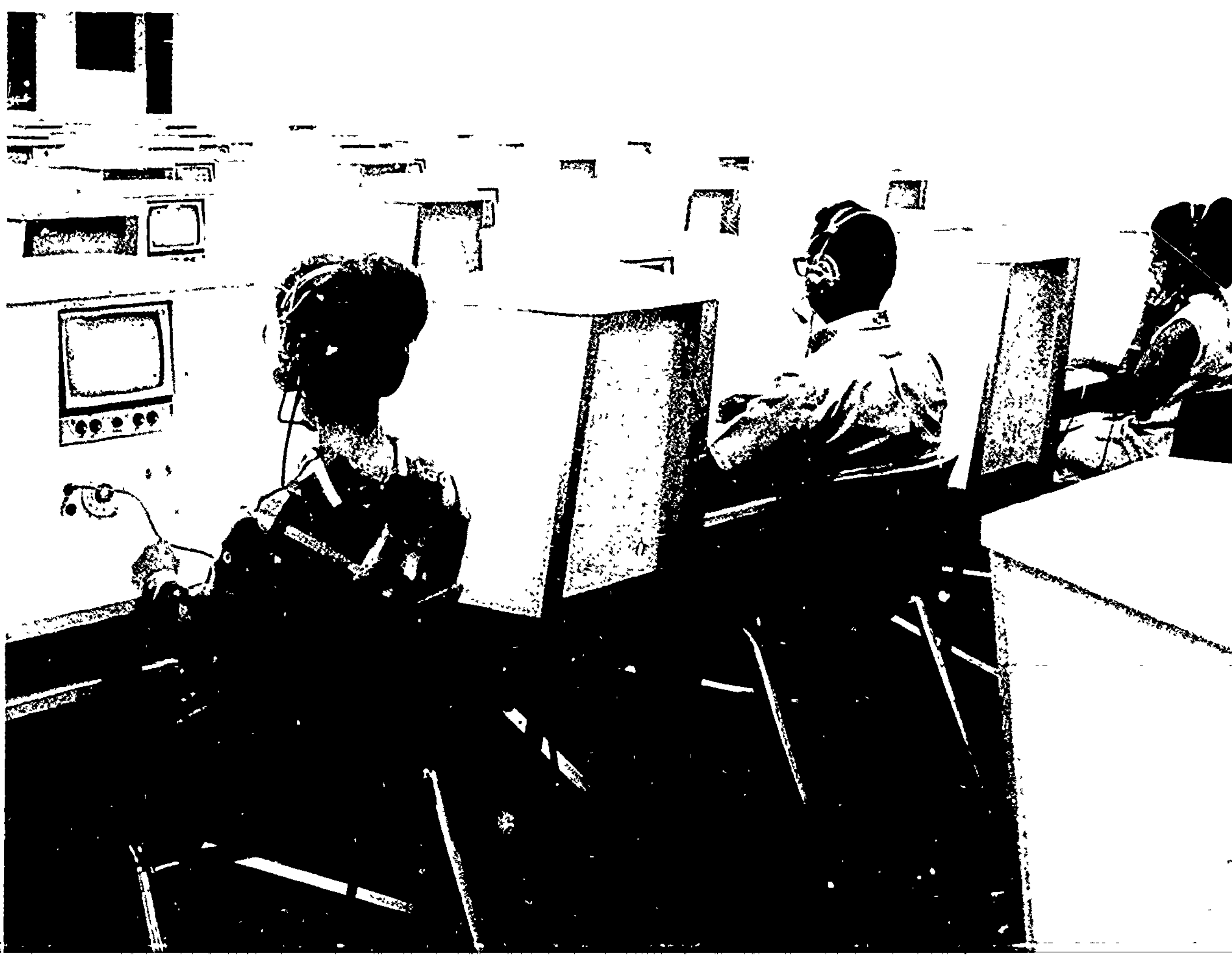

FIGURE 7.8. *Horizontal viewing angle*

during a telecast. At times the television teacher may want the viewers to do some pencil-and-paper work during the lesson. This and ordinary note-taking require adequate lighting. Glare on the television screen may be a problem, but this can usually be taken care of by proper placement of the set in the room. In the average classroom the best location for the set is probably in a corner on the side nearest the windows.

There is perhaps not much that can be done about acoustics in our older schools, although some teachers have noticed some improvement when the walls are covered by burlap or similar material. In planning new schools, thought needs to be given to the importance of good acoustics for televised instruction. Carpeting on the floors has proved to be advantageous and economically feasible in experiment after experiment.

The remarks in the preceding paragraphs pertain mainly to television instruction in a conventional classroom setting. Planners of educational facilities should consider carefully the opportunities for individualizing instruction which are afforded by a learning center such as that illustrated in Figure 7.9. A typical booth in this center is equipped with a small television screen, earphones, and a telephone-like dial. By dialing an assigned number a student gains access to a lesson recorded on videotape or film. Perhaps when centers such as this become commonplace—and creative programs become plentiful—television will finally begin to assume a vital role in education.

FIGURE 7.9. *Learning center at the University of South Florida, Tampa*



## OVERHEAD PROJECTION

The modern overhead projector has evolved from devices used by the military services in training programs during World War II. The early models had several disadvantages, the foremost being that they were large and cumbersome. Today's version is very compact yet rugged: it projects an image bright enough to be seen in a lighted room, and it is relatively inexpensive.

Mechanically speaking, an overhead projector is a simple device (Figure 7.10). It consists of a base and a projection head that is supported by a vertical post attached to the base. The base contains a light source and a horizontal projection stage. Light passes through the stage and is reflected onto a screen by a lens-mirror assembly in the head. Appearing on the screen is the image of whatever is on the stage. This might be a sheet of transparent plastic with written material on it (such a sheet is usually called a transparency), a transparent slide rule, or the like.

It has been noted that motion pictures and television have not been enthusiastically received by mathematics teachers. With the overhead projector the situation is just the opposite. It has won overwhelming acceptance in the past decade. Perhaps it is overused by some teachers—the author has been in classrooms where it seems that students are so accustomed to the overhead that it no longer commands their attention.

Some of the reasons for the popularity of overhead projection follow:

1. Mathematics teachers have always made extensive use of a chalkboard, and almost anything that can be displayed on a board can be shown better with an overhead projector. It is easier for most people to write on the horizontal stage of a projector than on a vertical board. Improved legibility and more accurate diagrams are the results.
2. When a teacher is presenting material on a chalkboard, he is turned away from the class much of the time. The teacher faces the class when using an overhead projector. He consequently has better control, and it is easier for him to gauge the receptivity of the class.
3. The fact that it is not necessary to darken the room means that students can take notes as the lesson is presented. It also means that attention can be shifted to a model, a chart, or other visual aid to be used in the lesson without changing the lighting in the room. (In connection with this, the teacher should remember an im-

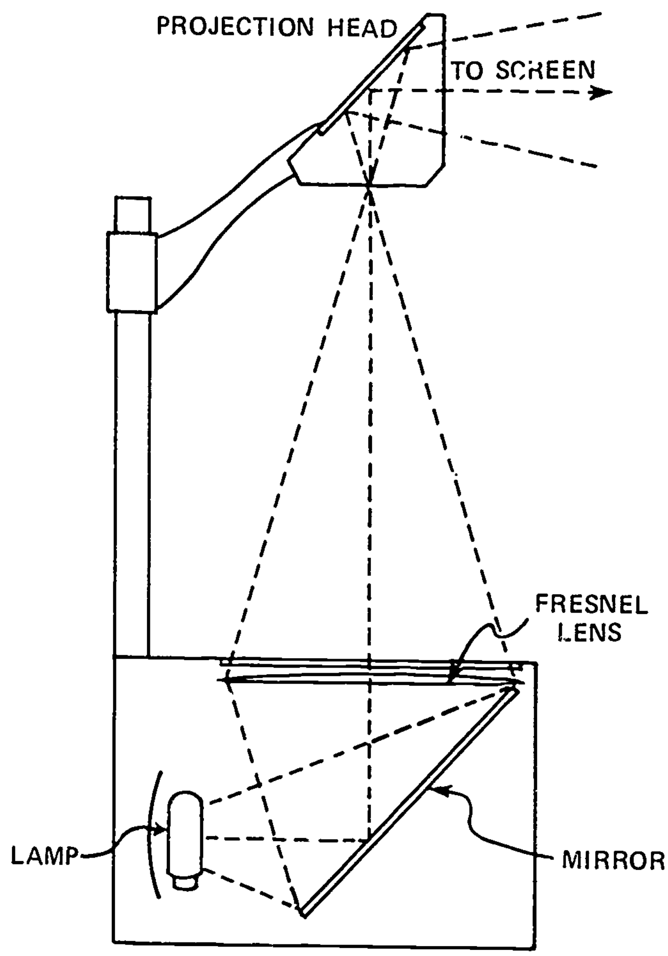



FIGURE 7.10
*Schematic diagram of typical overhead projector*

portant rule: When attention is to be shifted away from the screen, the projector lamp should be switched off.)

4. Materials can be prepared in advance and used repeatedly. The projector is certainly a great time-saver, since it enables a teacher to prepare a table, graph, diagram, or drawing of a complex mathematical figure and use it over and over.

## *Preparing Transparencies*

In the past few years, hundreds of commercially prepared transparencies have become available. They are being produced by book publishers, manufacturers of copying equipment, film makers, and many other companies. Several of the main sources of such transparencies are listed at the end of the chapter.

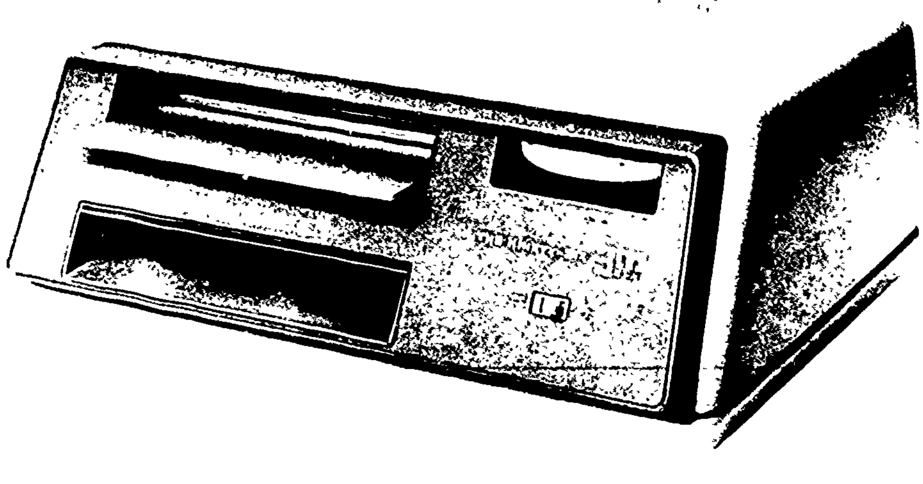
While some of the commercially prepared materials are excellent, many teachers find that the transparencies of most value to them are the ones they have made themselves. The simplest means of producing a transparency is to write or draw on a sheet of clear plastic with a grease pencil or a felt-tipped or nylon-tipped pen. This method should, however, be considered a temporary measure.

A much more professional-looking and more permanent transparency can be made with the aid of an infrared copying machine (Figure 7.11). The material to be shown is first drawn in pencil or ink on a piece of paper. A sheet of transparent film is placed in contact with this original, and both are fed into the copying machine. A transparency ready for projection is available in a matter of seconds.

For reproduction in an infrared copying machine, the original must be done in black ink or No. 2 pencil—the machine will not copy color. The original must be a sheet of paper that will pass through the machine. This means that it is not possible to copy from a bound volume with the infrared copier.

A photocopying machine is more versatile. It will produce a black and white transparency from a colored original and will also copy from a bound volume. But the photocopying process requires more time. It involves the production of a negative. Depending on the type of machine,

FIGURE 7.11
*Infrared copying machine*



*Courtesy of Minnesota Mining and Manufacturing Company*

the positive print is then developed either by heat or in a liquid solution.

Color can be added to transparencies in several ways. A simple but effective method is to apply any desired color with a felt-tipped pen. Another method utilizes pressure-sensitive colored transparent sheets. A teacher who has these sheets on hand can add color to a region of a transparency by cutting from a colored sheet a piece of the size and shape needed.

By using special film, an image in a single color on a clear background can be produced with an infrared copier. The special film comes in several different colors. It is naturally more expensive than film that produces a black image.

A transparency can also be produced in a single color through the so-called diazo process. This process requires special diazo film, an ultraviolet light source, and a development chamber containing ammonia fumes. Diazo film has a chemical coating that produces a brilliant color (diazo films are available in several different colors) on exposure to the ammonia fumes. The coating is, however, rendered inactive, and color will not form in the parts of the film which have first been exposed to ultraviolet light. To produce a colored transparency, a translucent or transparent "master" is produced first. This is then placed in contact with the diazo film. When the film is exposed to ultraviolet light, the opaque regions of the master protect the corresponding regions of the diazo film. These regions therefore become colored when the film is placed in the container of ammonia fumes, while the exposed regions remain clear. A complete description of the diazo process can be found in the booklet *The Overhead System: Production, Implementation, and Utilization* (25).

An illustration from a magazine can be transferred to a transparency through a color-lift process. As the term implies, in this process the ink is literally lifted from the magazine page and transferred to the film. This can be done only if the original was printed on a clay-coated paper. To determine if the paper is clay coated, one simply rubs a moistened finger on the margin of the page. If a white residue rubs off on the finger, the color can be lifted. (Most of the better-known magazines such as *Life* and *Look* are printed on clay-coated paper.) Of course, a transparency can be made only once from a single page. *The Overhead System* (25) gives full details on color lifting.

## *Illustrations of Transparencies*

A few examples will serve to illustrate the kinds of projectuals which the teacher can make for himself. One type that the author has found very useful is exemplified by Figures 7.12 and 7.13.

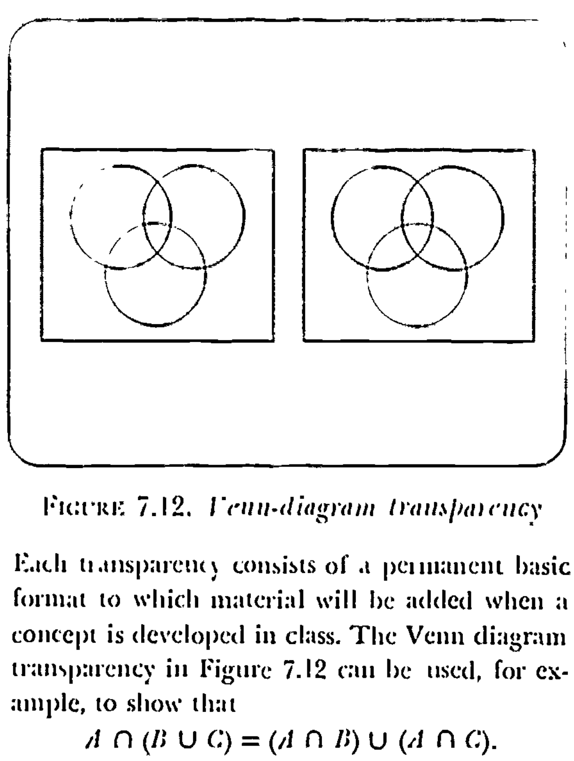


FIGURE 7.12. *Venn-diagram transparency*

Each transparency consists of a permanent basic format to which material will be added when a concept is developed in class. The Venn diagram transparency in Figure 7.12 can be used, for example, to show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

In step-by-step fashion, students can be shown that $A \cap (B \cup C)$ is represented by a certain region within the left-hand rectangle and that $(A \cap B) \cup (A \cap C)$ is represented by an identical region within the right-hand rectangle. The information added to the transparency in class is done with a grease pencil. It can easily be erased with a tissue or soft cloth, and the transparency

can immediately be used to show a different relationship.

A transparency like that shown in Figure 7.13 can be used in developing the addition and multiplication tables for a base-five numeration system or for a finite mathematical system. Rather than hastily sketching the table format each time it is needed, it makes sense to take the time to produce a permanent transparency.

The mathematics teacher can effectively utilize the technique of revealing a transparency a portion at a time. This works quite well, for example, in explaining the solution of an equation. All the steps in the complete process can

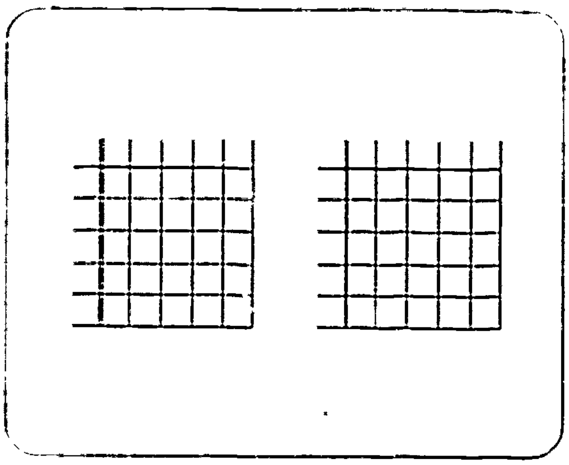


FIGURE 7.13
*Transparency for base-five numeration system*

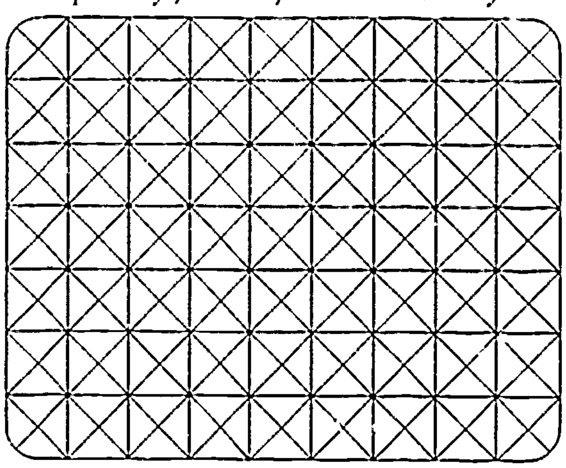


FIGURE 7.14
*Transparency for Pythagorean theorem*

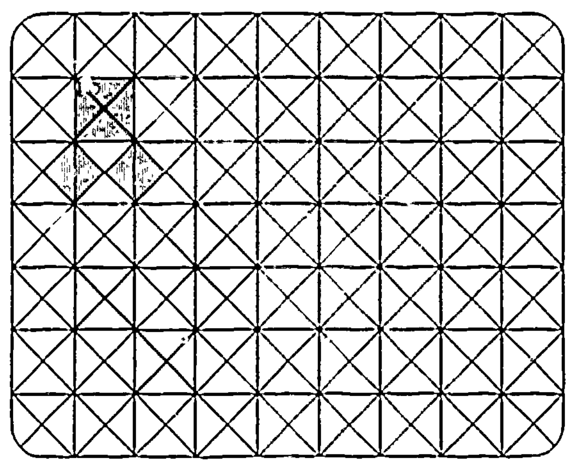


FIGURE 7.16
*Second overlay*

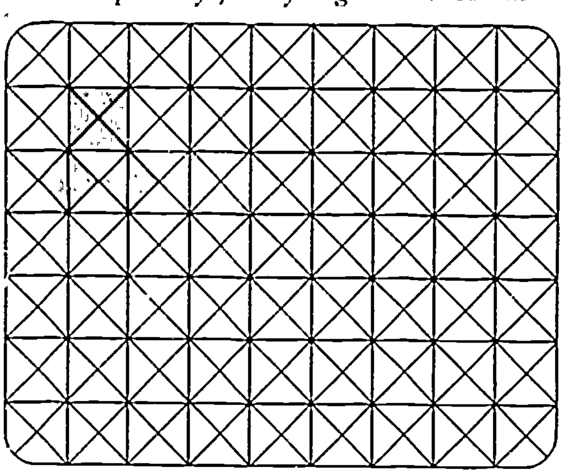


FIGURE 7.15
*Overlay for Pythagorean transparency*

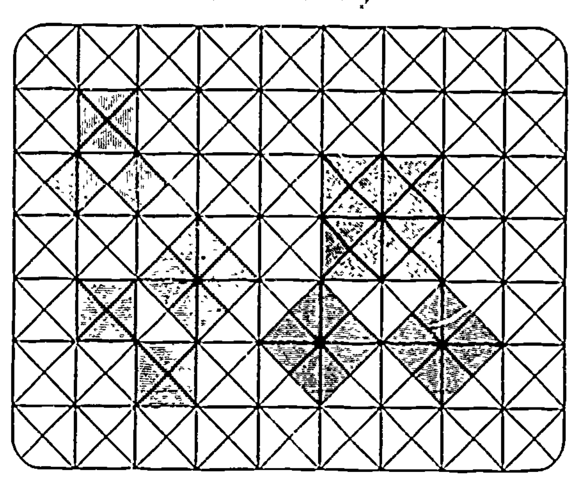


FIGURE 7.17
*Third overlay*

be printed on a permanent transparency. A sheet of paper (or anything opaque) can then be used to cover all but the original equation. As a teacher explains a step or as he elicits the explanation from the class, he reveals another line. This continues until the entire transparency is uncovered.

The revelation technique works as well in the presentation of a geometric proof. Some teachers also use it to administer an examination. By revealing one test item at a time, they set a pace for the student and make it likely that he will give some thought to each item.

While projectuals consisting of a single sheet are extremely useful, it is the overlay technique that has sold many mathematics teachers on overhead projection. By placing one or more transparencies on top of a first it is possible to develop a mathematical concept in stages.

The author uses the transparency shown in Figure 7.14 as the beginning phase of a discussion of a conjecture about the discovery of the Pythagorean theorem. Tiles have been found laid out in such a pattern in ruins in the Middle East. A second transparency is placed over the first to produce the shading shown in Figure 7.15. By counting triangular regions, it is easy to see that the area of the square on the hypotenuse of the isosceles right triangle is equal to the sum of the areas of the squares on the legs. When a third transparency is overlaid (Figure 7.16), attention is focused on a larger right triangle, and it can be seen that the same relationship among the squares exists. Another overlay fixes attention on a still larger triangle (Figure 7.17). The shadings on the three overlays are in three different colors.

Overlays can be used very effectively in developing graphical concepts. When working on the slope-intercept method of graphing, for example, the coordinate grid can be shown on a base transparency with a line of fixed slope on overlay. The overlay can then be moved up and down to show the effect of changing the $y$-intercept.

As another example, suppose one wishes to discuss the inverse of a function. Again the base transparency can contain the coordinate grid. The graph of a function is then drawn on an overlay that is attached to the base with tape hinges along the line $y = x$ (Figure 7.18). When the overlay is flipped over, the graph of the inverse appears.

Many variations of the above technique can be used. How does the graph of $y = |f(x)|$ compare to the graph of $y = f(x)$? Suppose that the graph of the latter is shown as in Figure 7.19. Since the absolute value of a number is never negative, the graph of $y = |f(x)|$ will not contain any points below the $x$-axis. A

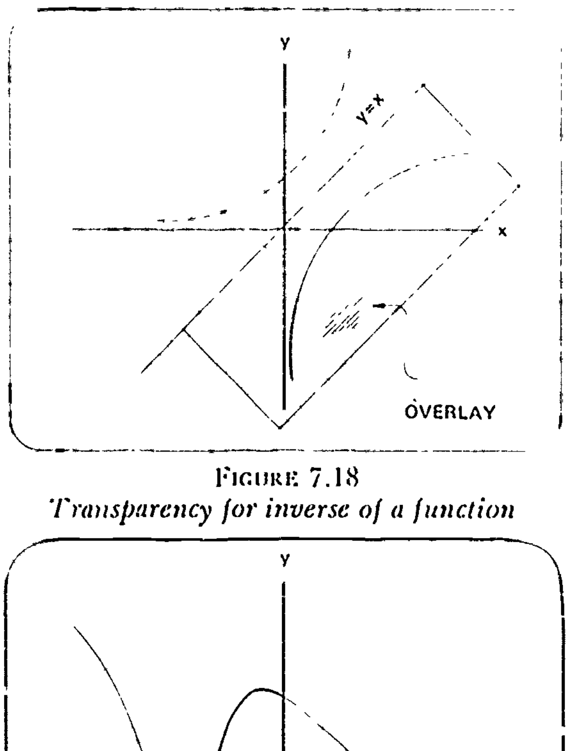



FIGURE 7.18
*Transparency for inverse of a function*

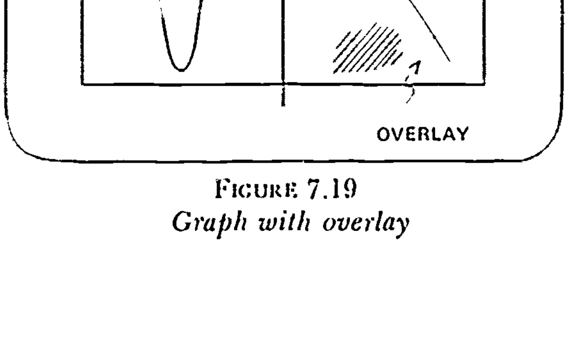



FIGURE 7.19
*Graph with overlay*

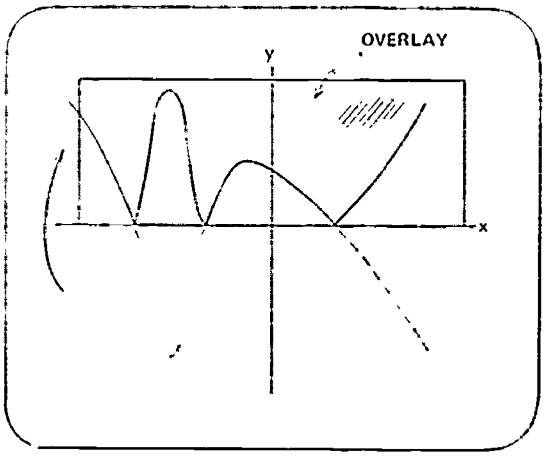



FIGURE 7.20
*Overlay flipped above x-axis*

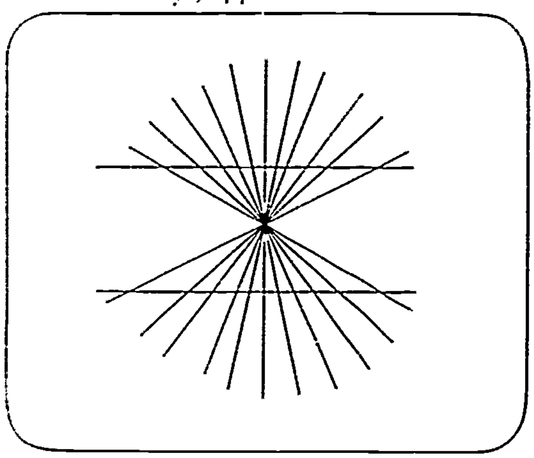


FIGURE 7.21
*Optical illusion*

moment's thought will convince one that the graph of $y = |f(x)|$ will be the same as the graph of $y = f(x)$ except that the part below the $x$-axis will be reflected above. This can be illustrated by hinging an overlay along the $x$-axis (Figure 7.20).

Figure 7.21 illustrates another way to use overlays. This well-known optical illusion can be projected with the two parallel lines on the base transparency and the family of intersecting lines on an overlay. When the overlay is removed, students see that the illusion of bending lines is destroyed. Optical illusions such as this sometimes help geometry students recognize the importance of not forming conclusions from the appearance of a figure.

For more examples of teacher-made transparencies, the reader is referred to excellent articles by Hansen, Osborne, and Unkrick (11; 18; 25) and to a booklet by Krulik and Kaufman (12). See also Chapter 4 of *The Teacher and Overhead Projection*, by Schultz (22).

If there is a general guideline for the teacher to follow in the design of a transparency, it is this: Keep it simple. Do not try to crowd too many diagrams or too much written material onto one transparency. The size of figures and lettering on a projectual is an important consideration. A teacher cannot expect to obtain a usable transparency by copying ordinary typewritten material—the letters are simply too small to be seen when projected in a typical classroom. Figure 7.22 shows the size of type that is

FIGURE 7.22

THIS IS THE SIZE OF TYPE THAT IS FOUND ON SPECIAL TYPEWRITERS FOR USE IN THE PRIMARY GRADES. IT IS RECOMMENDED FOR TRANSPARENCIES.

found on special typewriters designed for use by teachers in the primary grades. The image projected by this type is large enough to be seen by a person with normal vision at distances up to 40 feet from the screen. These typewriters are consequently highly recommended for the preparation of transparencies.

As indicated earlier, the overhead projector is not limited to the projection of material printed on transparent sheets (although these will undoubtedly be the main projectuals a teacher uses). Transparent rulers, protractors, slide rules, and other tools can be displayed very well with the overhead. Because of the size of the projected image, it is probably easier to explain the use of one of these tools by means of overhead projection than in any other way.

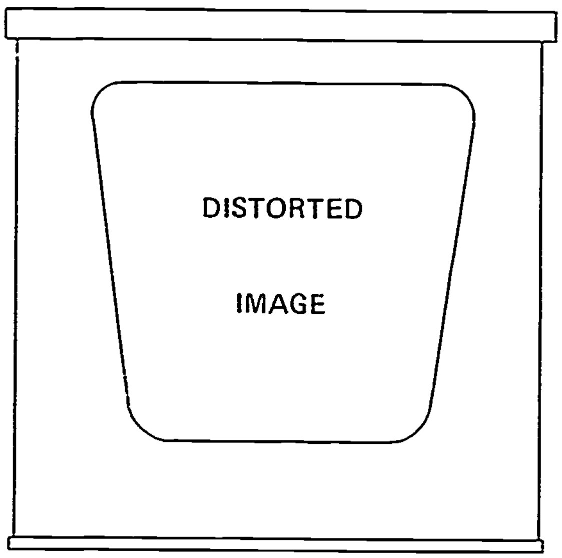



FIGURE 7.23. *Keystone effect*

### *Screen Placement for Overhead Projection*

The main consideration in placing the screen for overhead projection concerns the keystone effect. Keystoning is the image distortion (Figure 7.23) caused by the fact that one edge of the screen is farther from the projector than the opposite edge. The overhead projector is placed much closer to the screen than other kinds of projectors. This (combined with the fact that the screen must be high enough for students to see above the teacher) produces keystoning unless countermeasures are taken.

Keystoning is especially undesirable in mathematics because viewers may not be able to see what they are supposed to see. For example, a square will not project as a square unless keystoning is eliminated.

To avoid keystoning, the line determined by the center of the screen and the center of the projection head should be perpendicular to the screen. In most classrooms, the top of the screen will have to be tilted forward (Figure 7.24) in order to achieve this. It is probably best to have a screen permanently mounted in such a position. If a commercial screen cannot be so placed, a piece of hardboard will suffice. Many teachers have found that a piece of hardboard painted white will reflect a very satisfactory image.

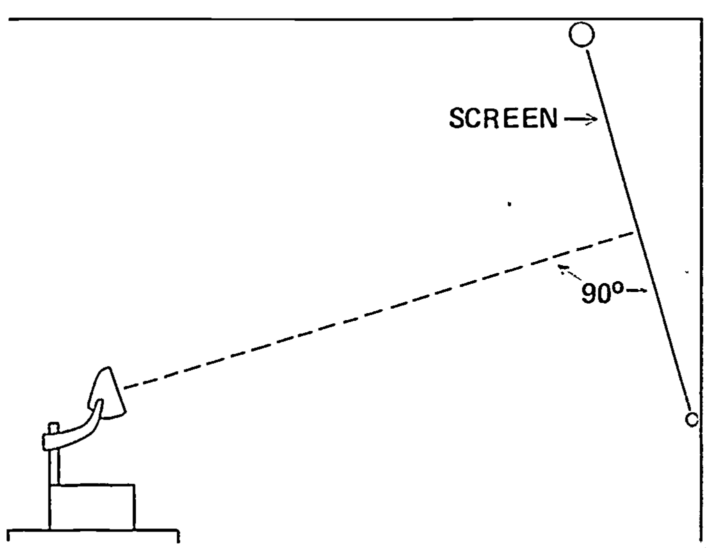



FIGURE 7.24
*Proper screen placement for overhead projector*

## OPAQUE PROJECTION

With the increasing popularity of the overhead projector, the opaque type seems to have lost some of its appeal. Tasks for which the opaque projector was usually utilized are now frequently performed with the aid of the overhead projector. Nevertheless, there are thousands of opaque projectors available in our schools, and teachers will continue to find uses for them.

An opaque projector is just what its name implies—a device for projecting an image of opaque material. As such, it projects light that is reflected from the material being displayed (Figure 7.25). Since the light is reflected, the projected image is not as brilliant as it is for a transparency projected on the overhead, and this means that the room must be darkened. The lighting efficiency of some of the newer models has, however, been improved to the extent that a moderately darkened room is adequate.

Other features of the newer opaque projectors are a down-draft cooling system that holds projected materials in place and a relatively light weight that makes for reasonable portability. The model shown in Figure 7.26 weighs 33 pounds.

The main advantage of an opaque projector is that materials can be shown with little before-

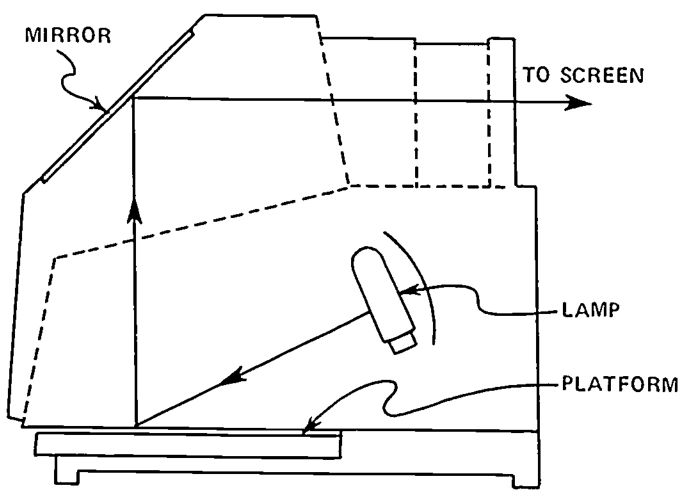
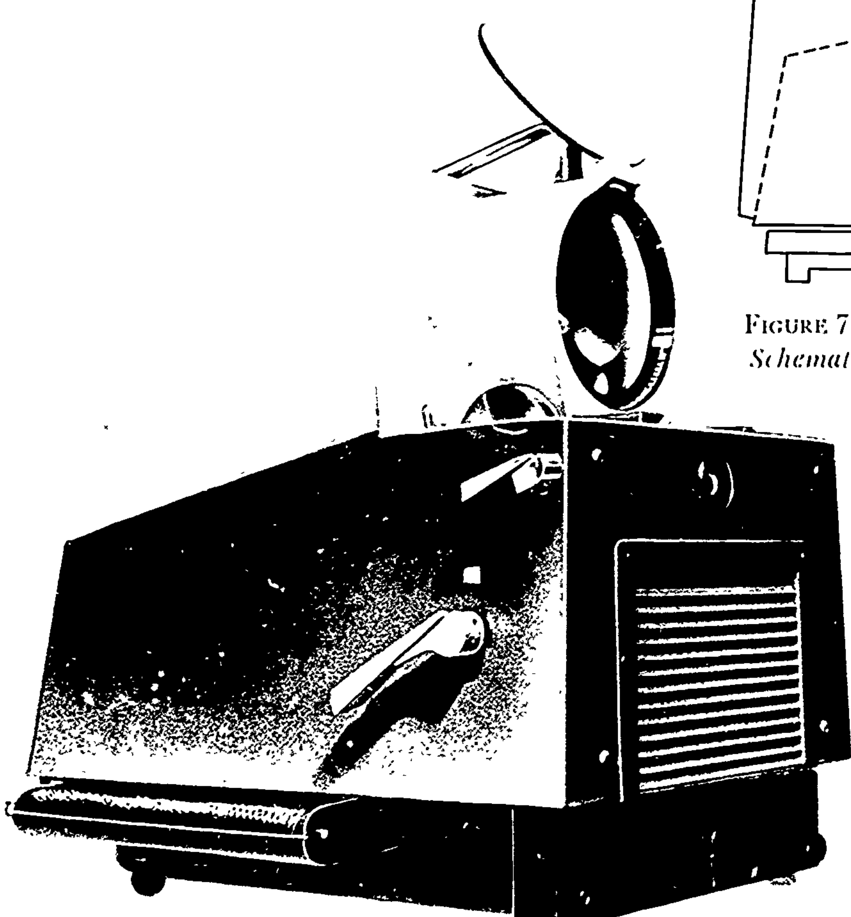



FIGURE 7.25
*Schematic diagram of typical opaque projector*



*Courtesy of Charles Beseler Company*

FIGURE 7.26
*Model of opaque projector*

hand preparation. If a teacher wishes to display a paper or a page from a book with an overhead, a transparency must be made first. But with an opaque projector any paper (or practically any object less than two inches thick) can be shown to the class almost immediately. And the projected image is in true color. Examples of pictures that come to the author's mind as he writes this are those found in the Life Science Library volume *Mathematics* and the ones in Heinrich Tietze's *Famous Problems of Mathematics*. Such exceptional colored illustrations can be transferred to a transparency through the color-lift process, but in so doing the original is of course destroyed. Hence, displaying them with an opaque projector is the sensible thing to do. The teacher will also find the opaque projector a handy device when he wishes to display a student's paper.

Still another use that some teachers make of the opaque projector is the enlarging and copying of intricate geometric figures. To copy a diagram, the image is projected on a chalkboard or poster board. It is then quickly and easily traced. For a discussion of this technique and other useful ideas, the teacher should refer to *The Opaque Projector* (24).

## SLIDES AND FILMSTRIPS

Slides and filmstrips can be treated together because of their similar characteristics. Commercially prepared slides come in two sizes, 3¼ by 4 inches and 2 by 2 inches, with the smaller size being by far the most popular. Teachers can and do make their own slides. An excellent reference for the teacher interested in trying his hand at making slides is *Production of 2 × 2 Inch Slides for School Use* (26).

A filmstrip is a collection of still pictures on a strip of 35-mm film. There are usually somewhere between 25 and 60 frames on a strip. Thousands of filmstrips are available, and perhaps more use is made of them in our schools than of any other kind of projected material. The *Index to 35-mm Educational Filmstrips* (17) lists nearly a thousand titles on mathematical topics.

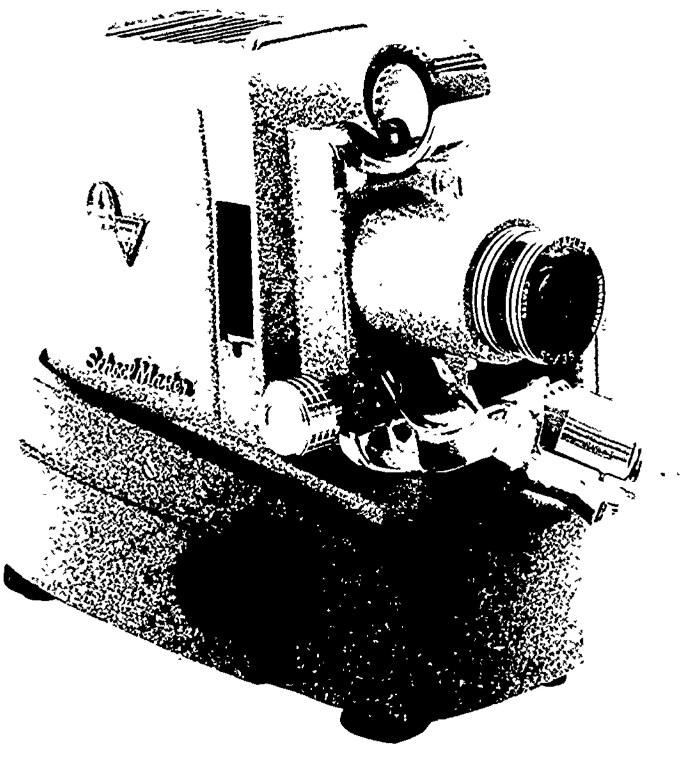
There are good reasons for the popularity of slides and filmstrips. They are relatively inexpensive, and because of their size they require little storage space. They thus can be purchased by schools and made readily available. The projection equipment is also low in cost, is easily portable, is simple to operate, and requires little maintenance.

Mathematics teachers have found filmstrips to their particular liking because so many topics in mathematics must by their very nature be discussed with the steps in a certain sequence. The proof of a geometrical theorem, the solution of an equation, the step-by-step process of the slope-intercept method of graphing, and the explanation of an algorithm all lend themselves well to presentation by means of filmstrips. Each frame can be viewed as long as necessary, the teacher can answer questions that arise, and the strip can, if desired, be backed up to a previous frame.

Many motion-picture producers put out filmstrips to accompany their motion pictures. In most cases the filmstrip consists of certain frames taken directly from the motion-picture film. A filmstrip of this sort provides a teacher with an excellent means of focusing attention on important points in the motion picture.

Projectors for slides and filmstrips are of several types. There are those that will project only filmstrips or only slides, but one of the most popular projectors is a combination type that will handle either filmstrips or 2 by 2 inch slides (Figure 7.27). One of the greatest stimulants to the use of slides has been the development of projectors that can be fitted with a cartridge holding a number of slides in proper order for projection (Figure 7.28). The cartridges also serve as storage boxes for the slides. A useful feature of some projectors is a remote-control attachment that enables the operator to change slides or advance the filmstrip from anywhere in the room.

Some filmstrips and slides are coordinated with



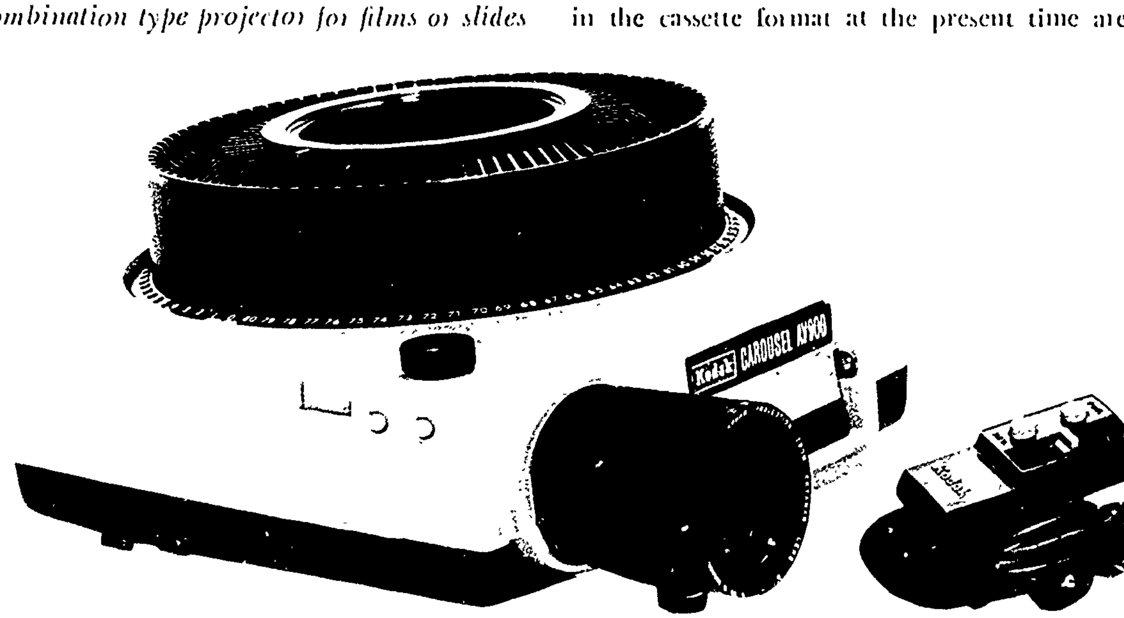

*Courtesy of Graflex, Incorporated*

FIGURE 7.27
*Combination type projector for films or slides*

sound recordings. Such combinations provide the teacher with another means of individualizing instruction. A recent development is a sound-slide system utilizing a slide holder that has a magnetic sound-disc encircling the slide (Figure 7.29). This system assures synchronization of sound and picture. Each disc will hold up to 35 seconds of sound. A teacher using the system can produce his own 35-mm slides and can do his own recording. New sound can be recorded over the old, and a slide can be replaced in the holder in a matter of seconds.

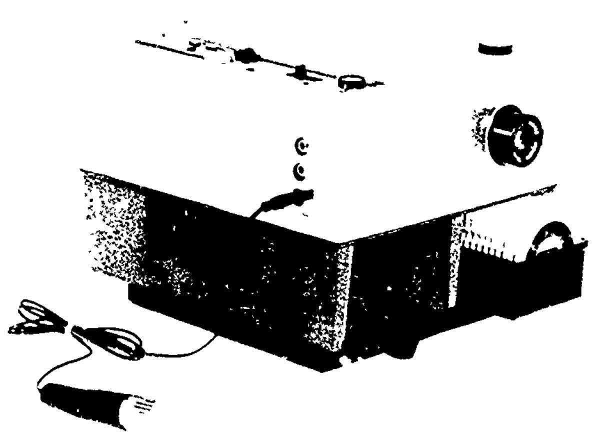
While talking about sound in connection with slides and filmstrips, we should mention a promising development that does not seem to fit into this yearbook anywhere else. This is the cassette audiotape system. The cassette system employs a reel-to-reel unit encased in a plastic container that measures about $2\frac{1}{2}$ by 4 inches. The big advantage of a cassette is the ease of use. All one has to do is snap one into place and press a button. Sound quality is good, and cassettes are so durable that they can be played thousands of times before wearing out.

Most of the mathematics audiotapes available in the cassette format at the present time are




*Courtesy of Eastman Kodak Company*

FIGURE 7.28. *Projector holding slides in order for projection*



*Courtesy of Minnesota Mining and Manufacturing Company*

FIGURE 7.29. *Sound-slide system*

designed to be used for drill and review. They cover such topics as addition, subtraction, multiplication, and division of whole numbers and rational numbers, problem solving, and elementary number theory. Students listen to the tapes and usually respond using pencil and paper. Some tapes are coordinated with study booklets. Teachers find that audiotapes help to reinforce and maintain skills and find them useful in diagnosing weaknesses. The tapes also present a way of providing for individual differences because a student can proceed at his own pace and listen to a single tape as many times as he wishes.

One other innovation that should be noted is the transparency cartridge projector (Figure 7.30). This combines some of the advantages of the overhead projector with those of a slide projector. It projects $3\frac{1}{4}$ by 4 inch slides that can be produced by the teacher in the same manner as transparencies for the overhead. As many as six slides can be made at one time from an 8 by $10\frac{1}{2}$ inch piece of transparency film. The slides are mounted in snap-together frames that interlock with one another to form cartridges of as many as forty frames. As with the overhead, the transparency cartridge projector can be used in a normally lighted room. Slides can be changed through a remote-control switch, so the teacher can remain at the front of the room.

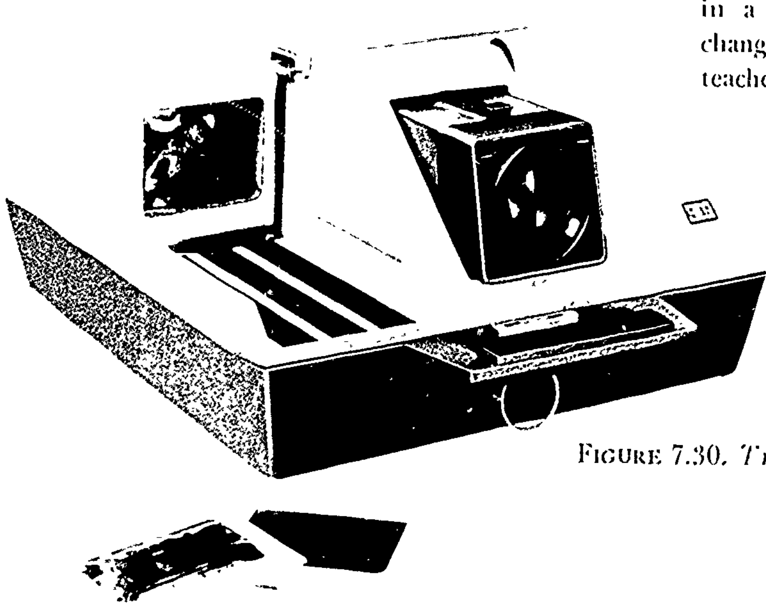


FIGURE 7.30. *Transparency cartridge projector*

*Courtesy of Minnesota Mining and Manufacturing Company*

## SOME MAJOR SOURCES OF PROJECTION EQUIPMENT AND MATERIALS

Addresses of producers and distributors are listed in the Appendix.

### MOTION PICTURE PROJECTORS

16-MM PROJECTORS

- Bell and Howell Co.
- Busch Film and Equipment Co.
- Eastman Kodak Co.
- Graflex, Inc.
- Harwald Co.
- Kalart Co., Inc.
- L-W Photo, Inc.
- Movie-Mite Corp.
- Paillard, Inc.
- RCA, Commercial Electronics Div.

8-MM CARTRIDGE PROJECTORS

- Eastman Kodak Co.
- Fairchild Camera and Instrument Corp.
- Technicolor, Commercial and Education Div.

### OVERHEAD PROJECTORS

- AO Instrument Co., Scientific Instrument Div.
- Applied Sciences, Inc.
- Bell and Howell Co.
- Charles Beseler Co.
- Buhl Optical Co.
- Graflex, Inc.
- Gregory Magnetic Industries, Inc.
- Keystone View Co.
- E. Leitz, Inc.
- Math-U-Matic, Inc.
- Minnesota Mining and Manufacturing Co., Visual Products Div.
- Ozalid Div., General Aniline and Film Corp.
- Projection Optics Co., Inc.
- Tecnifax Corp.
- H. Wilson Corp.

### OPAQUE PROJECTORS

- AO Instrument Co., Scientific Instrument Div.
- Bausch and Lomb Optical Co.
- Charles Beseler Co.
- Karl Heitz, Inc.
- Keystone View Co.
- Lacey-Luci Products, Inc.
- Projection Optics Co., Inc.
- Squibb-Taylor, Inc.

### SLIDE AND FILMSTRIP PROJECTORS

- Airequipt, Inc.
- AO Instrument Co., Scientific Instrument Div.
- Audio-Master Corp.
- Bell and Howell Co.
- Charles Beseler Co.
- Brumberger Co., Inc.
- DuKane Corp., Audio Visual Div.
- Eastman Kodak Co.
- Elco Corp.
- GAF Corp.
- Graflex, Inc.
- Honeywell, Inc., Photographic Products Div.
- E. Leitz, Inc.
- McClure Projectors, Inc.
- Mundus Co.
- Spindler and Sauppe
- Standard Projector and Equipment Co.
- Victor Animatograph Corp. (Div. of Kalart Co., Inc.)
- Viewlex, Inc.

### FILMS

*(There are many more distributors of educational motion pictures, but these appear to be the main sources of mathematics films.)*

16-MM MOTION PICTURES

- Association Films, Inc.
- Bailey-Film Associates
- University of California Extension Media Center
- Cenco Educational Aids
- John Colburn Associates, Inc.
- Coronet Films
- Encyclopaedia Britannica Educational Corp.
- Film Associates of California
- Indiana University Audio-Visual Center, Field Services Dept.
- International Film Bureau, Inc.
- State University of Iowa, Bureau of Audio-Visual Instruction
- Knowledge Builders
- McGraw-Hill Book Co., Text-Film Div.
- Modern Talking Picture Service, Inc., Modern Learning Aids Div.

Silver Burdett Co.
Universal Education and Visual Arts

8-MM CARTRIDGED FILMS

Audio-Visual Systems (Div. of Anglophoto. Ltd.)
Avis Films
Encyclopaedia Britannica Educational Corp.
Hester and Associates
Modern Learning Aids
National Film Board of Canada
National Instructional Films
Potter's Photographic Applications Co.
Universal Education and Visual Arts

FILMSTRIPS (35-MM)

Stanley Bowmar Co.
Cenco Educational Aids
John Colburn Associates, Inc.
Colonial Films, Inc.
Curriculum Materials Corp.
Curtis Audio Visual Materials
Educational Audio Visual, Inc.
Educational Projections, Inc.
Herbert M. Elkins Co.
Encyclopaedia Britannica Educational Corp.
Eye Gate House, Inc.
Filmstrip House
Jam Handy School Service, Inc.
McGraw-Hill Book Co., Text-Film Div.
Number Films
Popular Science Publishing Co., Inc., Audio-Visual Div.
Society for Visual Education, Inc.
Stipes Publishing Co.

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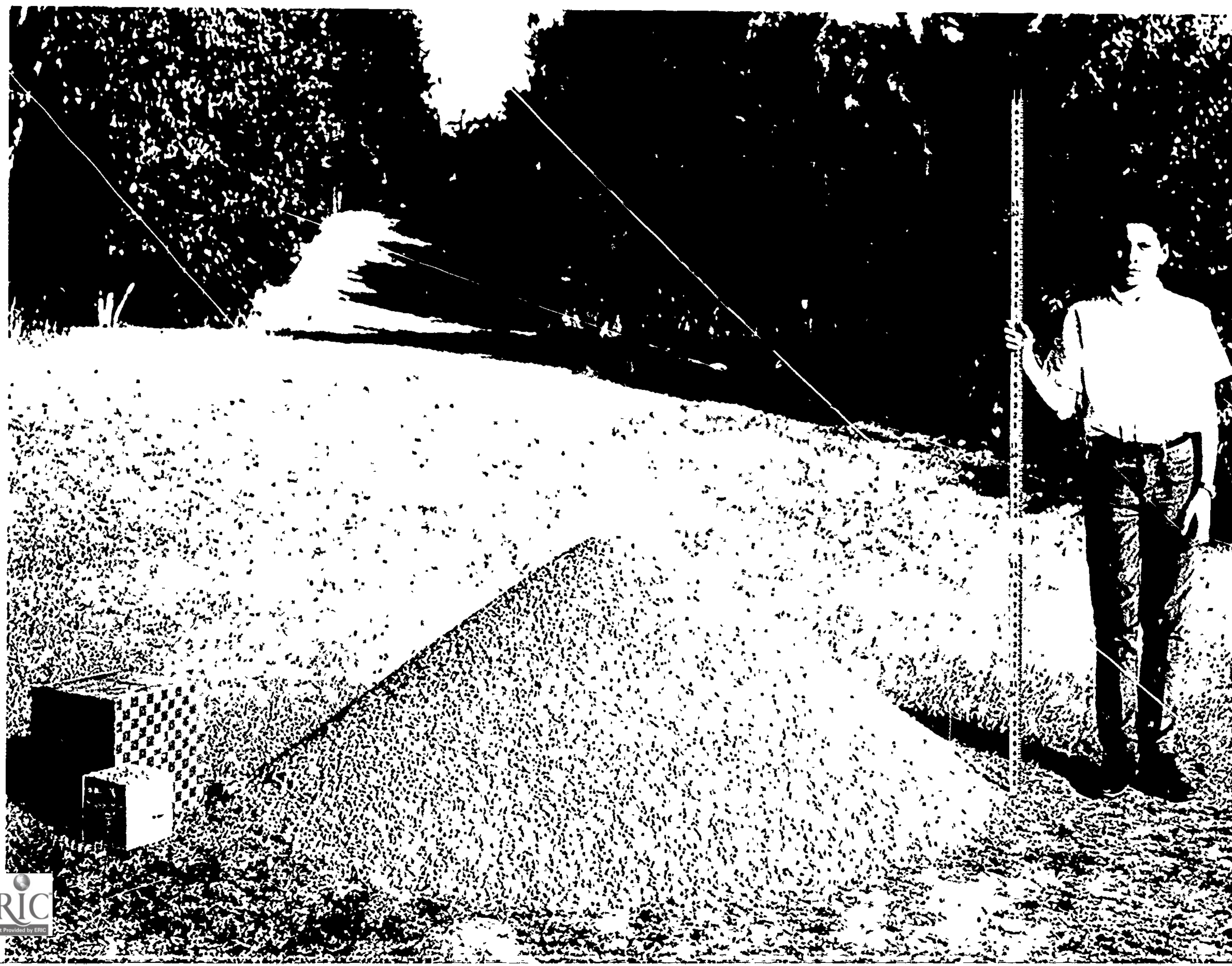
# 8. USING MODELS AS INSTRUCTIONAL AIDS

THE VOLUME OF THE GRAVEL PILE is approximately 1 cubic yard. Its shape is almost that of a right circular cone. The pile was formed by filling the small steel box (inside measurements 6 by 6 by 6 inches) shown at lower left 216 times and emptying the contents over the center of the base of the pile. Hence the volume of the pile should be 27 cubic feet, or 1 cubic yard. $[(216 \times 6 \times 6 \times 6) \div 1728 = 27.]$

The box with faces marked off in square inches has the shape of a cube 12 inches on edge. Its purpose in the picture is to help the reader make comparisons of size. The height of the boy is 60 inches. The "2" inside the target on the vertical ruler indicates that the height of the pile is 2 feet 2 inches. If the cone-shaped pile could have been topped (i.e., put in the form of a perfect cone), its height would be 28 inches. The radius of the circular base of the pile is 40 inches. If the very small error incurred by taking 28 inches as the height of the cone is ignored, the volume of the pile, computed by using the formula $V = \frac{1}{3}\pi r^2 h$, is 27.11 cubic feet, which is approximately 1 cubic yard. $[(\frac{1}{3} \times 3.14 \times 40 \times 40 \times 28) \div 1728 = 27.11.]$



CHAPTER 8

# USING MODELS AS INSTRUCTIONAL AIDS

*by*

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The principal concern of Chapter 8 is the role of models in the teaching and learning of mathematics. There are illustrations to show how models can be used to promote discovery, to enhance class discussion, and to encourage independent investigations by students. There are also illustrations to show how models can be used to generate interest in new topics, promote enjoyment of mathematics, build appreciation, improve retention, and teach applications. Suggestions are given for making effective use of models, promoting student production of models, building a school model collection, and selecting models for purchase.

# 8. USING MODELS AS INSTRUCTIONAL AIDS

Teachers and students of mathematics have used models as instructional aids since ancient times. Using models helps to enlarge the totality of sensations and to improve the quality of sensations received by the learner. These two phenomena contribute to improvement in perception, which is basic to improvement in learning.

Models may be made of paper, plastic, wood, metal, or other kinds of construction material. They may be in the form of three-dimensional objects; or they may be pictures, diagrams, or graphs.

The word *model* is used in two ways in this chapter. It is used in the usual sense to refer to concrete representations of mental constructs or ideas. For example, a geometric figure is a mental construct; and a model of the figure is a concrete representation of the mental construct. The word *model* is also used in a generic sense—that is, to refer to the various kinds of instructional materials listed below.

1. Demonstration devices (large concrete representations, including charts)
2. Manipulative devices
3. Computational devices
4. Instruments for measuring and drawing
5. Kits
6. Games and puzzles
7. Science apparatus
8. Realia (ordinary objects)

## THE ROLE OF MODELS IN THINKING

Models are constantly being used in thinking. Consider how scholars in different fields use models to solve problems. The geologist uses maps and charts to locate a new ore field. The chemist represents atoms with models made of little balls and connectors. The architect designs a building by using a scale drawing. The economist uses graphs to find the most efficient production process. And the psychologist uses a flow chart to display processes in learning theory.

Because much of mathematics is concerned with abstractions, there is a special need for models in mathematics. Mathematicians recognize this when they create new mathematics.

Euler's development of certain topological properties of geometric configurations is a good example of a mathematician's use of models in creating new mathematics. Euler sought a solu-

*Pythagoras making a model*

tion to the Königsberg bridge problem (Figure 8.1). The challenge in the problem was to find a pedestrian path that would cross each of the seven bridges exactly once.

Euler identified those features in the concrete situation that were essential to the problem. These he represented in a diagram called a network (Figure 8.2). In this network points rep-

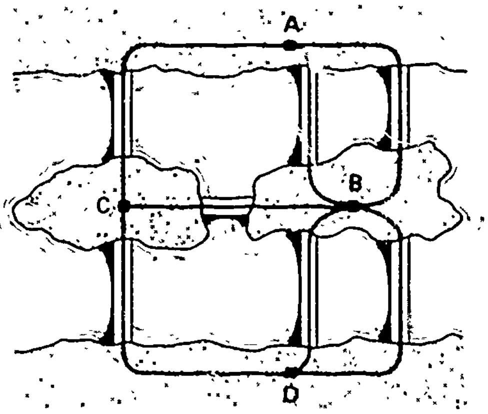



FIGURE 8.1

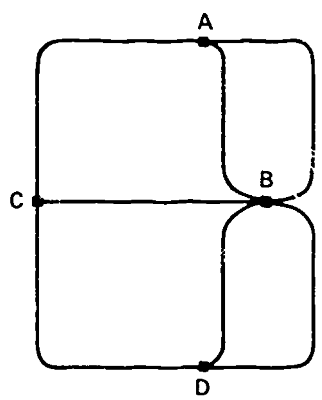



FIGURE 8.2

resent land and arcs represent bridges. By using this network as a model Euler discovered several generalizations. Later on he extended these generalizations to networks (involving regions, vertices, and arcs) that had no physical counterparts. In this way topology developed into a mathematical structure that was completely independent of bridges, networks, or applications to reality.

## THE ROLE OF MODELS IN THE TEACHING AND LEARNING OF MATHEMATICS

This section describes nine general ways in which models can be used by the teacher, by the learner, or by both. In each case the point of view presented is illustrated with examples of lesson techniques.

### *Models Provide a Setting for Discovery of Concepts*

Consider how students can discover Euler's formula for polyhedra. First, each student makes sketches of plane networks and records the number of vertices $V$, edges $E$, and regions $R$ for each network as shown in the table.

| Plane Network | Number of Vertices $V$ | Number of Edges $E$ | Number of Regions $R$ |
|---|---|---|---|
| A | 3 | 3 | 2 |
| B | 4 | 5 | 3 |
| C | 5 | 7 | 4 |
| D | 7 | 10 | 5 |
| E | 8 | 12 | 6 |

After discovering and informally proving that the formula $R + V - E = 2$ holds for plane networks the students are given several models of polyhedra (Figure 8.3) to examine, or they are asked to construct their own out of cardboard. This time the students record the number of vertices, edges, and faces (instead of regions). The formula $F + V - E = 2$, which is almost identical to the formula for plane networks, is found to apply in three-dimensional space. Students may be helped to prove the formula by studying the proof contained in the book *What Is Mathematics?* by Courant and Robbins (15).

When the students are satisfied that the formula $F + V - E = 2$ holds for all polyhedra, the teacher presents a new model—still a polyhedral model, but different from the models presented earlier in that it has a hole through it (Figure 8.4). Substitution of the number of faces, vertices, and edges of the new model in the

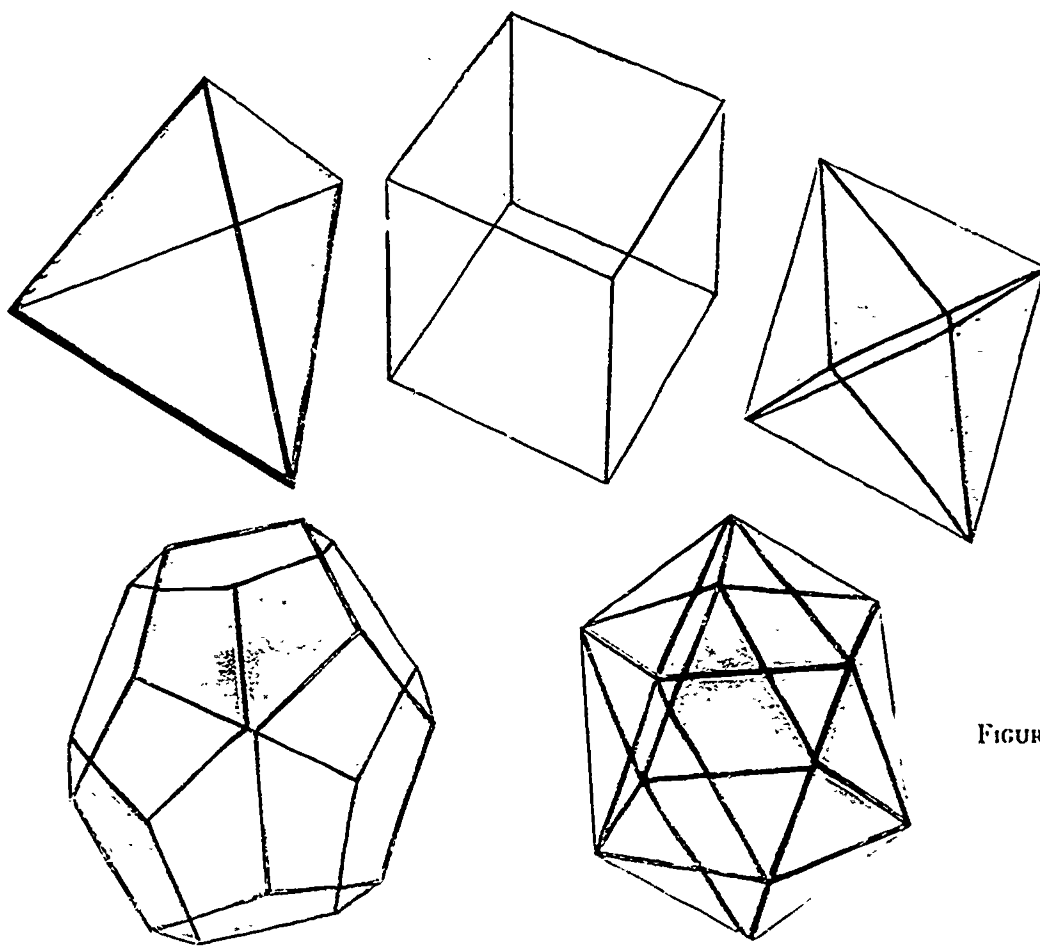


FIGURE 8.3

"proven" formula leads to a contradiction. A lively discussion ensues. Can the same statement be proved and disproved? An important aspect of the scientific method becomes evident. Students can (1) reject the "proven" formula, (2) restrict it to eliminate contradictory cases, or (3) extend it to include the new case. Eventually, the students recognize that the formula $F + V - E = 2$ is true when properly limited but is only a special case of the more general formula $F + V - E = 2 - 2H$, where $H$ is the number of holes in the polyhedron.

Successful discoveries prompted by manipulating or just contemplating physical models give students a sense of independence, a feeling of active participation, and the exhilaration of success.

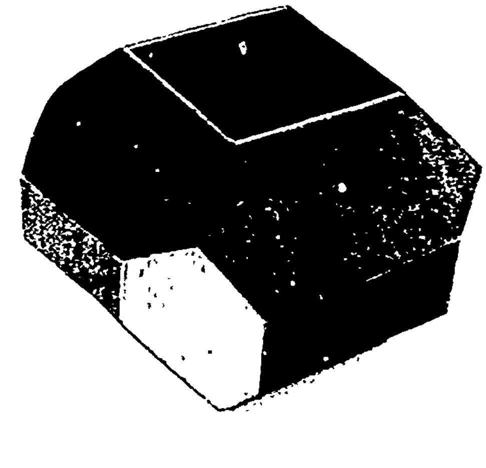


FIGURE 8.4

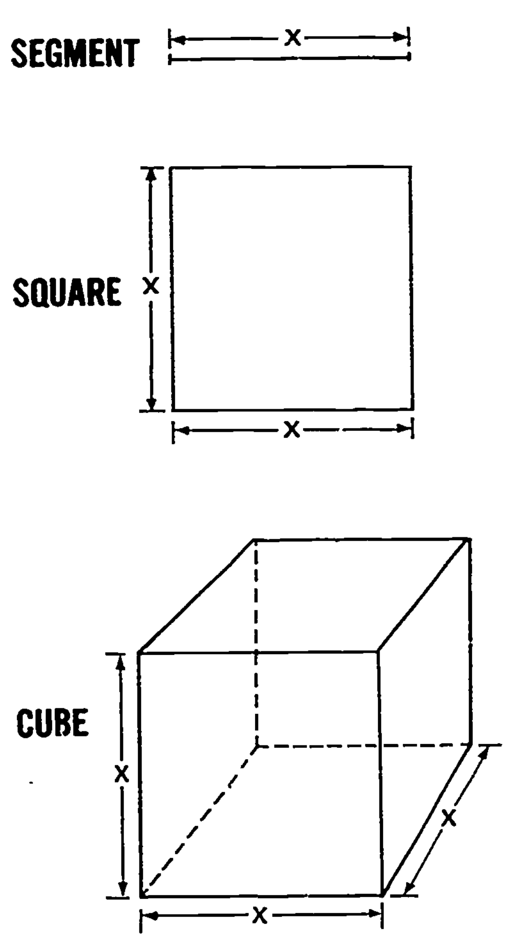
### *Models Can Be Used to Focus Attention on Ideas That Are under Discussion*

How models can be used to focus attention on ideas that are under discussion is perhaps best illustrated with a description of a lesson.

A hearty class discussion involving far-reaching ideas can be started by showing students a chart like the one displayed in Figure 8.5 and pursuing the following line of questioning:

1. What does $x$ represent in each figure?
2. What expression can be used to represent the area of the square?

FIGURE 8.5




3. What expression can be used to represent the volume of the cube?
4. How many dimensions does a line segment have? a square? a cube?
5. How many dimensions are suggested by each of the following algebraic expressions: $x$, $x^2$, $x^3$?

Using the chart helps students focus attention on the possibility of relating the measures listed below on the left with the algebraic expressions listed on the right.

| | |
|---|---|
| Length of a segment | $x$ |
| Area of a square | $x^2$ |
| Volume of a cube | $x^3$ |

Using the chart also helps students focus attention on developing a correspondence between the algebraic expressions $x$, $x^2$, and $x^3$ and familiar one-, two-, and three-dimensional figures. But let us proceed with the lesson and a few more questions.

6. How many dimensions does the algebraic expression $x^4$ suggest?
7. Can a geometric figure have more than three dimensions?
8. Does the *drawing* of the cube in the chart have three dimensions?
9. Is the *drawing* of the cube in the chart a two-dimensional figure?
10. Is the *drawing* of the cube in the chart a two-dimensional representation of a three-dimensional figure?

The last few questions should give the reader a hint of the reason for starting this lesson with a chart rather than with three-dimensional models.

At this point in the discussion, the class should be ready to consider the question: "Who can explain how to build a three-dimensional model of a four-dimensional figure?" If the question draws a blank, then the time is at hand for the teacher to unveil a three-dimensional model of a tesseract (Figure 8.6). A tesseract is also referred to as a fourth-dimensional analog of a cube or as a hypercube.

Representing a tesseract with a three-dimensional model is analogous to representing a cube

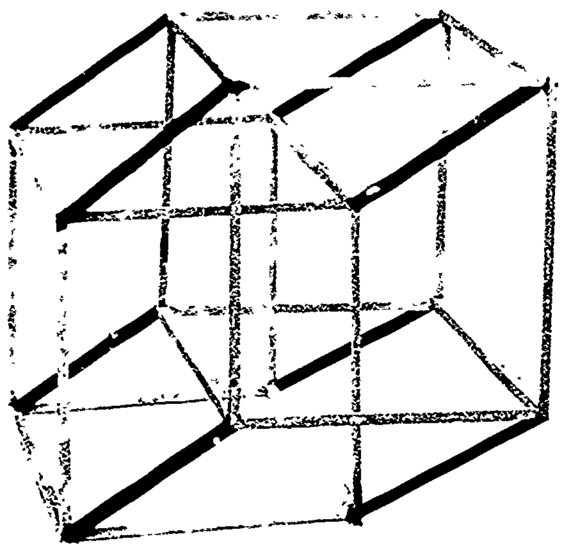


FIGURE 8.6

with a two-dimensional drawing. One way of representing a tesseract in three dimensions is by joining corresponding vertices of an upper cube and a lower cube with parallel oblique rods as shown in Figure 8.6.

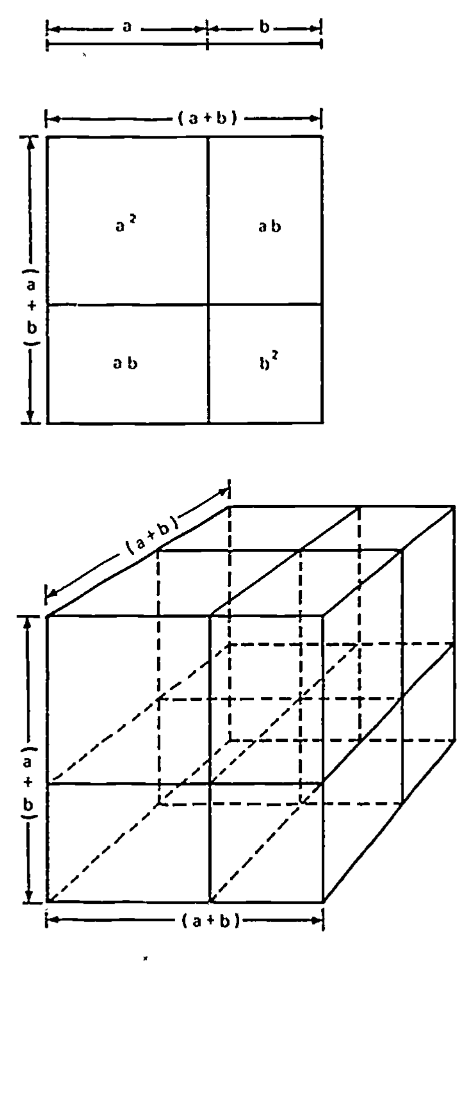
Once the imaginations of students make the leap to the idea of four-dimensional space, some eager student can usually be counted on to raise the question of how to give concrete representation to a figure in five-dimensional space. To keep the imaginations of students in orbit, students may need to have a clue to help them with the new problem. A useful clue can be given by having students focus attention on the key to the solution of the problem—that is, by asking them to count the edges joined to one vertex of the tesseract and to do the same thing for each figure displayed in the chart. Consideration of the abstract idea of $n$-dimensional space is only a step away. Using models to help students focus attention on ideas that are under discussion can lead to some very sophisticated mathematics.

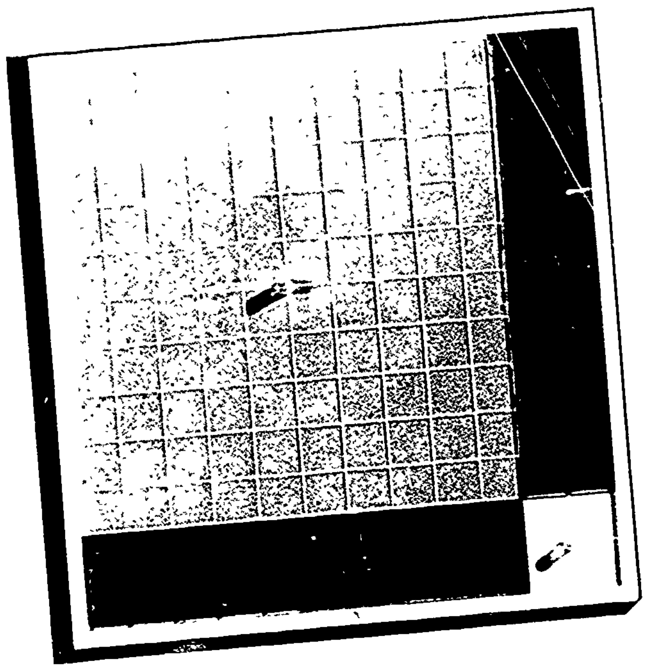
As a sequel to the lesson outlined above, class discussion involving demonstrations of special cases of the binomial theorem might be anticipated. Such demonstrations are really proofs of the special cases. The technique recommended is the same as described above—class discussion motivated by questions that refer to figures represented by diagrams or three-dimensional models.

The first diagram in Figure 8.7 shows a segment of length $a$ joined to a segment of length $b$ to form a segment of length $a + b$. Certainly, $(a + b)^1 = a + b$.

In the second diagram in Figure 8.7 each side of a square is composed of two segments whose combined length is $a + b$. The diagram

FIGURE 8.7





*Courtesy of LaPine Scientific Company*

FIGURE 8.8

shows how a square may be separated into two smaller squares with areas $a^2$ and $b^2$, respectively, and two rectangles, each with area $ab$. Thus, the diagram shows that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

The dissectible model shown in Figure 8.8 can also be used for this demonstration.

The bottom diagram in Figure 8.7 shows a cube with each edge composed of two segments that have a combined length of $a + b$. More importantly, the diagram shows how a cube with edge $a + b$ can be separated into the eight three-dimensional figures described below.

1 cube with volume $a^3$

3 rectangular prisms, each with volume $a^2b$

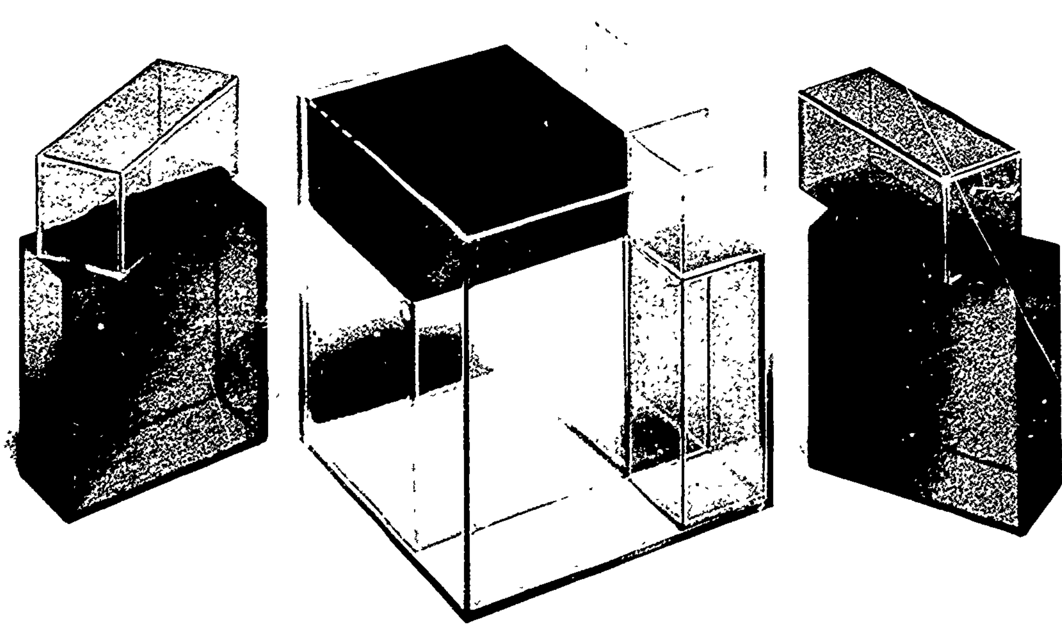
3 rectangular prisms, each with volume $ab^2$

1 cube with volume $b^3$

Therefore, the diagram shows that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

FIGURE 8.9



*Courtesy of LaPine Scientific Company*

For some students a three-dimensional dissectible model is more helpful with this demonstration than a diagram. Figure 8.9 pictures a dissectible plastic model that is available commercially. Students can also make a model like this. (See Figure 10.20 in Chapter 10.)

Figure 8.10 shows a tesseract with each edge separated into two segments having lengths $a$ and $b$, respectively. The entire tesseract is separated into the fourteen subspaces described below.

1 tesseract associated with $a^4$

4 hyperprisms, each associated with $a^3b$

6 hyperprisms, each associated with $a^2b^2$

4 hyperprisms, each associated with $ab^3$

1 tesseract associated with $b^4$

Thus, the tesseract in Figure 8.10 can be used to show that

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Focusing attention on ideas that are under discussion is one of the most important functions of models. The examples described thus far illustrate uses that have been carefully planned. However, this particular function of models sometimes operates dramatically even without preplanning. Here is an account of an episode that actually happened.

A geometry class was asked to prove that the volume of a triangular pyramid equals one-third the product of its base and altitude. The class had available a chalkboard diagram similar to the one in Figure 8.11. The usual way to do this proof is to start with the triangular pyramid $A\text{-}BDF$, construct a prism $ACE\text{-}BDF$ on the base of the pyramid, and then show that planes $ACF$ and $ADF$ dissect the prism into three pyramids with equal volumes. The crux of the proof is to show that there are two pairs of equivalent pyramids. (The word *equivalent* is used in the same sense as in Cavalieri's theorem.) A few students in the class were able to do this proof with the aid of the chalkboard diagram, but most of them were unable to identify and keep track of the pairs of equivalent pyramids that enter into the proof. No amount of talk by the

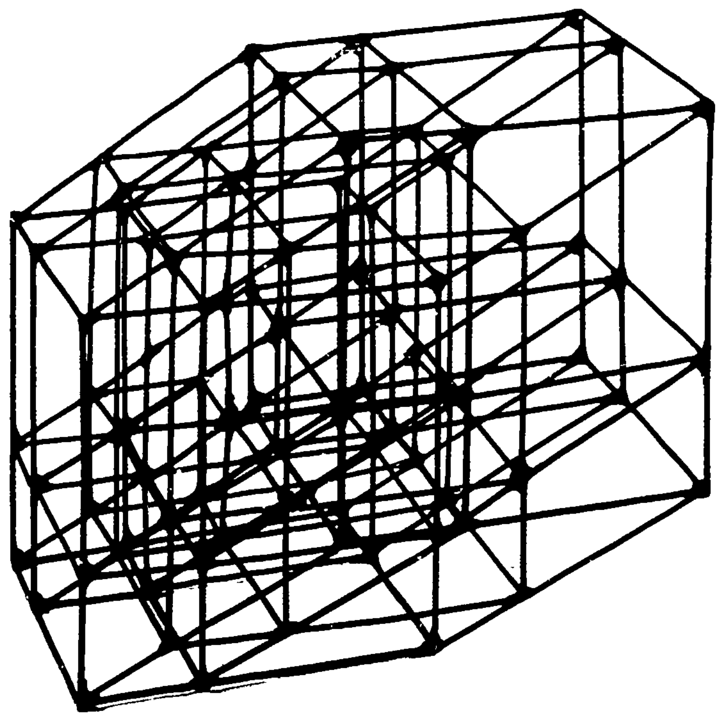


FIGURE 8.10

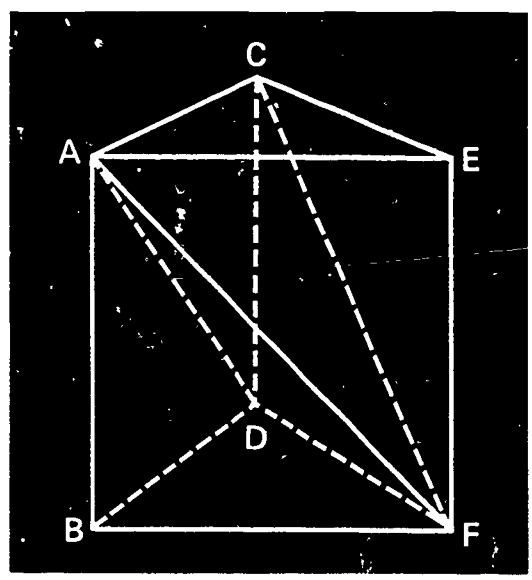


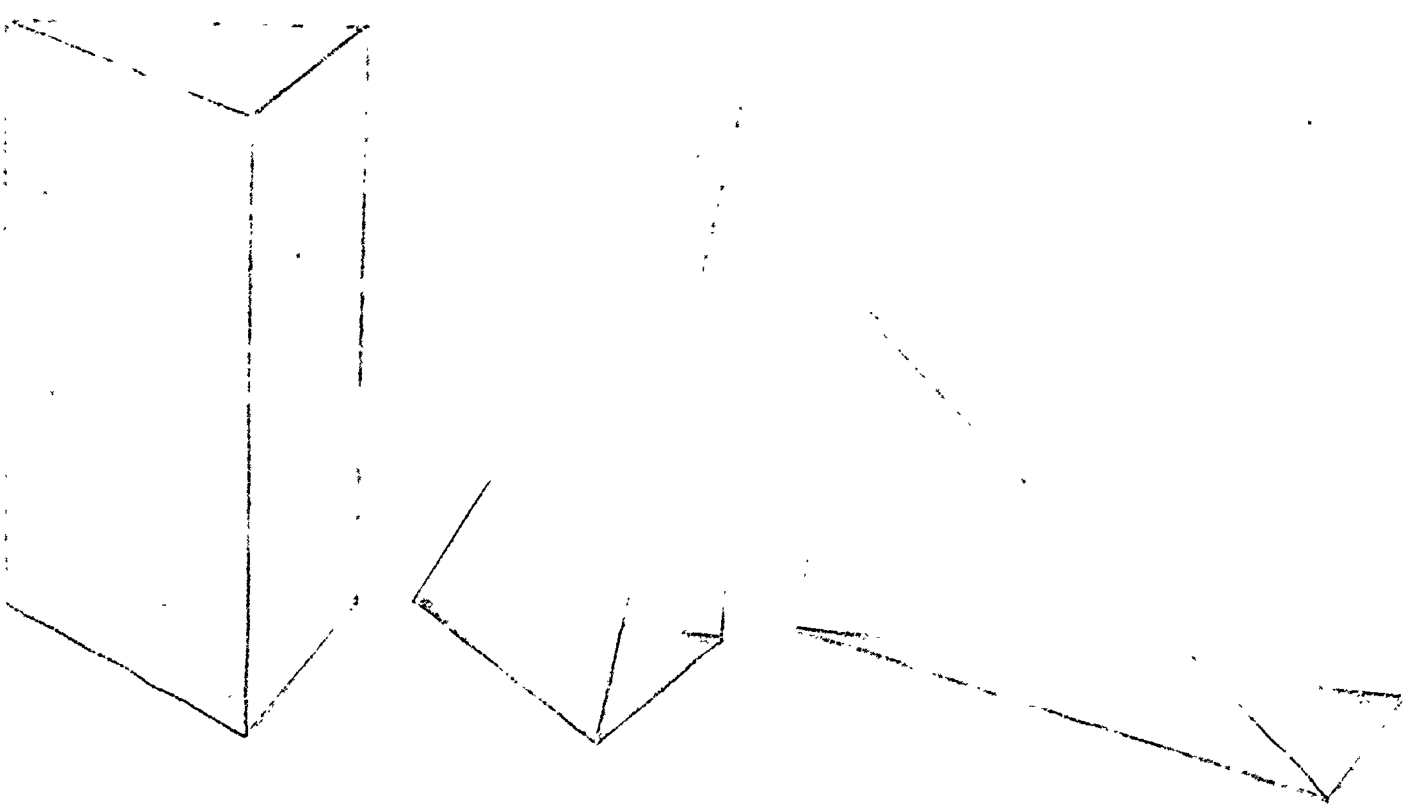

FIGURE 8.11

teacher helped, either. However, when the teacher introduced a three-dimensional dissectible model of a triangular prism (Figure 8.12), the proof came into focus for most students in class.

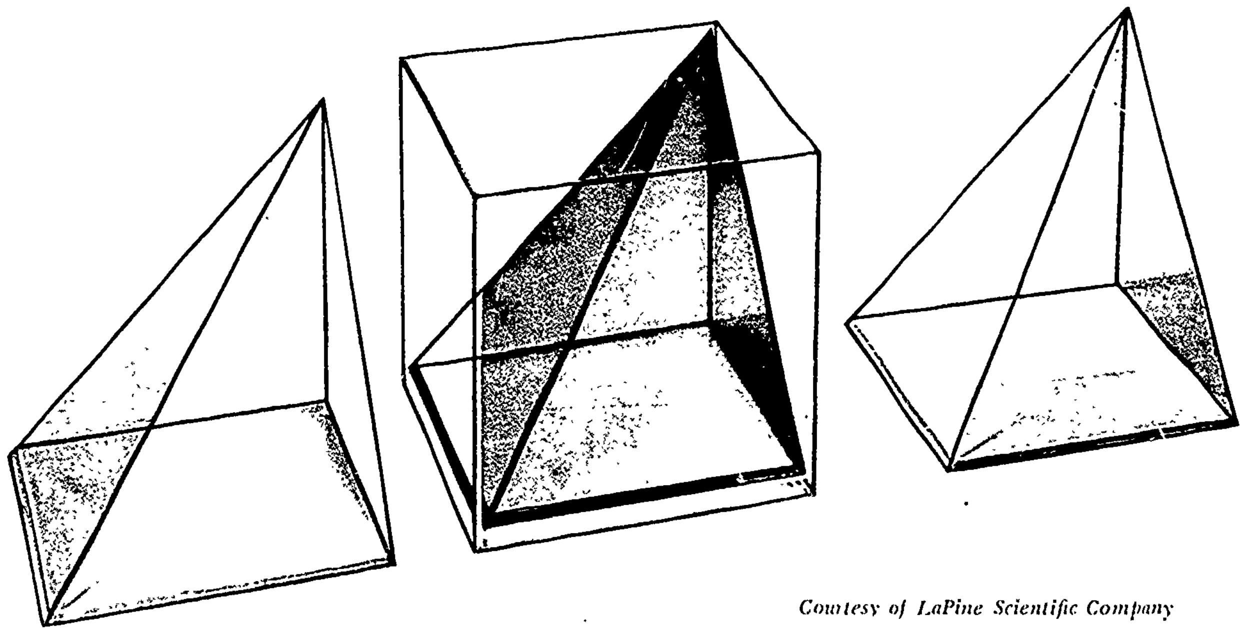
One reason students had difficulty with this proof is that the chalkboard diagram put a premium on selective attention. There was simply too much in the diagram to keep track of. Using a three-dimensional dissectible model enabled students to focus attention on fewer relationships at a time.

Another reason students had difficulty with this proof is that the usual dissection of a triangular prism yields three pyramids that are not congruent. This last difficulty cannot, of course, be completely removed, even by using a concrete dissectible model of a triangular prism. The difficulty is inherent in the problem. So it is suggested that when students are asked to consider the proof about which we have been talking they be given an opportunity to examine a three-dimensional model of a cube that is dissected into three *congruent* pyramids (Figure 8.13). The proof that the volume of each of these pyramids equals one-third the product of its base and altitude is both easy and convincing.

FIGURE 8.12



*Courtesy of LaPine Scientific Company*



*Courtesy of LaPine Scientific Company*

FIGURE 8.13. *The models of the three congruent pyramids can be put together to form a model of a cube that just fits inside the cubical-shaped box.*

### *Models Provide a Means for Making Independent Investigations*

Students can find a rational approximation to the value of $\pi$ by comparing the circumference and diameter of each of several circular objects such as a tin can, a plate, and a hoop. One way of finding the circumference of a circular object is to roll it like a wheel for one complete turn and then measure the distance traveled. Another way is to measure the circumference directly by wrapping a tape measure around the object.

In carrying out investigations, students should record in a table the diameter $D$ and circumference $C$ for each object and then compute the difference $C - D$, the product $C \times D$, and the quotient $C \div D$ for each object. By comparing the results of these computations, students should be able to see that the quotient $C \div D$ is constant for all circles and is close to 3.14. (See the table.)

The fact that $\pi$ is approximately equal to 3.14 can be verified by computing the perimeter of a many-sided regular polygon inscribed in a circle of unit radius. Archimedes found the value of $\pi$ to be between $3\frac{10}{70}$ and $3\frac{10}{71}$ by computing perimeters of 96-sided regular polygons inscribed in and circumscribed about a circle with unit radius.

| Object | Diameter $D$ in Inches | Circumference $C$ in Inches | $C - D$ | $C \times D$ | $C \div D$ |
|---|---|---|---|---|---|
| Tin Can | 3 | 9.4 | 6.4 | 28.2 | 3.13 |
| Hoop | 26 | 82 | 56 | 2132 | 3.15 |
| Plate | 10 | 31 | 21 | 310 | 3.10 |

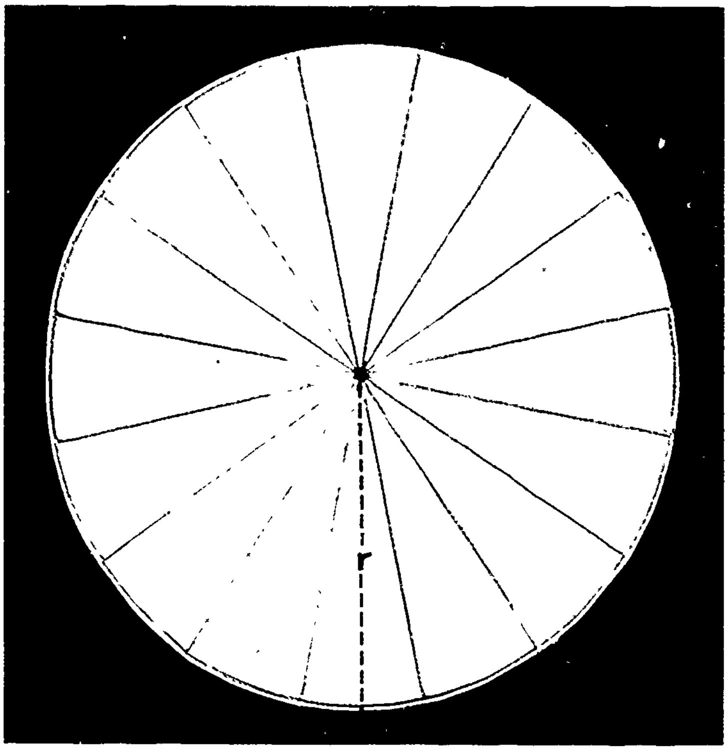
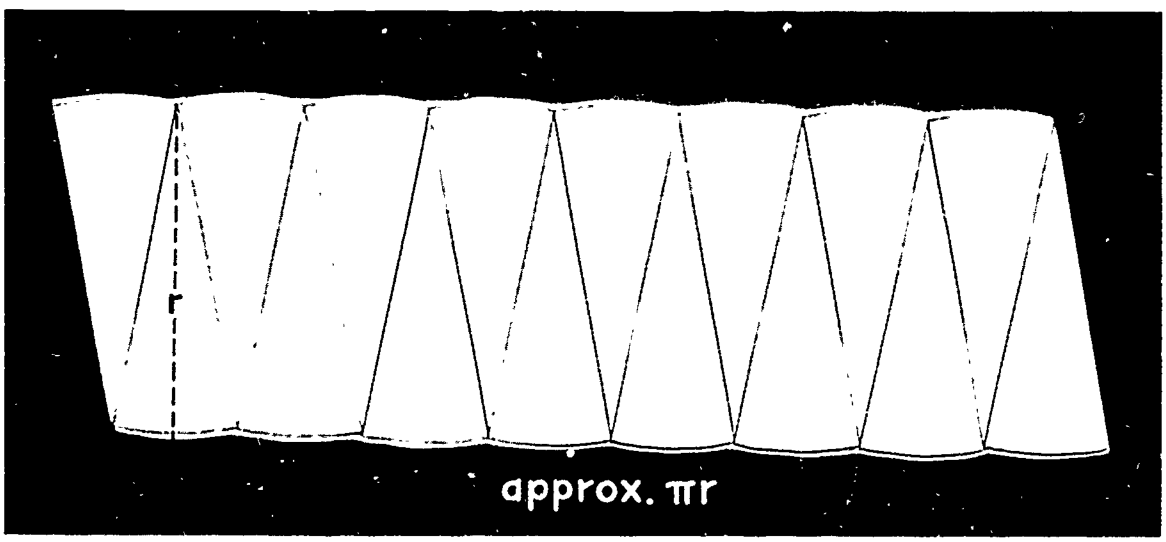



FIGURE 8.14 *Courtesy of Yoder Instruments*

A formula for the area of a circle can be determined experimentally by the following procedure. Cut a circular disc of radius $r$ into an even number (16 or more) of congruent circular sectors (Figure 8.14). Fit the sectors together as shown in Figure 8.15. The shape of the resulting figure is somewhat like a parallelogram. The sum of the lengths of the arcs along the bottom (or along the top) equals half the circumference of the circular disc. Since the radius of each sector is $r$, the height of the figure is approximately $r$ units. Therefore, an estimate of the area of the figure shaped like a parallelogram is one-half the circumference of the disc times the radius or $\pi r^2$. Since this figure has the same area as the circular disc, it is reasonable to accept $A = \pi r^2$ as a formula for the area of a circle.

A formula for the surface area of a sphere can be obtained experimentally by using a grapefruit. Slit the skin of the grapefruit from each pole toward the equator, peel it off, and make an outline of the skin on a grid (Figure 8.16). Also draw a circle with the same radius as the grapefruit on the grid. Next, count the squares in the interior of the region bounded by the outline of the grapefruit skin. The number of squares in this region may be considered an approximation of the area of the grapefruit skin.

FIGURE 8.15



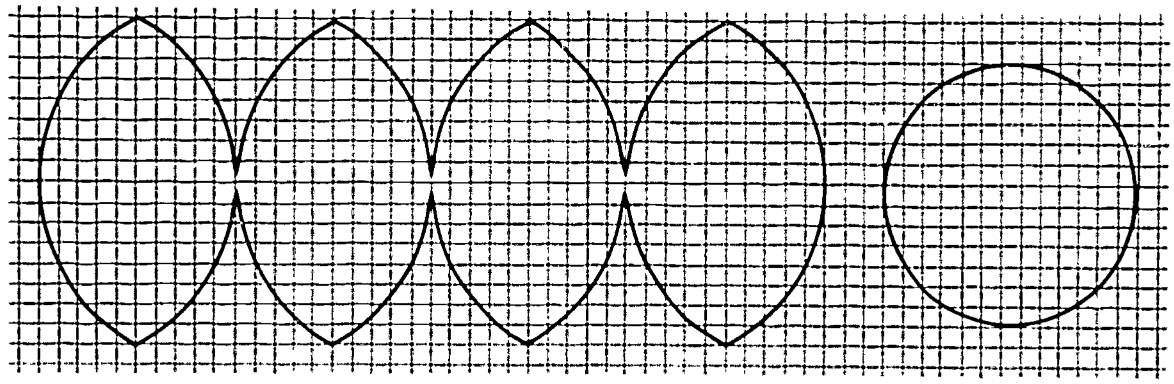


FIGURE 8.16

A comparison of this area with the area of the circle that has the same radius as the grapefruit suggests that the area of the grapefruit skin is about four times the area of the circle. Thus it seems reasonable to accept $A = 4\pi r^2$ as a formula for the area of a sphere.

There is another interesting way to determine a formula for the area of a sphere using concrete materials. Divide a croquet ball into two parts by sawing it through the center. Cover the hemispherical surface of one part with a spiral of venetian-blind cord (Figure 8.17). Then cover the flat surface of the other half of the croquet ball with a spiral of venetian-blind cord (Figure 8.18). The length of the cord needed to cover the surface in each case may be considered a measure of the area of the surface covered. If the work is carefully done, a comparison of the lengths of the two pieces of cord should show that the area of the hemispherical surface is twice the area of the circle. Thus it appears that $A = 2\pi r^2$ is a suitable formula for the area of a hemisphere; and this leads, again, to $A = 4\pi r^2$ as a formula for the area of a sphere.

The result of the following experiment can be used to obtain a formula for the volume of a sphere. The three plastic models shown in Figure 8.19 are constructed so that $D = H$

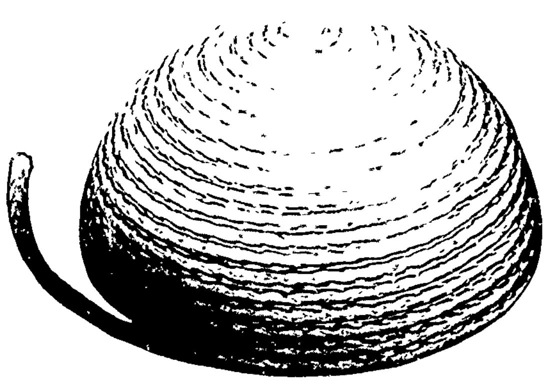


FIGURE 8.17

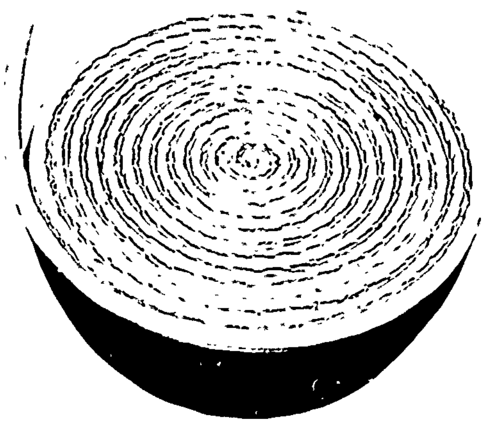


FIGURE 8.18

for all three models. If the cylinder is filled with water and the sphere put inside and forced to the bottom, the water that remains in the cylinder will just fill the cone. This result suggests the following derivation of a formula for the volume of a sphere, expressed in terms of the radius.

$$V = \pi\left(\frac{D}{2}\right)^2 H - \frac{1}{3}\pi\left(\frac{D}{2}\right)^2 H.$$

$$V = \frac{\pi D^3}{4} - \frac{\pi D^3}{12}.$$

$$V = \frac{\pi D^3}{6}.$$

Let $D = 2r$,
where $r$ represents the radius of the sphere.
Then

$$V = \frac{4}{3}\pi r^3.$$

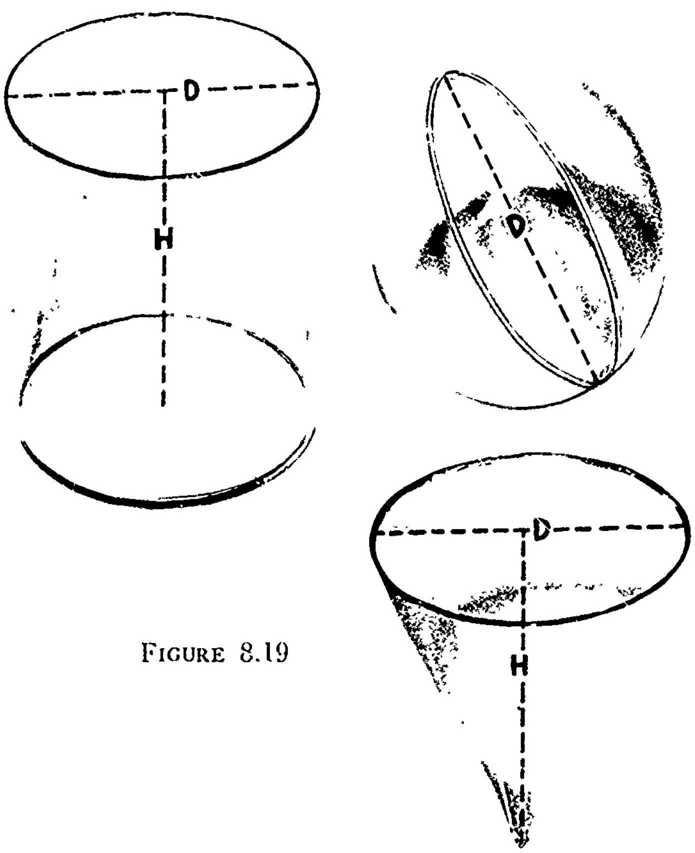



FIGURE 8.19

*Courtesy of Ideal School Supply Company*

A more sophisticated way of obtaining a formula for the volume of a sphere is suggested by the possibility that a sphere can be dissected into spherical pyramids (Figure 8.20). Either an orange or a grapefruit can be used as a model. The

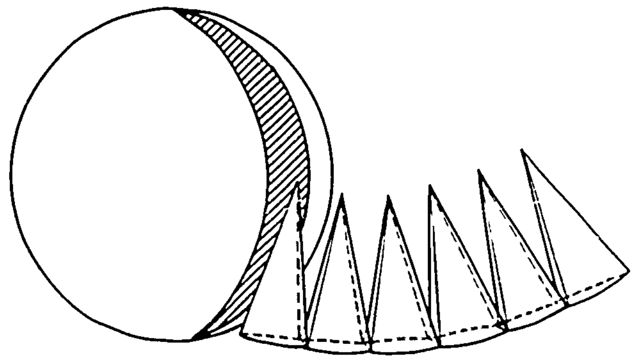


FIGURE 8.20

development of the formula relies on the following assumption: *A sphere is a polyhedron with an infinite number of small faces; with these faces as bases and the center of the sphere as the common vertex, the sphere can be separated into an infinite number of pyramids.* These pyramids all have the same altitude $r$, and the sum of their bases is the area of the sphere—that is, $A = 4\pi r^2$. Since the volume of a pyramid equals $\frac{1}{3}$ the product of its base and altitude, the volume of the sphere is $\frac{1}{3}r \cdot 4\pi r^2$, or $\frac{4}{3}\pi r^3$.

### *Models Can Be Used to Provide for Individual Differences*

Students differ so drastically in their cultural experiences, in their ability to deal with abstract ideas, in their interests, and in their mathematical backgrounds that it becomes necessary to tailor instruction to provide for these differences.

The slow learner needs concrete representation. He needs to make and deal with objects because he usually reads poorly and has difficulty understanding verbal instruction. In the case of the slow learner, there is no substitute for experiences with concrete materials.

The abler student usually progresses rapidly and independently. Many of today's accelerated

programs were developed with the gifted student in mind. Unfortunately, acceleration has often meant limited instruction, lengthened assignments, and complex text material. The result has been frustration, hostility, and rejection of a future in mathematics. Even for bright students instruction needs to be enriched with experiences that develop interest, arouse curiosity, and build appreciation. Experiences with concrete materials often enhance attainment of these goals better than do direct appeals to abstract ideas and formal assignments.

Between the slow learners and the bright students there is a large group of average students who also need varied experiences to ensure learning.

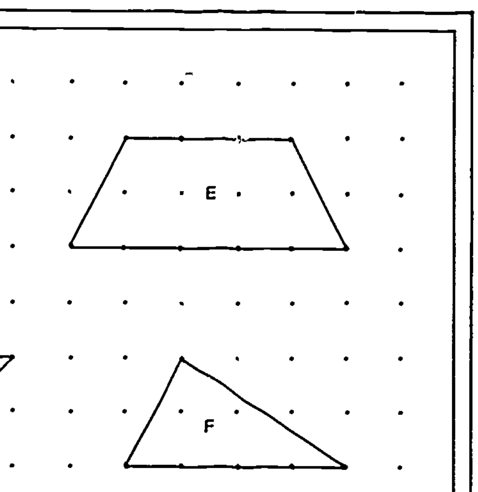
A pegboard on which the pegs represent a square lattice is an example of a model that can be used to find areas of polygons in several different ways depending on the ability level of the student. The diagram in Figure 8.21 shows how elastic thread can be used to locate the vertices and boundaries of polygons on a pegboard that has the desired construction.

The slow learner can obtain approximations of the areas of the polygons shown in the diagram by counting squares in the interiors. The average student can probably discover formulas for the areas of polygons A, D, E, and F but may need to count squares to find the areas of polygons B and C. The abler student should be encouraged to explore the possibility of obtaining a formula for the area of a polygon in terms of the number of lattice points on the perimeter and in the interior. The table contains the pertinent data for polygons A, B, C, D, E, and F.

| Polygon | Perimeter Points $P$ | Interior Points $I$ | Area $A$ |
|---|---|---|---|
| A | 12 | 3 | 8 |
| B | 11 | 2 | $6\frac{1}{2}$ |
| C | 14 | 4 | 10 |
| D | 8 | 1 | 4 |
| E | 10 | 4 | 8 |
| F | 6 | 2 | 4 |

By examining specific cases, the discerning student should be able to discover the area formula, $A = \frac{1}{2}P + I - 1$. This is Pick's formula. The formula holds for simple polygons under the condition that vertices are lattice points. The proof of the formula leads to some rather advanced mathematics (48).

FIGURE 8.21



Models made by folding paper are another kind of model that can be used at different ability levels. If waxed paper is used, the folds stand out readily and can be projected on a screen by an overhead projector. This technique is effective in helping the slow learner learn basic ideas about perpendicular lines, parallel lines, concurrent lines, angle bisectors, segment bisectors, and congruent figures. The same technique is effective in helping the average student discover properties of secondary lines of triangles, diagonals of a parallelogram, chords of a circle, and central angles and arcs of a circle. The abler student often finds real excitement in folding paper to obtain representations of envelopes of parabolas, ellipses, and hyperbolas, and then investigating properties of these curves (Figures 8.22, 8.23, 8.24). For further information about paper folding, see *Paper Folding for the Mathematics Class* by Donovan Johnson (35).

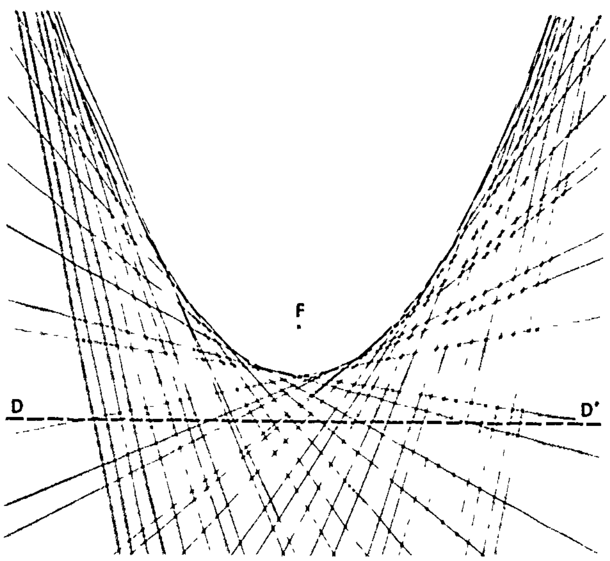



FIGURE 8.22. *Each full-drawn line represents a crease formed by folding paper so that a point of line* DD′ *coincides with a point* F *that is not in line* DD′. *Thirty-seven full-drawn lines are shown in the diagram. The set of all such lines is the envelope of a parabola with focus* F *and directrix* DD′. *Each line is tangent to the parabola.*

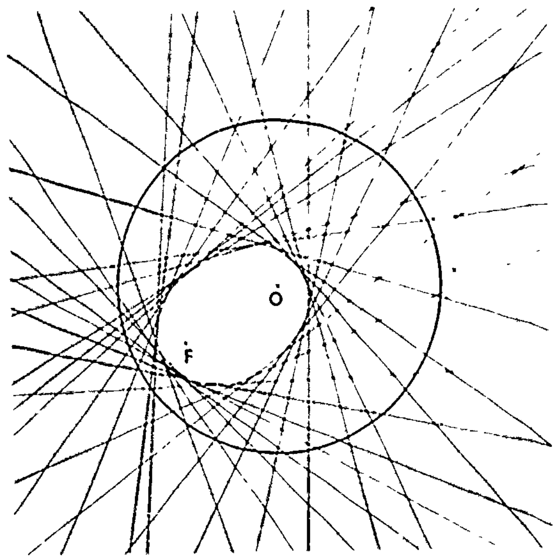
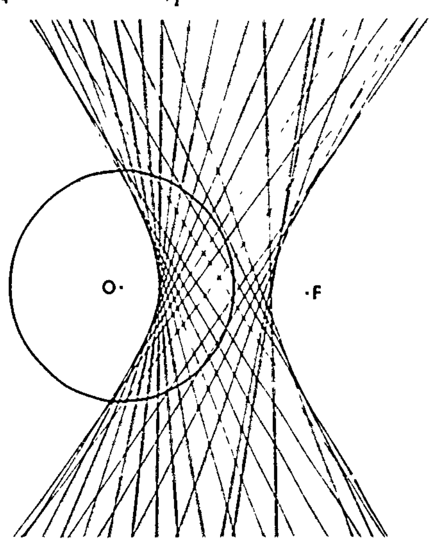



FIGURE 8.23. *Each line represents a crease formed by folding paper so that a point of the circle that has* O *as center coincides with point* F *in the interior of the circle. Twenty-eight lines are shown in the diagram. The set of all such lines is the envelope of an ellipse with points* O *and* F *as foci. Each line is tangent to the ellipse.*

FIGURE 8.24. *Each line represents a crease formed by folding paper so that a point of the circle that has* O *as center coincides with point* F *in the exterior of the circle. Twenty-seven lines are shown in the diagram. The set of all such lines is the envelope of a hyperbola with points* O *and* F *as foci. Each line is tangent to the hyperbola.*



### *Models Can Be Used to Generate Interest in a New Topic*

Students frequently have interests in mathematical topics that are related to the regular curriculum but are not part of it. Teachers can encourage further development of such interests by offering suggestions for doing independent projects. (See Chapter 10.) In addition, teachers should seek to kindle interest in topics that may be completely new to students. One way of doing this is to start with a familiar idea and then expand it to a new topic. Having students manipulate familiar concrete materials is an example of this technique. Another way to stimulate interest in new topics is to use a novel device.

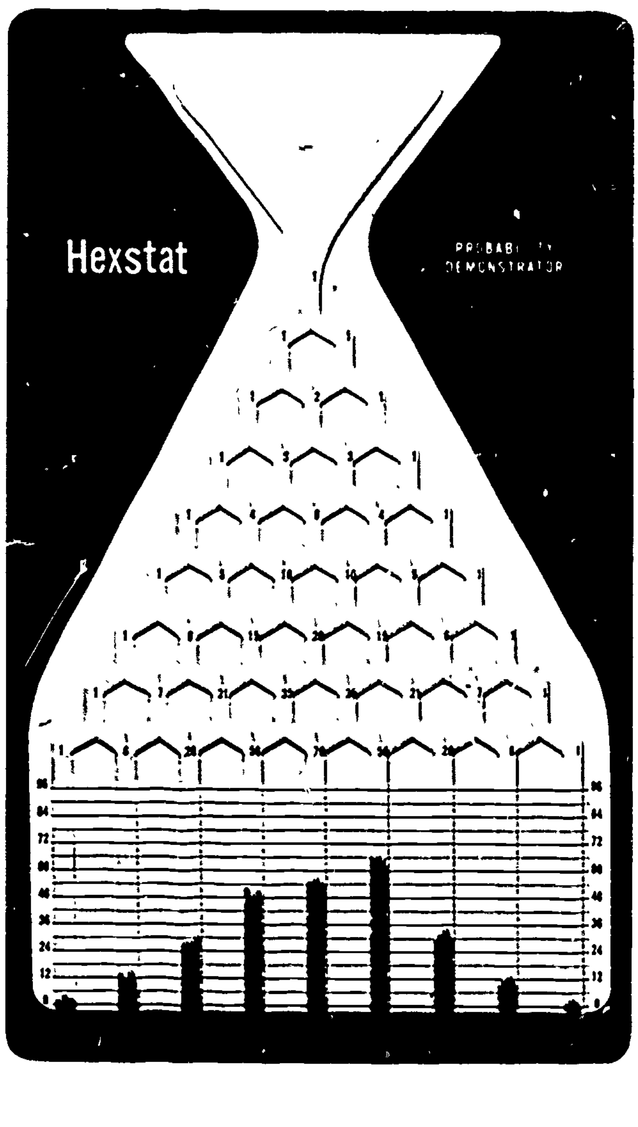
Consider how experiments involving tossing of coins, dice, or thumbtacks can help students become interested in probability. Students often become so absorbed with experiments like these that they willingly keep records of results for 100 tosses, or even 1,000. Record keeping of this kind can be used to introduce such ideas as a set of all possible outcomes, a distribution, a graph of a distribution, and so forth. These ideas lead naturally to definitions of events as sets of outcomes, mutually exclusive and independent events, and empirical and theoretical probabilities.

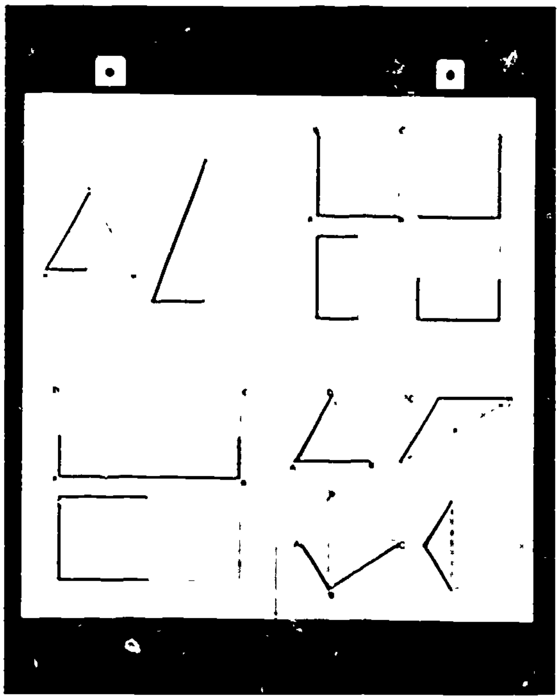
The simple experiments mentioned above can be enlarged upon with the aid of a device known as Galton's Quincunx. A modern adaptation of Galton's device, called a Hexstat, is pictured in Figure 8.25. Approximately 256 small steel balls are trapped inside the device. The picture shows the position of the balls after being poured down the channels from the reservoir at the top. The distribution of results obtained is an approximation of a binomial distribution, which in turn is an approximation of a normal distribution. Thus, the Hexstat may be used to introduce important and engaging ideas from probability.

It is interesting to note the parallel between pouring 256 balls down the channels of the Hexstat and tossing 8 coins 256 times. These experiments are equivalent in the following sense: The number of balls expected to fall into each of the nine channels at the bottom, counted in order from either end, corresponds respectively to the number of times 0 heads, 1 head, 2 heads, and so forth, are expected to occur if 8 coins are tossed 256 times.

FIGURE 8.25

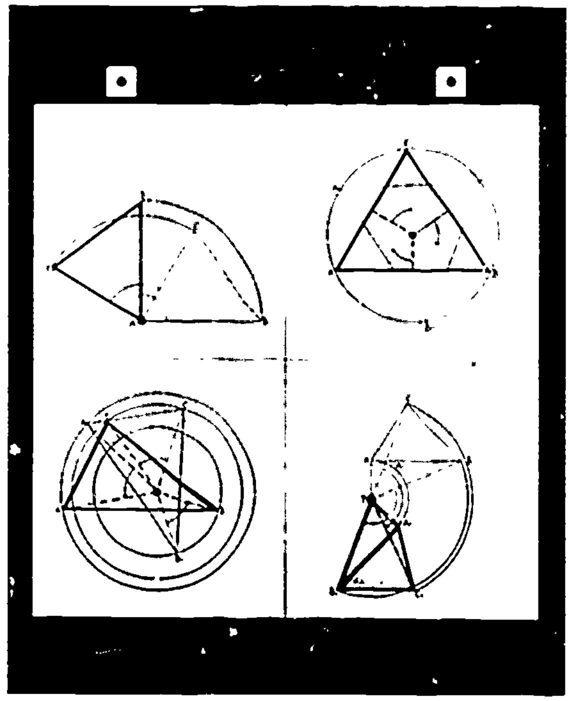
*Courtesy of Harcourt Brace Jovanovich, Inc.*





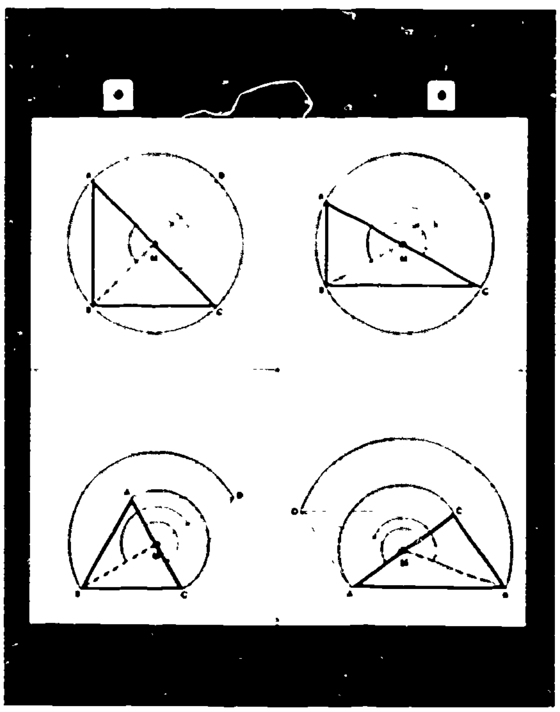
Courtesy of Günter Herrmann

FIGURE 8.26. *Reflection board*



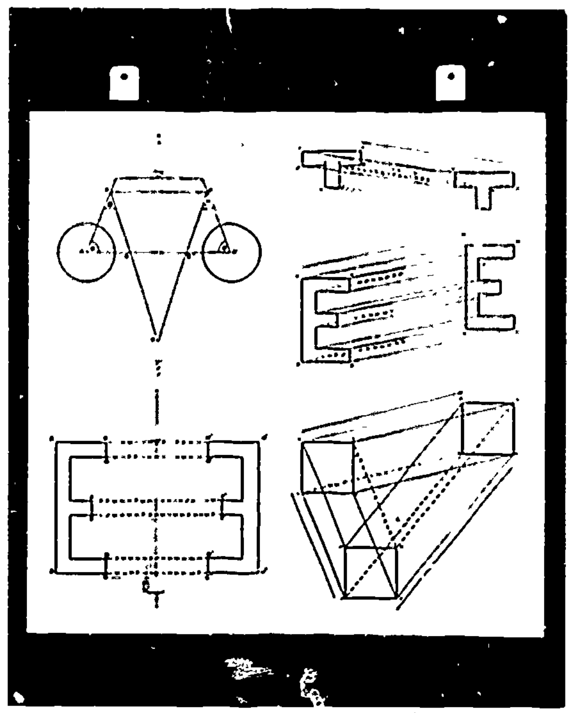
Courtesy of Günter Herrmann

FIGURE 8.27. *Rotation board*



Courtesy of Günter Herrmann

FIGURE 8.28. *Half-turn board*



Courtesy of Günter Herrmann

FIGURE 8.29. *Reflection and translation board*

Another topic in which student interest can be generated by using models is geometric transformations. The animation boards pictured in Figures 8.26, 8.27, 8.28, and 8.29 can be used to advantage in helping students perceive what is meant by such transformations as reflections, rotations, half-turns, and translations.

Logic is still another topic in which student interest can be developed with the aid of models. Shown in Figure 8.30 is an electronic logic trainer that can be used to clarify the basic postulates and theorems of logic and set theory and the meanings of the operations "and," "or," and "not." The logic trainer shown in the illustration can also be used to demonstrate what is meant by Boolean algebra and switching circuits.

Arousing interest in a topic can be critical to the future of our culture. Consider this startling statement, which is contained in the opening paragraph of the Ford publication *Worldwide Use of Measuring Systems.*

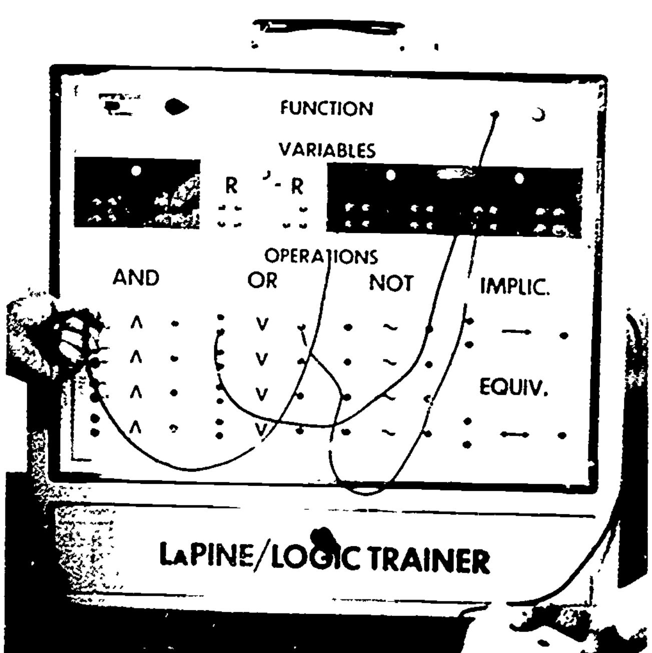
> The metric system has been adopted by 82 of the 135 nations of the world. Britain and the United States are the only highly industrialized nations that continue to use the inch-pound system, and Britain has announced its intention to convert to the metric system. [63, p. 3]

The following excerpt from the foreword of the same publication indicates why today's students should have adequate opportunity *now* to gain familiarity with the metric system.

> Emphasis in this publication has been given to the development, growth and current status of the metric system in anticipation that the United States and the automotive industry may move to expand the use of metric units in the foreseeable future. [63, p. 2]

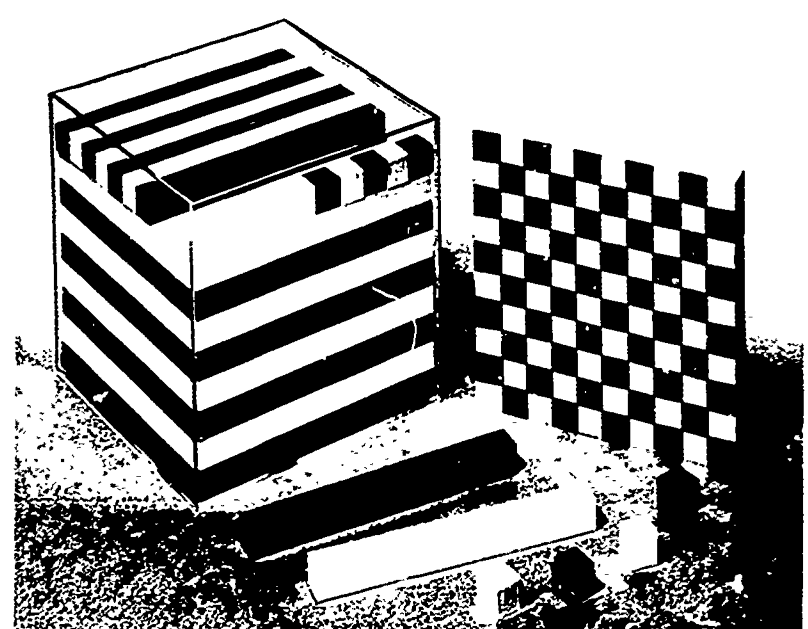
Since the United States lives by the inch-pound system, teachers of mathematics do not have a feeling of urgency about teaching the metric system. The result is that the topic often gets cursory treatment. But the topic is import-

FIGURE 8.30




*Courtesy of LaPine Scientific Company*

FIGURE 8.31



*Courtesy of LaPine Scientific Company*

ant, and it is likely to become more important, so the present lukewarm interest in the metric system should be heated up. To achieve this, the following equipment should be available for use by every medium-sized class group: a half-dozen meter sticks, a metric liquid measuring set, a dissectible liter block (Figure 8.31), and several inexpensive balances that are calibrated in grams. Procurement of this equipment, however, is only a start. The really useful materials for firing up interest in the metric system will come from a collection of realia that can be obtained from the kitchen cupboard, the grocery store, or the gasoline filling station.

The packaged products shown in Figure 8.32 are part of the contents of a kitchen cupboard that one of the writers raided. By actual count, 30 out of 90 items found in the cupboard were labeled in both English *and* metric units. The English and metric weights or volumes printed on the labels of the packaged products shown in the picture are displayed in "boxes." Presenting a collection like this to students is certain to generate interest in the metric system. Students who have an opportunity to examine such a collection will no longer think of the metric system simply as "the other system" they have to know in science class.

FIGURE 8.32

### *Models Can Be Used to Promote Enjoyment of Mathematics*

Most students are intrigued by games, puzzles, and novel problems and willingly spend time on them. The resulting learning experiences can be both challenging and entertaining.

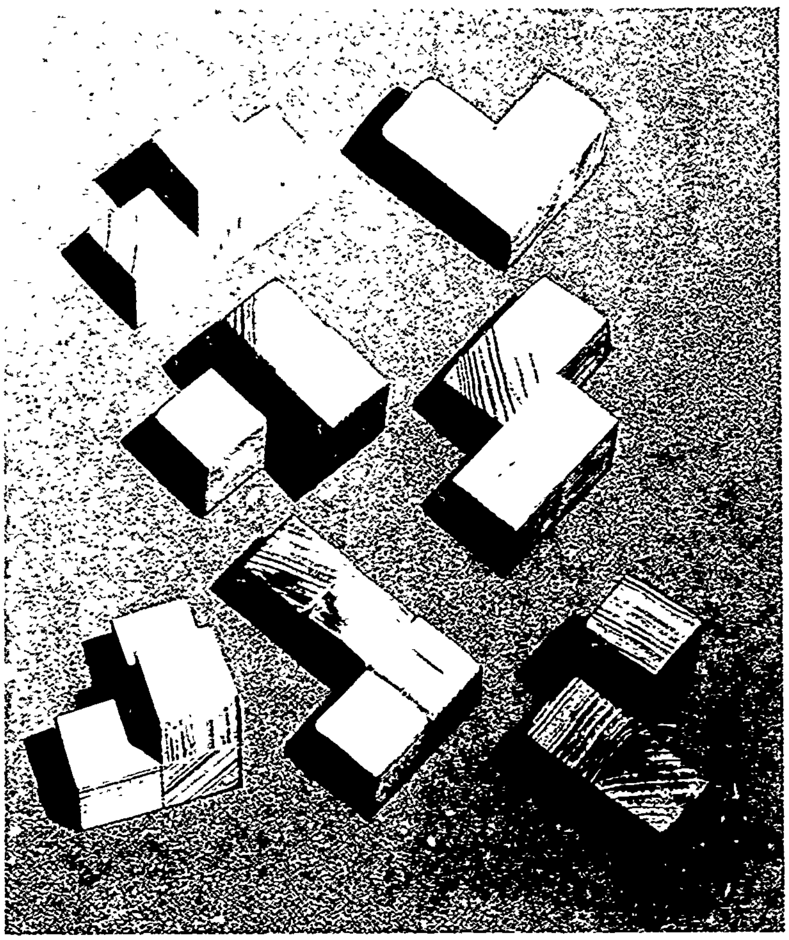
Games contribute to a student's enjoyment of mathematics because they are fun. A well-structured game also helps students learn mathematics. Following the rules of a game is not unlike pursuing a programed instruction unit. Ordinarily the rules of a game promote active participation and provide for immediate reinforcement of learning. Usually, there is also a certain allowance for individual pacing. Games that are structured so that the learner encounters increasingly complex situations as the game progresses have an added feature that is similar to controlled adaptation of learning materials in programed instruction. For a further discussion of the analogy between games and programed instruction, see "Games and Programmed Instruction" by Allen (1).

Teachers may wish to construct their own games, but there are plenty of well-constructed games available from commercial producers. One example of a versatile and creative game is Equations (Figure 8.33). This game is designed to provide stimulating and entertaining opportunities for practice in addition, subtraction, multiplication, division, exponentiation, and the square root operation in various number bases.

Puzzles contribute to a student's enjoyment of mathematics because they are intriguing. One example of an exciting puzzle involves the Soma cube. This is a cube cut into seven pieces,


FIGURE 8.33

*Courtesy of WFF 'N PROOF*

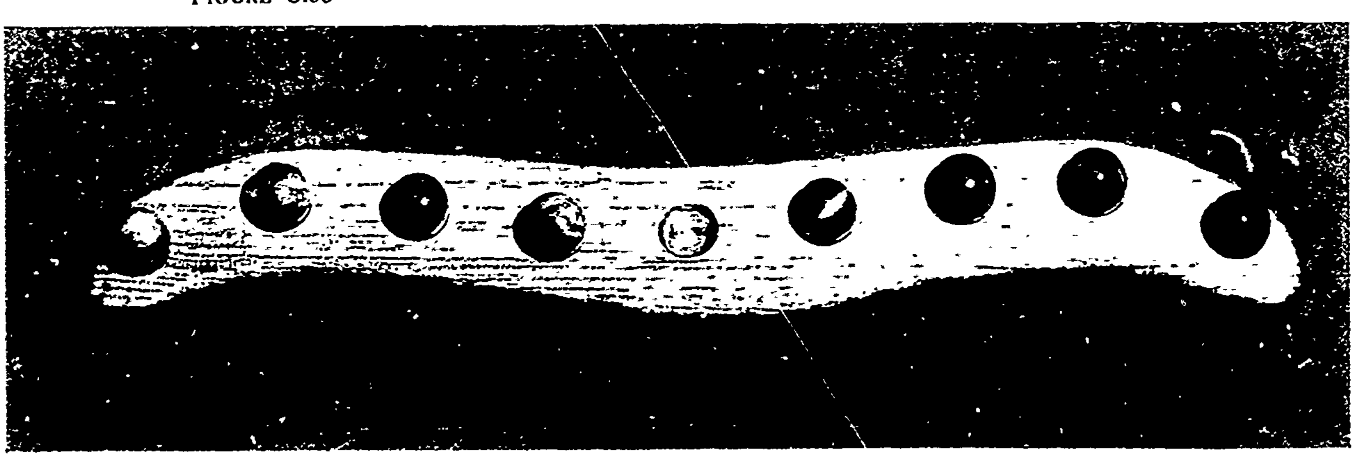


no two of which are alike (Figure 8.34). The puzzle is to put the seven pieces together to form a cube and to make other shapes. The pieces of this puzzle are easily constructed by gluing together cubical counting blocks. For a fuller description of this puzzle, see *Mathematical Models* by Cundy and Rollett (14).

Another puzzle is the oxbow puzzle shown in Figure 8.35. This puzzle is somewhat like a game, because there are rules to observe. The object of the puzzle is to exchange the marbles on the two sides of the oxbow using a minimum number of legal moves. A marble can be moved either one or two spaces in one move, but no more. If the marble in hole D is moved into the open hole E, this leaves the D hole open. Then either the marble in hole C or the marble in hole F can be moved into the open hole D. This process is continued until the marbles on the two sides are exchanged.

FIGURE 8.34

FIGURE 8.35



Students also find enjoyment in mathematics by working on intriguing problems, or just contemplating them. Rademacher and Toeplitz's delightful book *The Enjoyment of Mathematics* (52) is filled with exciting problems and their solutions. Listed below are the titles of several sections that are particularly rich in solved problems that can be tested with the aid of models.

"Traversing Nets of Curves"
"On Closed Self-Intersecting Curves"
"The Four-Color Problem"
"The Regular Polyhedrons"
"Producing Rectilinear Motion by Means of Linkages"
"The Figure of Greatest Area with Given Perimeter"
"Curves of Constant Breadth"

The section entitled "Producing Rectilinear Motion by Means of Linkages" contains a number of inviting problems that suggest the use or production of models. Peaucellier's cell is one of several linkages that are described.

Peaucellier's cell (Figure 8.36) consists of a hinged rhombus $ADCE$ and two rods $AB$ and $BC$ that are hinged to opposite vertices of the rhombus. The two rods $AB$ and $BC$ are equal in length and are hinged together at $B$. Because of the symmetry of the linkage, $B$, $D$, and $E$ are always collinear. By applying the Pythagorean theorem it can be proved that $(BD)\cdot(BE) = (BA)^2 - (AD)^2 = k^2$, a constant.

Therefore, $D$ and $E$ are said to be inverse to each other with respect to a circle with center $B$ and radius $k$, where $B$ is referred to as the center of inversion. It can also be proved that the image of a circle that passes through the center of inversion is a straight line. Thus $E$ will travel a straight line if $D$ is constrained to move along a circle which passes through $B$. In Figure 8.37, bar $DH$ has been added to the Peaucellier cell, and the positions of $H$ and $B$ fixed so that the distance $BH$ is equal to the length of bar $DH$. As $D$ moves it will follow the arc $DB$, which goes through point $B$, the center of inversion. This will force $E$ to move along a straight line perpendicular to $BH$.

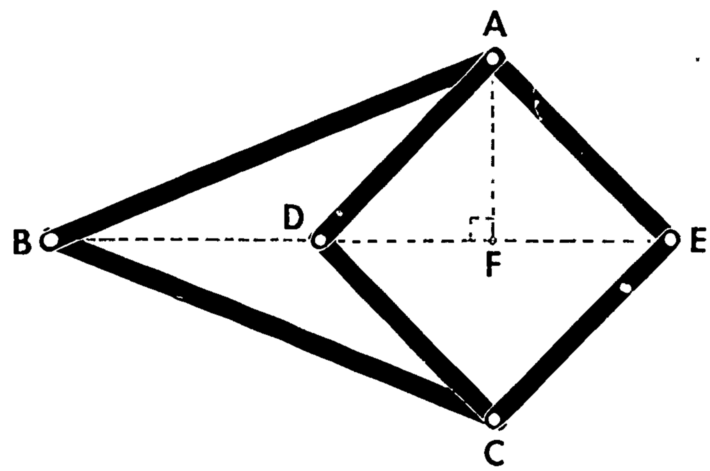



FIGURE 8.36

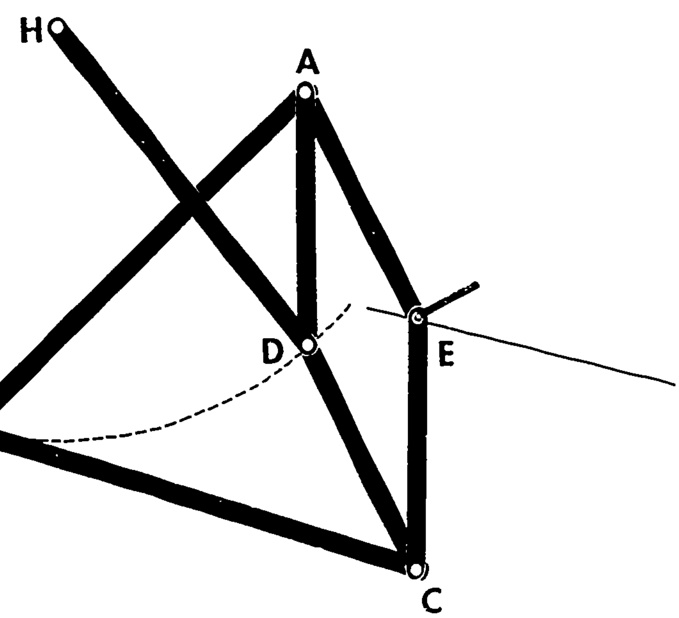



FIGURE 8.37

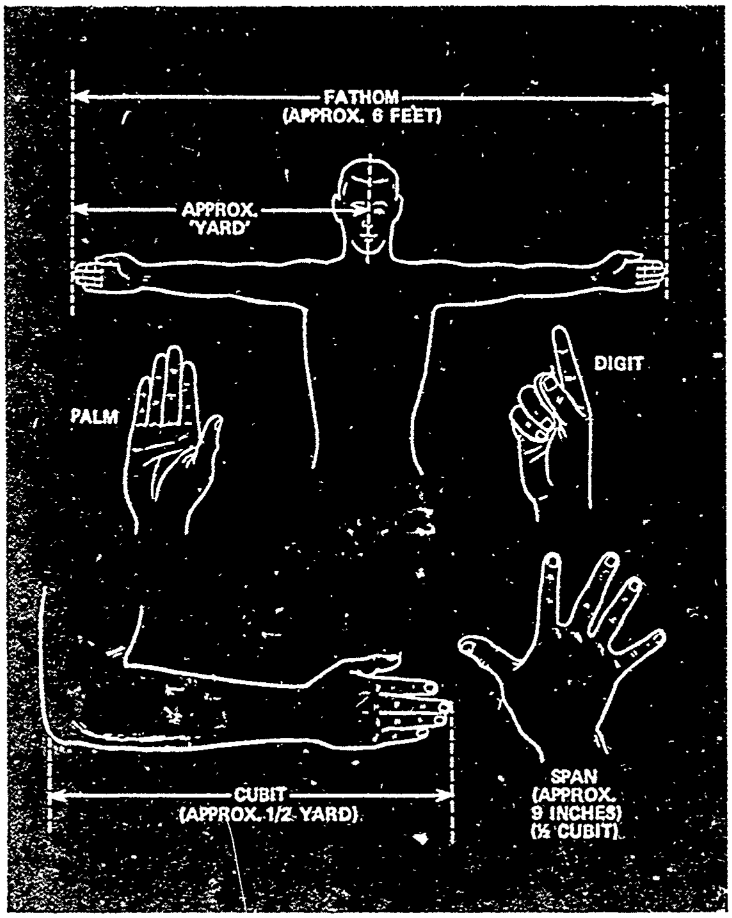



FIGURE 8.38

### *Models Can Be Used to Build Appreciation for Mathematics*

The following statement by Benjamin Franklin gives an indication of the importance he placed on both the practical and the aesthetic aspects of mathematics.

> It would be well if they [the students] could be taught everything that is useful and everything that is ornamental; but art is long and their time is short. It is therefore proposed that they learn those things that are likely to be most useful and most ornamental.[1]

If students are to "learn those things that are likely to be most useful and most ornamental," they need to have experience in making subjective appraisals of the utility, beauty, and other aspects of the subjects they study. Such appraisals may be quite informal and will probably involve feelings.

To build appreciation for mathematics, students should be presented with situations or given opportunities to participate in activities in which they are likely to develop positive feelings for some aspect of mathematics. The examples that follow illustrate how models can be used or how they function in helping students develop such feelings.

As a prelude to work with standard units of measurement, students may be introduced to such units as the fathom, cubit, palm, digit, and span and invited to make measurements using their own limbs as units of measure (Figure 8.38). Students who engage in this activity usually develop a sincere sense of respect for the utility, convenience, and precision of the ordinary foot ruler and yardstick.

Mathematicians are always talking about the beauty of mathematics, but few seem able to elaborate on what they mean. One mathematician whom we queried thinks beauty in mathematics refers to the elegance of a proof. He considers a proof to be elegant if the results contain an element of surprise or if the techniques used are clever.

1. This statement appears in *The Process of Education*, by Jerome Bruner (10, p. 4).

An example of a proof that has at least some of these qualities is Legendre's proof of the Pythagorean theorem. This proof is unusual for its brevity. There are only three steps. This is certainly clever.

Following are the steps of Legendre's proof and the accompanying diagram (Figure 8.39), as excerpted from Loomis's book *The Pythagorean Proposition* (44, pp. 23–24). The proof attributed to Legendre is contained in a discussion of the shortest possible proof of the Pythagorean theorem. The equations in the first two steps can be obtained with the aid of the diagram.

$a:y = h:a \quad \therefore a^2 = hy.$

$b:h - y = h:b \quad \therefore b^2 = h^2 - hy.$

Adding these gives $h^2 = a^2 + b^2$.

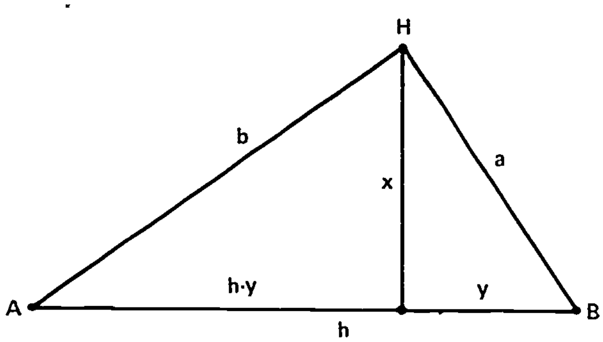



FIGURE 8.39

For many students of geometry, a three-piece set of overlay models made of plywood, cardboard, or plastic will be needed if they are to see the similar triangles on which Legendre's proof is based. An appropriate set of models is shown in Figure 8.40.

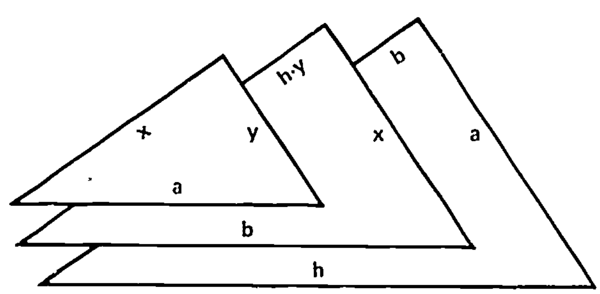
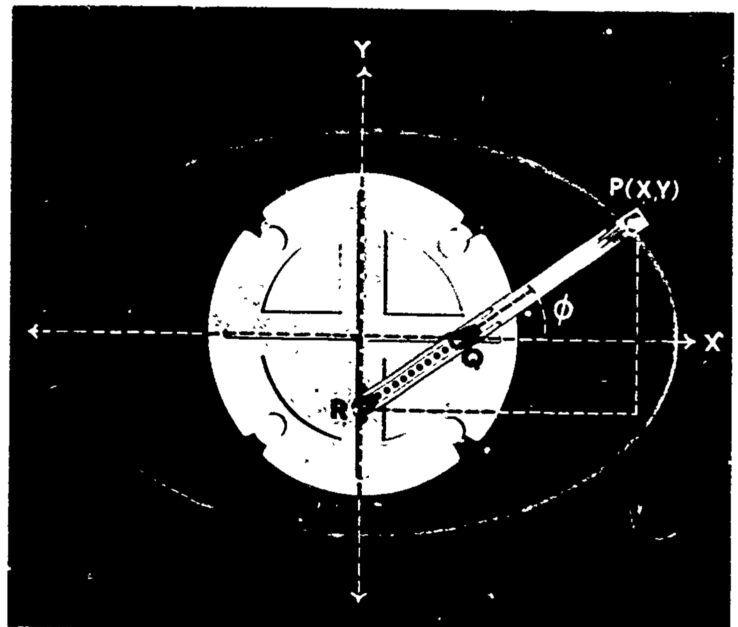



FIGURE 8.40




*Courtesy of LaPine Scientific Company*

FIGURE 8.41

While the last paragraph was being written, a visitor came into the office. She spied the ellipsograph and the full drawn ellipse pictured in Figure 8.41 on the office blackboard. (The ellipsograph was held in place with rubber suction cups.) She contemplated the situation for a moment and then asked: "You mean you can draw an ellipse with the device fastened to the blackboard?" "Yes," was our reply. "But how do you know that the figure is an ellipse?" she wanted to know. Hurriedly we sketched the dotted lines shown in the picture, labeled points with capital letters, and then took up a proof.

Let $d(PR) = a$, let $d(PQ) = b$, and let $\phi$ represent the measure of the angle that the crossbar makes with the positive $x$-axis.

Then

$$\frac{x}{a} = \cos \phi, \text{ and } \frac{y}{b} = \sin \phi;$$

and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi + \sin^2 \phi.$$

So

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

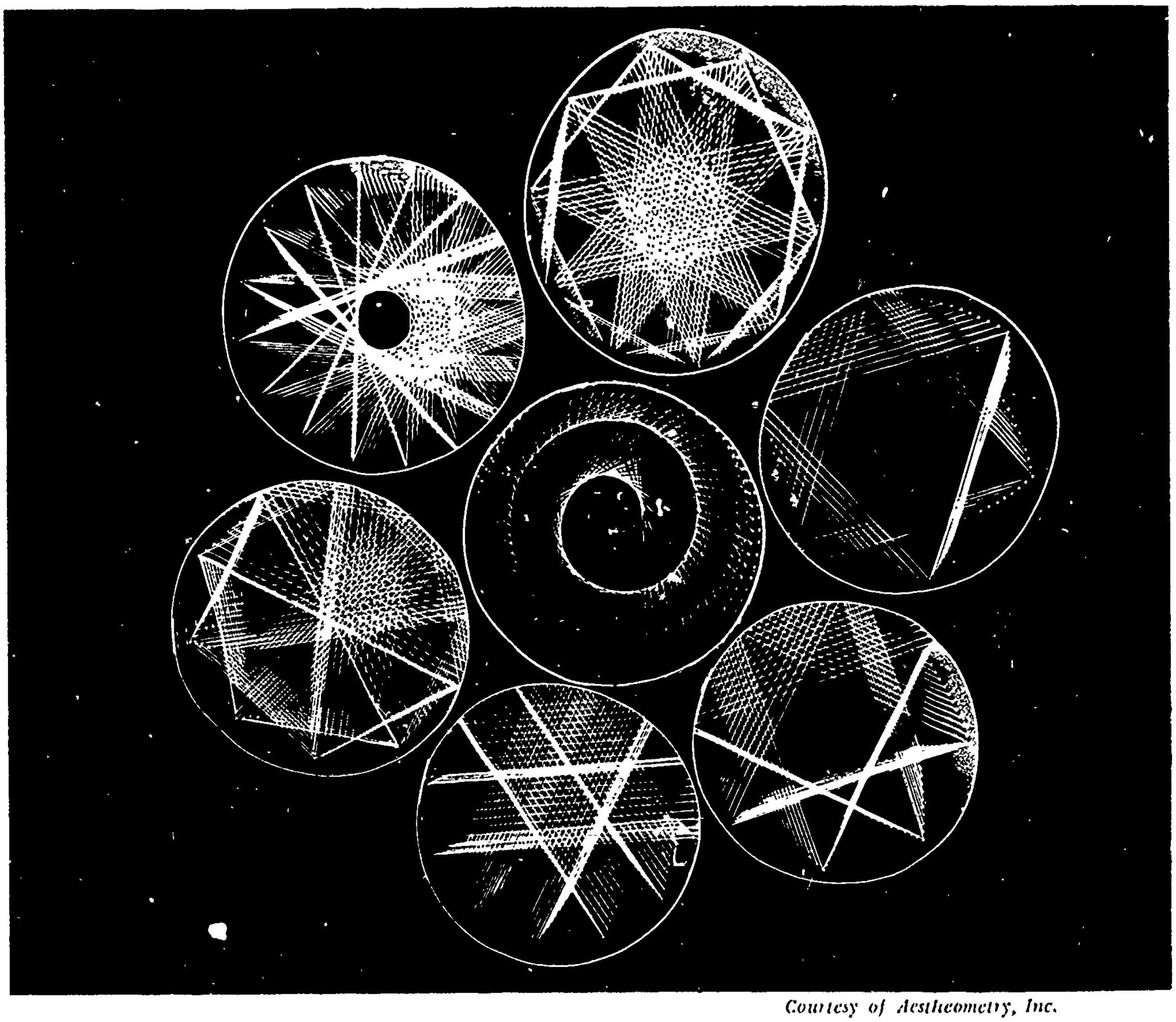
"That's elegant," said our visitor; then she exclaimed, "No, it's beautiful!"

Students, and adults, too, are usually fascinated by the many figures and forms of geometry that lend themselves to the arts and sciences. Pictured in Figure 8.42 are string models of stellar polygons constructed within a cylinder. String models of this kind can conceivably be constructed within the walls or ceilings of buildings by using wire or other materials. Constructing three-dimensional models like these helps students develop a real appreciation for the beauty of geometric forms. Detailed instructions for building the models shown in the picture may be found in the book by Anderson, *Space Concepts through Aestheometry* (2).

### *A Major Function of Models is Their Positive Effect on Retention*

If you were asked to recall the most memorable mathematics lesson during your education, it would probably be a lesson involving the use of a model, a transparency, or a film or a lesson

FIGURE 8.42



*Courtesy of Aestheometry, Inc.*

involving an experiment. It is not likely that it would be a lesson involving a lecture on an abstract theorem. Learning based on work with models or learning that results from experiments is usually remembered long after the learning period ends.

Students often have difficulty learning and remembering mensuration equivalents. The set below is typical of those we have in mind.

| | |
|---|---|
| 12 inches | 1 foot |
| 144 square inches | 1 square foot |
| 1728 cubic inches | 1 cubic foot |

When there are no physical referents to work with, students can have difficulty learning and remembering that the measures in each line are equivalent. However, if a student is given a foot ruler, a picture of a square foot ruled into square inches, and a cubical box one foot on edge with faces ruled into square inches, he can "see" the equivalent measures and make associations that will assist him in learning and remembering them. Another way of assisting the student in learning and remembering that the measures in each line are equivalent is by having him make cardboard models of squares that are one inch on a side and then use these models to find the dimensions and areas of objects in the classroom. If the last line in the foregoing table proves troublesome, have the student make a model of a cubic foot with 1728 blocks, each one inch on edge. The student who does this is not likely to forget that 1728 cubic inches and 1 cubic foot are equivalent measures.

Let us consider how concrete referents might be used to help students learn and remember another set of mensuration equivalents.

| | |
|---|---|
| 3 feet | 1 yard |
| 9 square feet | 1 square yard |
| 27 cubic feet | 1 cubic yard |

Giving the student an opportunity to make measurements with a yardstick will usually ensure his learning and remembering that 3 feet and 1 yard are equivalent measures of length. Having the student form a model of a square yard with 9 square-shaped cork tiles, each 1 foot on edge, is ordinarily all that is needed for him to learn and remember that 9 square feet and 1 square yard are equivalent measures of area. Only the volume equivalents in the last line give students much trouble, but unfortunately this trouble is usually persistent and there are plenty of adults who do not know that 27 cubic feet and 1 cubic yard are equivalent measures of volume.

Here is a way of teaching this fact so that it will be remembered. Have each student in class cut out 6 cardboard models of squares 1 foot on a side, as a homework assignment. The next day have each student use his 6 cardboard squares to make a model of a cubic foot using masking tape to join the edges. Then have the students in the class build a model of a cubic yard. Experience indicates that if there are between 20 and 30 students in class, every student soon realizes that 27 is the magic number. Students (and teachers, too) who participate in this activity rarely forget that 27 cubic feet and 1 cubic yard are equivalent measures of volume.

To illustrate how to use physical referents to help students remember what they learn we deliberately picked examples in which remembering is a matter of practical concern. However, we could just as easily have illustrated our point of view with other models.

If a student learns the meaning of similarity by preparing a scale drawing of a floor plan, he is likely to use the idea of similarity in the future. If a student constructs his own slide rule (Figure 8.43), he is not likely to forget that slide rule scales are logarithmic scales. If a student explores the variation of trigonometric functions with a device such as a Trig-Aide (Figure 8.44), he will usually be able to recall the variation of each function.

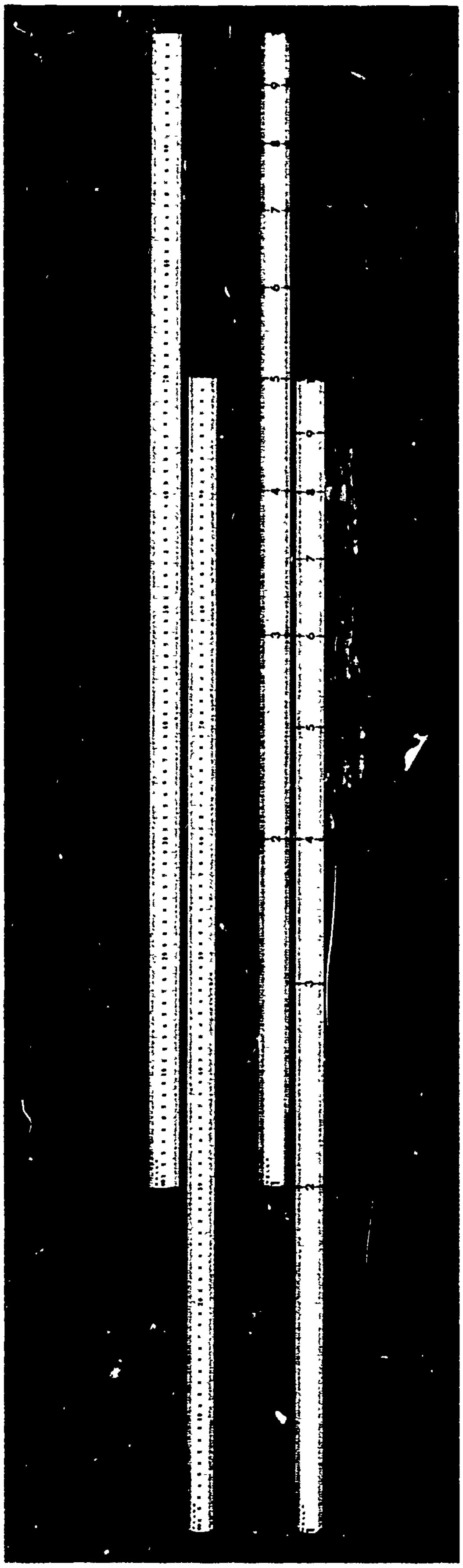
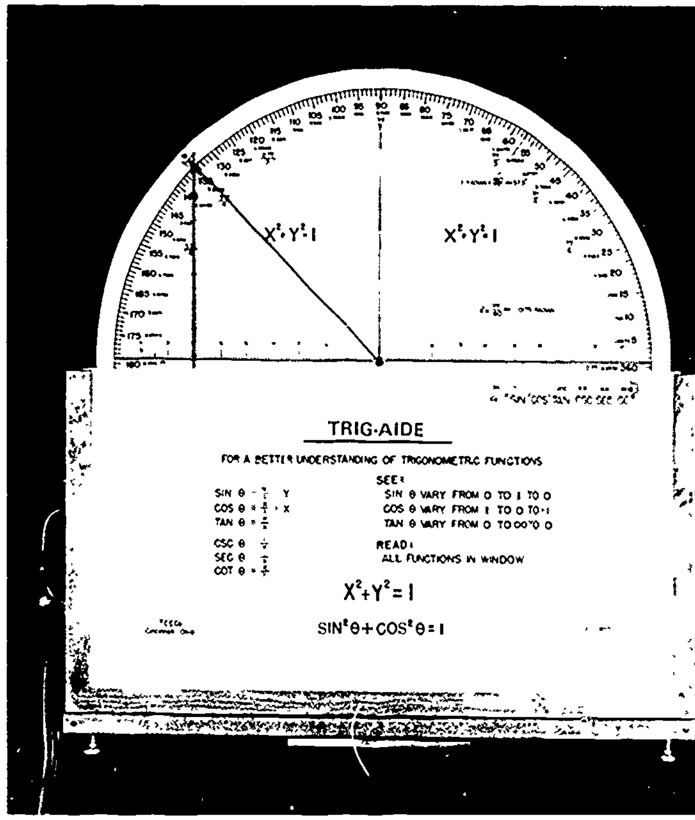


FIGURE 8.43. *Slide rule made with meter sticks*

FIGURE 8.44




*Courtesy of Brooks Manufacturing Company*

### *Models Can Be Used to Teach Applications*

The teaching of applications is complicated by two factors. First of all, most students do not have backgrounds in which significant applications have meaning. At the same time, students have a tendency to become bored with the ordinary.

Neither fact lessens the teacher's obligation to teach applications. What is to be done? One suggestion is to make applications catchy or even offbeat. Unfortunately, thinking up offbeat applications is not easy. Occasionally inspiration comes from unusual questions asked by students and grown-ups. Following are three such questions:

1. "The super market has started selling beef stroganoff in a box that has trapezoids for bases and rectangles for faces. What's their angle—to take my mind off the calories?"
2. "The landscape man just dumped a load of black dirt on our lot. He says it's 'five yards.' How do I know his truck holds 'five yards'?"
3. "How do you find the area of a flower bed that has the shape of an ellipse?"

The first question points the way to a host of application-type problems about boxes used by distributors of food and other products.[2] Circular cylinders and rectangular solids are still the most popular shapes of boxes used for packaging products, but frustums of circular cones, frustums of pyramids, and right prisms of all kinds are being used more and more (Figure 8.45).

2. In this discussion the word *box* is used both to convey its usual meaning and in a generic sense when referring to cartons, cans, containers, and so on.

FIGURE 8.45

Fourteen boxes are identified below in the column on the left. A description of the shape of each is given in geometrical language in the column on the right. Having students give such descriptions can be an exciting learning activity.

| BOXES | DESCRIPTION |
|---|---|
| 1. Corsage box | 1. Right hexagonal (nonregular) prism |
| 2. Camembert soft-ripened cheese box | 2. Right cylinder with a semicircular base |
| 3. Iodized salt carton | 3. Right circular cylinder |
| 4. Timex watch box | 4. A convex solid formed by a closed rectangular prismatic surface with a frustum of a rectangular pyramid enclosing each end |
| 5. Cottage cheese carton | 5. Frustum of a right circular cone |
| 6. Strawberry box | 6. Frustum of a regular square pyramid |
| 7. Talcum powder can | 7. Right elliptical cylinder |
| 8. Seth Thomas metronome box | 8. Frustum of a regular square pyramid |
| 9. Dobbs hatbox | 9. Right octagonal (not regular) prism |
| 10. Lady Esther face powder box | 10. Right regular octagonal prism |
| 11. Easy window points box | 11. Right regular triangular prism |
| 12. Orthodontic retainer box | 12. Frustum of a regular octagonal pyramid mounted on a right regular octagonal prism |
| 13. Gold Label Shamrock cigar box | 13. Right trapezoidal prism |
| 14. Hormel chicken cacciatore dinner box | 14. Right rhombic prism |

Having a box collection available makes it possible to pose all kinds of application-type problems. The following are suggestive:

1. A salt carton has the shape of a right circular cylinder (Figure 8.45). The diameter of the base is $3\frac{1}{4}$ inches and the height is $5\frac{1}{2}$ inches. To what height is the carton filled when it contains 2 cups of salt? What is the length of the longest spoon that can be hidden in the carton?
2. A strawberry box that has a capacity of 1 quart has the shape of a frustum of a regular square pyramid. If the edges of the top and bottom bases are 4 and 3 inches, respectively, what is the slant height of the box?
3. A box of bake cups (Figure 8.45) has the shape of a right regular hexagonal prism. If the perimeter of each base is 14 inches and the height of the box is 5 inches, what is the total area of the outer surface of the box?
4. How long is the piece of string used to tie up the box shown in Figure 8.46? Allow 12 inches for the bow.

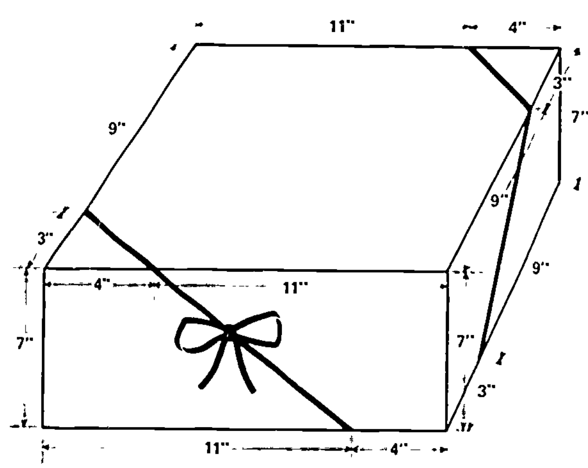



FIGURE 8.46

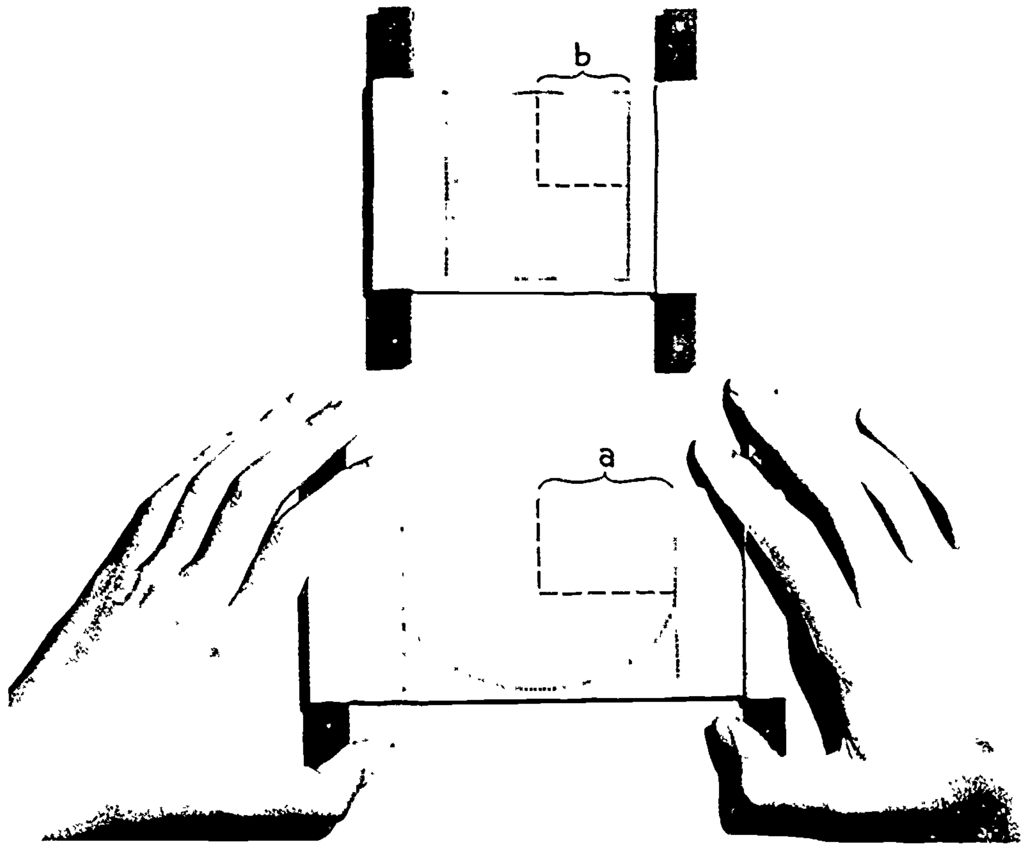



FIGURE 8.47. *The picture shows two identical devices. The device at the top is nailed to a wall. The bottom device is stretched horizontally and held against the wall.*

Now let us look at the problem of the person who has doubts about his landscaper's veracity or about the capacity of his truck. It is easy to give instructions telling how to compute the volume of a truck box in cubic yards; but doing this will probably not satisfy a skeptic who is confronted with a pile of dirt that has been dumped on the ground. The problem is that he probably has no notion of the size of a pile of dirt that contains five cubic yards. As a matter of fact, very few people do. That's why the question comes up.

Here is a suggestion that deserves being tried. Obtain one cubic yard of dirt and put it in a pile that has a shape like that of a right circular cone. Then build a tent that has the size and shape of the pile. This tent can be set up inside the school or on the playground to exhibit the size of a cubic yard of dirt.

The person who asked how to find the area of a flower bed that has the shape of an ellipse was interested in the cost of fertilizing the soil. Finding the area of an ellipse is not, of course, an offbeat problem if one knows calculus. But a layman who asks how to find the area of an ellipse usually doesn't know calculus.

One way of finding a plausible formula for the area of an ellipse, without appealing to calculus, is by using a rectangular piece of elastic with sticks fastened at the ends. A device of this kind with a circle and a square drawn on the elastic is shown in the upper part of Figure 8.47. The square is circumscribed about a circle of radius $b$. The area of the circumscribing square is $4 \times b \times b$, and the area of the circle is $\pi \times b \times b$.

By stretching the elastic horizontally the square is transformed into a rectangle (lower part of Figure 8.47). It follows that:

$$\text{Area of square} \rightarrow \text{Area of rectangle.}$$
$$4 \times b \times b \rightarrow 4 \times a \times b.$$

The picture also suggests that stretching the elastic transforms the circle into an ellipse and that:

$$\text{Area of circle} \rightarrow \text{Area of ellipse.}$$
$$\pi \times b \times b \rightarrow \pi \times a \times b.$$

Therefore, we surmise that the area of the ellipse shown in Figure 8.47 is $\pi ab$.

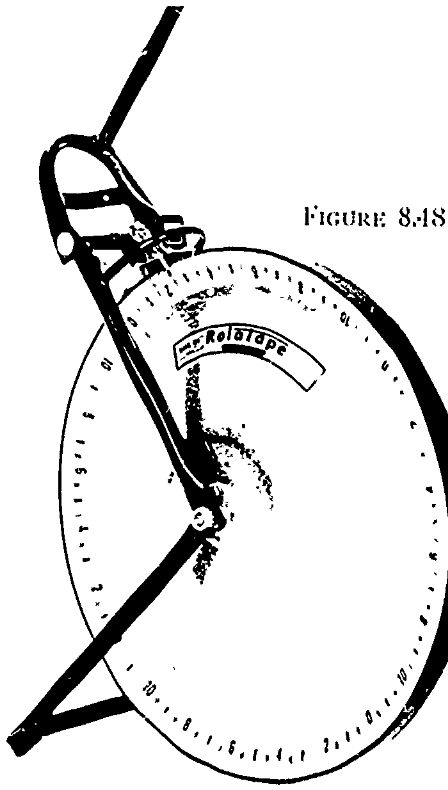
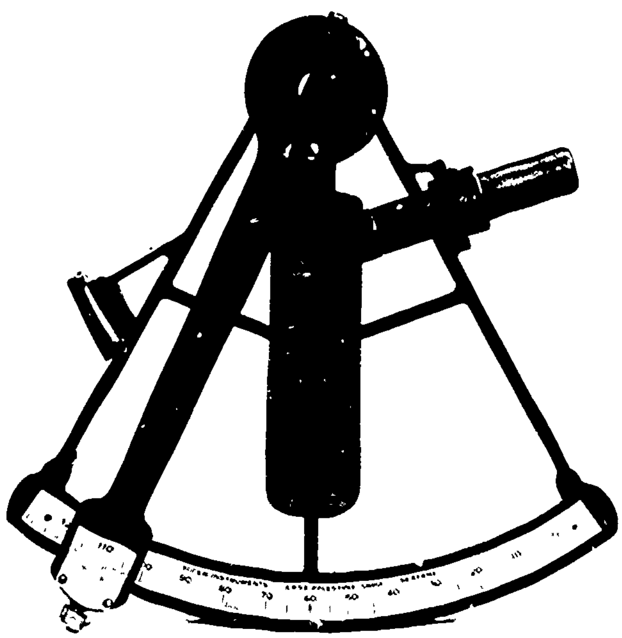



FIGURE 8.48

*Courtesy of Rolatape, Inc.*

FIGURE 8.49



*Courtesy of Yoder Instruments*

Now let us consider some applications that may be a little more practical than those described thus far. Pictured in Figure 8.48 is the Rolatape measuring wheel, an instrument for measuring distances. This instrument was used at Dulles International Airport when the runways were paved. The circumference of the Rolatape wheel is 4 feet; its diameter is about $15\frac{1}{4}$ inches. One side of the wheel is calibrated to $\frac{1}{2}$ inch, and the other side to $\frac{1}{10}$ inch. Attached to the instrument is an automatic counter to record revolutions. Distances of all kinds, including the lengths of chords and arcs of circles, can be measured quickly and accurately with this instrument. Using it is as easy as walking.

To measure angles, students can use such instruments as the sextant (Figure 8.49) or field protractor. With the aid of the Rolatape and an appropriate angle-measuring instrument, students can undertake such activities as finding the areas of parking lots and floors of large buildings, finding the heights of tall buildings, and making maps. The speed, convenience, and accuracy of the Rolatape in measuring fairly long distances will add new excitement to practical problems involving the measurement of distances.

## SUGGESTIONS FOR USING MODELS EFFECTIVELY

How and when to use models are matters that involve tastes and preferences. It is, therefore, inappropriate to propose a hard-and-fast guide to be followed by all teachers. There are, however, some suggestions that seem generally applicable.

First of all, the particular needs of a class should be watched for cues that indicate whether or not models are to be used. When interest lags, when student participation wanes, when students fail to make the right associations, it may be time to use a model. The model may be a chalk box, a coin, a piece of string, a lamp shade, or a commercial device that will add needed reality to the idea being discussed. However, when stu-

dents respond immediately to direct appeals to abstract ideas, when students are eager to spend large blocks of time on challenging problems, when students want to explore new ideas independently, then it may be time to withdraw models that are being used or omit some whose use has been planned. A teacher who is sensitive to the needs of his students and who gives careful attention to what works best for him will soon learn when models should be part of the total approach to a lesson and when they should be withheld.

Secondly, advance planning should precede the use of models. Plans should be made well ahead of presentations so that the best models can be procured and time will be available for the teacher to familiarize himself with the models he intends to use.

Third, strategies for using models should be developed. Merely holding up a model before a class or pointing to a model resting on a shelf is not enough. If necessary, teaching techniques should be modified and the strategy for content development changed in order to incorporate desirable uses of models in a lesson. This means, of course, that models are to be incorporated with other instructional aids such as bulletin board displays, films, chalkboard diagrams, textbooks, and so on. When models are used frequently, using them becomes spontaneous, and they become an indispensable part of the total instructional procedure.

Fourth, there must be a willingness to use models without insisting on total justification before attempting to use them.

Fifth, an effort should be made to involve students in discovery activities. Whenever possible a model should be put in the hands of each student so that an idea that is under discussion can be explored by him.

Sixth, guide sheets should be available for use by students who are investigating ideas independently. Providing guide sheets helps a maximum number of students make discoveries and formulate generalizations during the time that is av[ail]ble for independent investigations. A sample guide sheet is included later in this chapter.

A basic reference tool in the area of instructional aids in mathematics is the pamphlet *A Guide to the Use and Procurement of Teaching Aids for Mathematics*, by Berger and Johnson (6). This publication lists seven steps to follow in using teaching aids effectively (6, p. 5). These apply especially to the use of models.

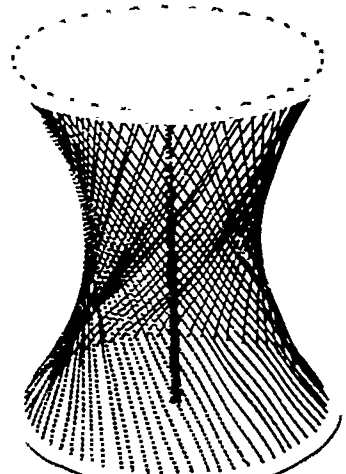
1. *Select the proper aid to use in order to attain the desired objective.* Select wisely from the wealth of materials that is available, or develop precisely the type of item that will suit your purposes best.
2. *Prepare yourself to use the teaching aid that has been selected.* Become familiar with techniques for making effective use of the item that you plan to use. If it is a film, preview it; if it is a dynamic device or model, familiarize yourself with the manipulations upon which its use as a learning aid depends. If preparatory materials such as study guides or descriptive literature are available, consult them.
3. *Prepare the classroom.* Check to insure that all needed materials are on hand and in "running order" before the class meets.
4. *Prepare the class.* Acquaint students with the purpose of the learning aid, what may be expected of it, why it is being used, what they can hope to learn from the experience, and how they should apply the information which they may gain. Often a written learning guide placed in the hands of students increases the effectiveness of the learning experience.
5. *Use the teaching aid selected in the most effective manner possible.* Relate the aid to the lesson in progress and to the symbolic representation that will be used later.
6. *Provide follow-up activities.* Discussions, readings, reports, projects, tests, and repeat performances are needed if maximum learning is to result. Whenever possible, provide opportunity for application of the information learned.
7. *Evaluate the effectiveness of the aid.* On the basis of your experience with the learning aid, it is wise to make a notation regarding its usefulness. A card file system should be employed so that over a period of time you will know what the best aids for each lesson are.

## THE KEY TO EFFECTIVE USE OF MODELS

The key to effective use of models is understanding the connection between a model and the mathematics it depicts.

One teacher uses models naturally, effectively, and with confidence. Another teacher in a similar situation scarcely uses models: and, when he does, his manner is hesitant and unsure. How can this difference be explained? One explanation is that the first teacher understands the connection between a model and the mathematics it depicts and the other doesn't.

FIGURE 8.50

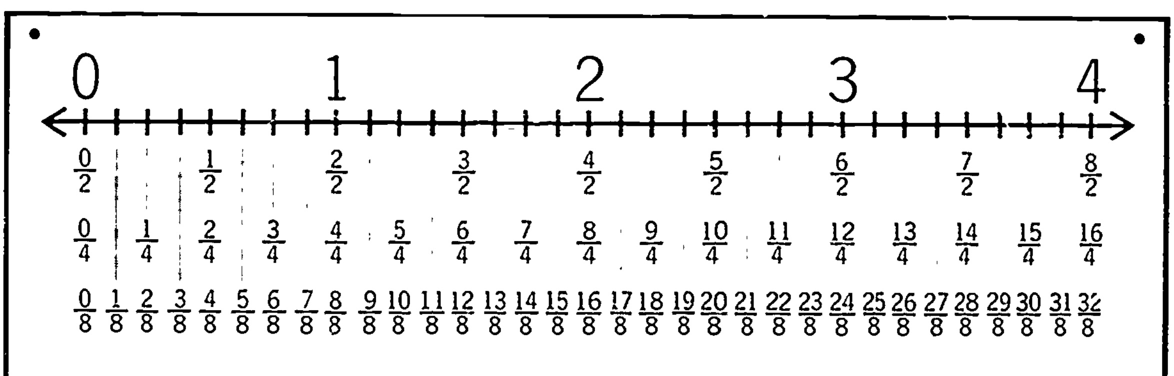


*Courtesy of LaPine Scientific Company*

The successful user of models is aware that mathematics is an abstract subject that makes contact with the real world through physical representations that are approximations of abstract ideas. For example, he views the string model in Figure 8.50 as an approximation of a hyperboloid and the number line in Figure 8.51 as an approximation of a correspondence between a set of points and a set of numbers. He uses these approximations to develop the ideas depicted by the models.

This is similar to the way in which mathematicians create new mathematics. They make successive approximations of abstract ideas, each less concrete than the one before. That is, they develop hierarchies of abstractions by taking successive steps away from reality. Euler's development of topology, described earlier, is an illustration of this principle. Having students arrange the sectors of a circular disc in the form of a "parallelogram" to help them find a formula for the area of a circle is an illustration of the same principle. The principle involved is one of transition from the concrete to the abstract. In the case of the mathematician this transition is usually spontaneous. But it is not always so for students, except perhaps for the very bright. Therefore, the teacher may have to do some prompting to get students to make the desired transition. The teacher who understands the connection between models and the mathematics they depict will recognize when insights occur to individual stu-

FIGURE 8.51

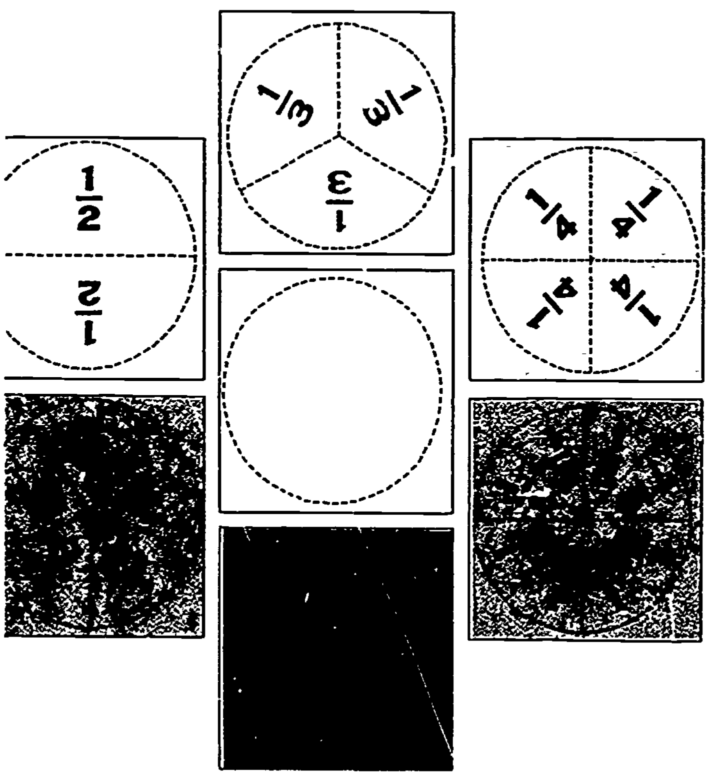



*Courtesy of Ideal School Supply Company*

dents. He will know when to move students away from concrete models and toward the abstract level. He will also sense when a student needs additional work with concrete models in order to reach the abstract level and will make needed provision on an individual basis. Below is a description of a lesson that illustrates what a teacher might do when helping students make the transition from the concrete to the abstract.

A seventh-grade teacher of remedial mathematics has been using paper circle cutouts (Figure 8.52) for several days to help students learn how to add rational numbers. Each student has made his own set of circle cutouts and has used them in solving simple addition problems. Now the students have moved on to a more abstract level and are starting to compute sums by using mathematical sentences. The teacher walks from desk to desk and occasionally asks a student to use cutouts to check his computation. The teacher is spotting weaknesses and allowing regression to the use of concrete models by students who still need this assistance. This technique gives each student the security he needs to make the desired transition at various stages of learning as he moves from the concrete to the abstract.

FIGURE 8.52




*Courtesy of Franklin Publications, Inc.*

## USING GUIDE SHEETS IN CARRYING OUT EXPERIMENTS WITH MODELS

To help the individual student organize his efforts when using a model for experimentation, it often proves fruitful to supply him with a guide sheet. Doing so will usually improve the student's chances of making hoped-for discoveries and formulating generalizations during the period of time that is available for experimentation. Following is a description of a unit that illustrates the technique.

The purpose of this unit is twofold: (1) to lead students to discover and state generalizations for the relationships that exist between measures of angles that intersect a circle and measures of the intercepted arcs, and (2) to have students prove the generalizations they state. The guide sheet presented in this section is organized into seven cases. Each case involves one general way in which an angle can intersect a circle.

The work of the unit can be started with a motivational activity that is designed to set the stage for individual experimentation. One way of starting is to have each student make drawings of all possible ways in which an angle can intersect a circle.

After each student has had time to make drawings of all intersections that occur to him, he should be given an opportunity to compare his set of drawings with a set displayed on a screen with an overhead projector (Figure 8.53).

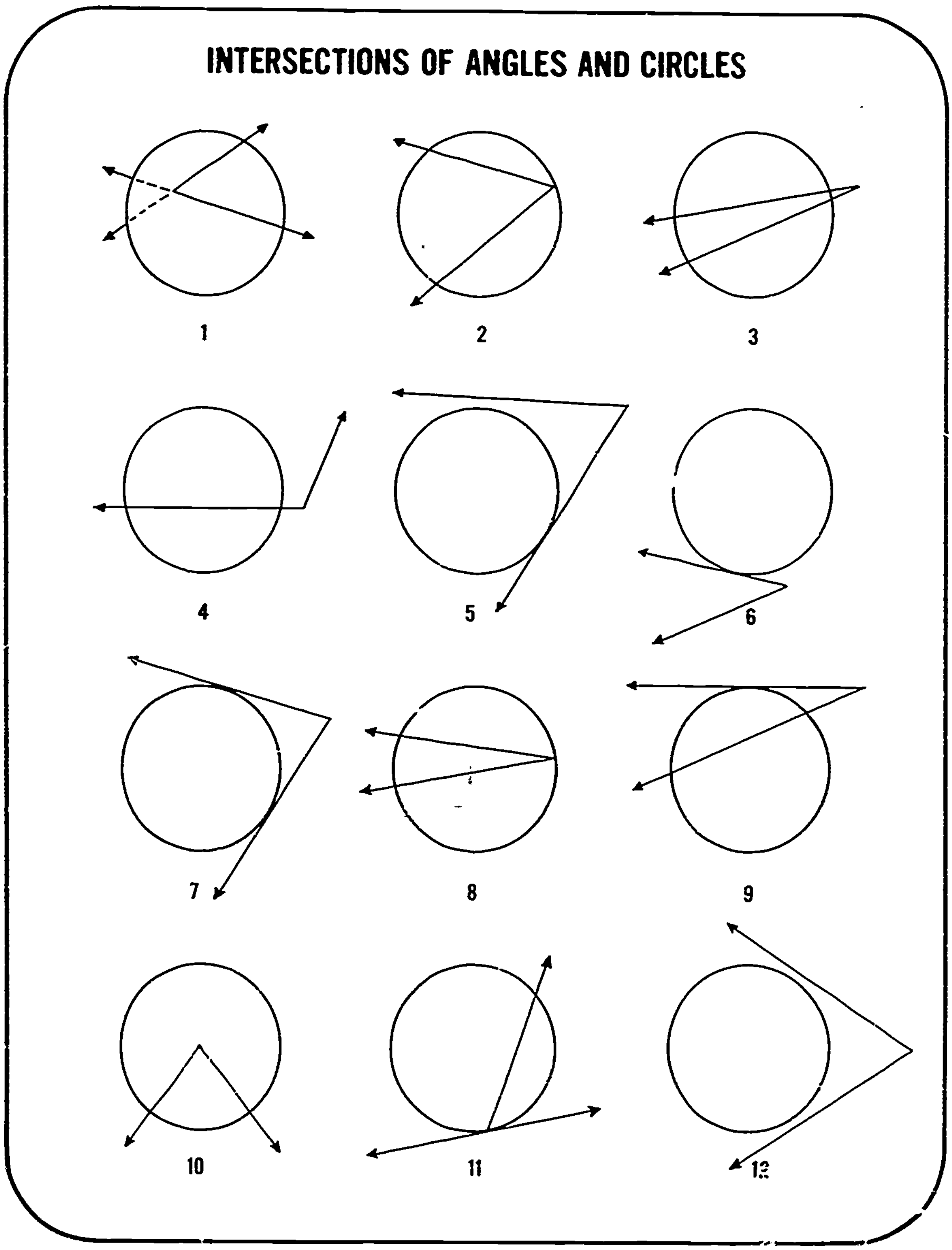


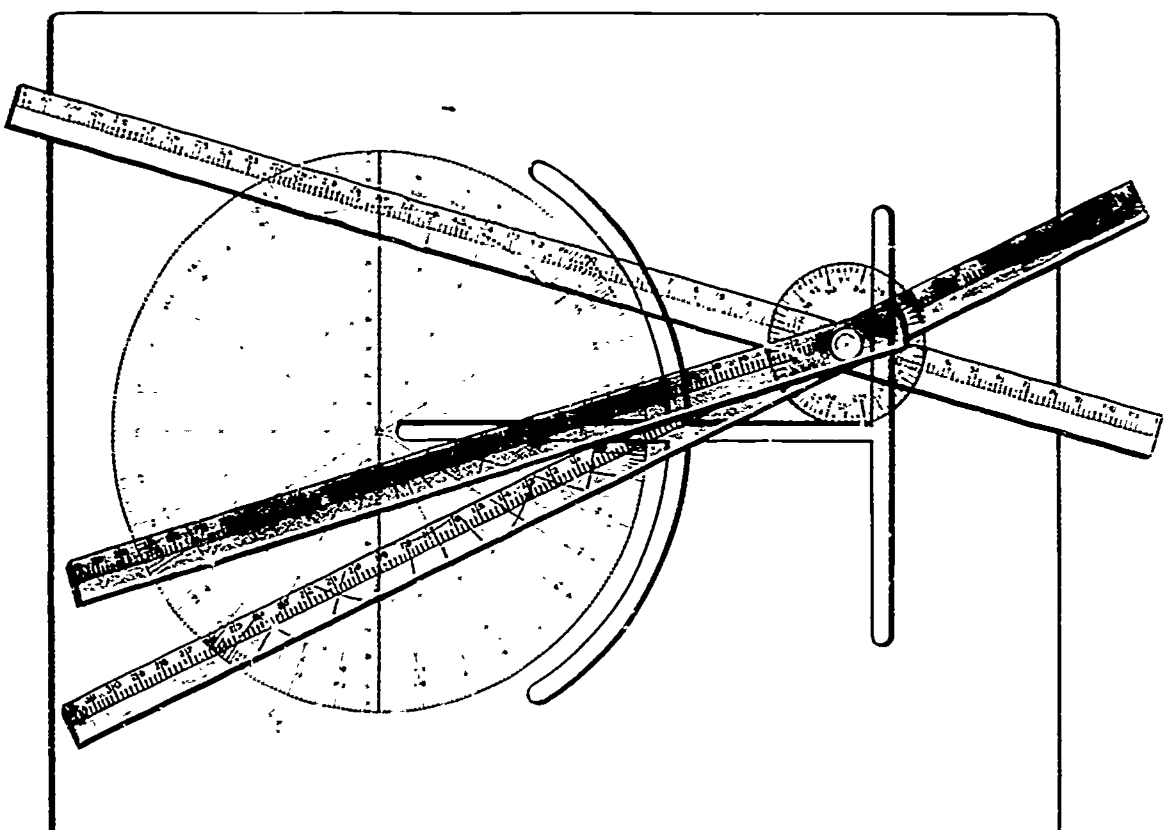

FIGURE 8.53

The overhead visual can be used to clarify what is meant by "an arc intercepted by an angle." Students need to have a clear understanding of what this expression means in order to experiment on their own. Formulating a definition can serve as a summarizing activity for the introductory motivational phase of the unit. (An arc of a circle is intercepted by an angle if and only if each side of the angle contains at least one endpoint of the arc, and the arc, except for its endpoints, is contained in the interior of the angle.)

To give each student an opportunity to experiment on his own, he should be provided with a circle device like the one pictured in Figure 8.54. This device has a circle marked off in degrees of arc for measuring intercepted arcs. Three plastic bars and a small circular protractor are attached to a pin that can be moved to points in the interior of the circle, on the circle, or in the exterior of the circle. The purpose of the protractor is to measure angles formed by the bars. In different positions the bars represent radii, chords, secants, or tangents.

The guide sheet that follows should be available to help the students in making appropriate manipulations with the circle device, making and recording measurements, and formulating generalizations. After the student has formulated a generalization for one of the cases specified in the guide sheet, the teacher should ask him whether there are conditions under which his generalization does not apply. This question calls for a proof, which may be the next step. Of course, there are variations to this procedure. Some teachers prefer to have students complete all experiments outlined in the guide sheet first and then attempt proofs of the generalizations they have formulated.

FIGURE 8.54

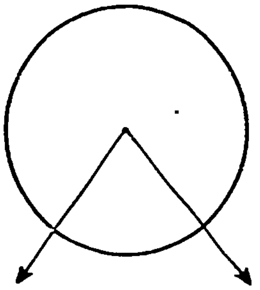


*Courtesy of Sargent-Welch Scientific Company*

## GUIDE SHEET

*Purpose: To find the relationship between the degree measure of an angle that intersects a circle and the degree measures of the arcs intercepted by the angle.*

CASE 1. The angle has its vertex at the center of the circle.



*Directions:*

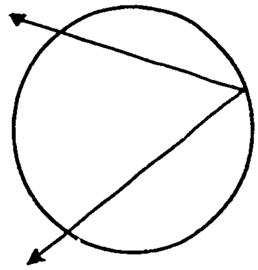
1. Set the device so the bars intersect at the center of the circle.
2. Choose one of the angles formed by a pair of bars and read its degree measure from the small circular protractor. Record this measure in the table below.
3. Read the degree measure of the arc intercepted by the angle you chose. Record this measure in the table below.
4. Repeat steps 1–3 for additional settings.

| Setting | Angle Measure | Arc Measure |
|---|---|---|
| A | 60° | |
| B | 90° | |
| C | 120° | |
| D | 150° | |
| E | 15° | |
| F | | |

*Interpretation:*

1. How does the degree measure of each angle compare with the degree measure of its intercepted arc?
2. Write a sentence describing this relationship.
3. What sources of error may account for minor inconsistencies between your data and the relationship you stated?

CASE 2. The angle has its vertex on the circle.



*Directions:*

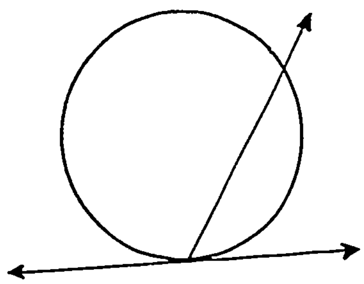
1. Set the device so that the bars intersect at a point on the circle and each bar intersects the circle in another point.
2. Choose one of the angles formed by a pair of bars and read its degree measure from the small circular protractor. Record this measure in the table below.
3. Read the degree measure of the arc intercepted by the angle you chose. Record this measure in the table below.
4. Repeat steps 1–3 for additional settings.

| Setting | Angle Measure | Arc Measure |
|---|---|---|
| A | 15° | |
| B | 30° | |
| C | 60° | |
| D | 90° | |
| E | 120° | |
| F | | |
| G | | |

*Interpretation:*

1. How does the degree measure of each angle compare with the degree measure of its intercepted arc?
2. Write a sentence describing this relationship.
3. Write a formula for the relationship.

CASE 3. The angle is formed by a tangent and a secant. The vertex of the angle is on the circle.



*Directions:*

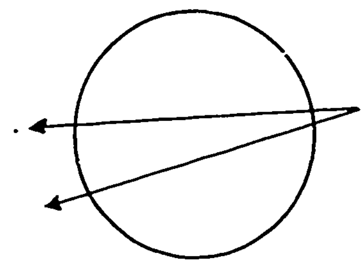
1. Set the device so that the bars intersect on the circle. Use one bar to represent a secant and a second bar to represent a tangent.
2. There are two angles for each setting. Record the degree measure of each angle and the degree measure of its intercepted arc in the table below.
3. Repeat steps 1 and 2 for additional settings.

| Setting | Angle Measure | Arc Measure |
|---|---|---|
| A | 60 | |
| | 120 | |
| B | 150 | |
| | 30 | |
| C | 131 | |
| | 49 | |
| D | 105 | |
| | 75 | |

*Interpretation:*

1. How is the degree measure of each angle related to the degree measure of its intercepted arc?
2. Write a sentence describing this relationship.
3. Write a formula for the relationship.

CASE 4. The angle is formed by two secants. The vertex of the angle is in the exterior of the circle.



*Directions:*

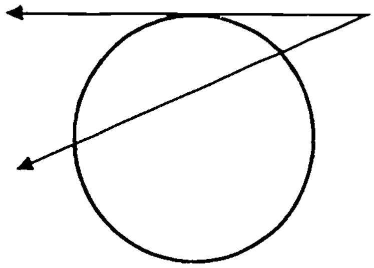
1. Set the device so that the bars intersect in the exterior of the circle. Use two of the bars to represent secants.
2. Record the degree measure of the angle formed by the two secants in the table below.
3. Record the degree measure of the two intercepted arcs in the table below.
4. Repeat steps 1–3 for additional settings.

| Setting | Angle Measure | Smaller Arc Measure | Larger Arc Measure |
|---|---|---|---|
| A | 30° | | |
| B | 50° | | |
| C | 45° | | |
| D | 60° | | |
| E | | | |
| F | | | |

*Interpretation:*

1. How is the degree measure of each angle related to the degree measure of its two intercepted arcs?
2. Write a sentence describing this relationship.
3. Write a formula for the relationship.

CASE 5. The angle is formed by a tangent and a secant. The vertex of the angle is in the exterior of the circle.



*Directions:*

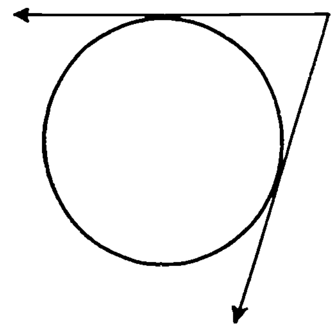
1. Set the device so that the bars intersect in the exterior of the circle. Use one bar to represent a secant and a second bar to represent a tangent.
2. Record the degree measure of the angle formed by the tangent and the secant in the table below.
3. Record the degree measures of the two intercepted arcs in the table below.
4. Repeat steps 1–3 for additional settings.

| Setting | Angle Measure | Smaller Arc Measure | Larger Arc Measure |
|---|---|---|---|
| A | 30° | | |
| B | 50° | | |
| C | 75° | | |
| D | | | |
| E | | | |
| F | | | |

*Interpretation:*

1. How is the degree measure of each angle related to the degree measures of its intercepted arcs?
2. Write a sentence describing this relationship.
3. Write a formula for the relationship.

CASE 6. The angle is formed by two tangents. The vertex of the angle is in the exterior of the circle.



*Directions:*

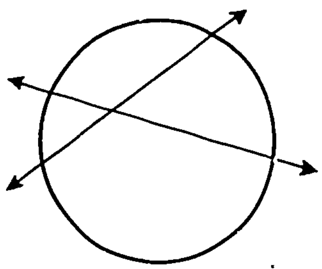
1. Set the device so that the bars intersect in the exterior of the circle. Use two of the bars to represent tangents.
2. Record the degree measure of the angle formed by the two tangents in the table below.
3. Record the degree measures of the two intercepted arcs in the table below. One arc is a major arc and the other is a minor arc.
4. Repeat steps 1–3 for additional settings.

| Setting | Angle Measure | Measure of Major Arc | Measure of Minor Arc |
|---|---|---|---|
| A | 60° | | |
| B | 30° | | |
| C | 15° | | |
| D | 100° | | |
| E | | | |
| F | | | |

*Interpretation:*

1. How is the degree measure of each angle related to the degree measures of its intercepted arcs?
2. Write a sentence describing this relationship.
3. Write a formula for the relationship.

CASE 7. The angle has its vertex at a point in the interior of the circle but not at the center.



*Directions:*

1. Set the device so that the bars intersect in the interior of the circle and each long bar intersects the circle twice.
2. Choose one of the angles formed by the two long bars and read its degree measure from the small circular protractor. Record this measure in the table below.
3. Read the degree measure of the arc intercepted by the angle you chose and the degree measure of the arc intercepted by the opposite angle. Record these two measures in the table below.
4. Repeat steps 1–3 for additional settings.

| Setting | Angle Measure | Measure of One Arc | Measure of Second Arc |
|---|---|---|---|
| A | 60° | | |
| B | 25° | | |
| C | 108° | | |
| D | | | |

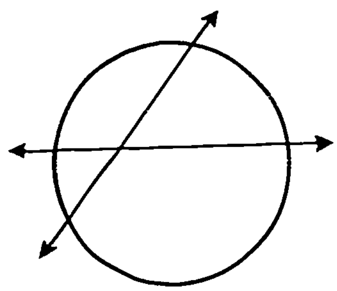
*Interpretation:*

1. How is the degree measure of each angle related to the degree measures of the two intercepted arcs?
2. Write a sentence describing this relationship.
3. Write a formula for the relationship.

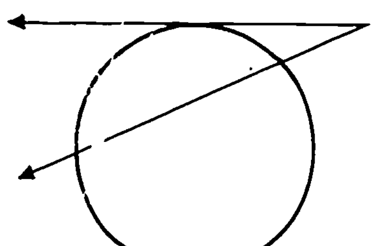
## ANOTHER GUIDE SHEET

Since the bars of the device pictured in Figure 8.51 are marked off in centimeters, the device can be used to help students discover, state, and prove generalizations for relationships concerning products of various segments of two intersecting chords of a circle, of two secants drawn from the same external point, and of a secant and a tangent drawn from the same external point. Accordingly another guide sheet can be organized around the following three cases:

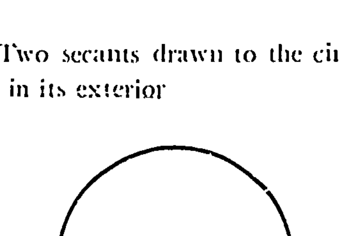
CASE 1. Two chords that intersect in the interior of the circle



CASE 2. A tangent and a secant drawn to the circle from a point in its exterior



CASE 3. Two secants drawn to the circle from a point in its exterior



## STUDENT-MADE MODELS

As has already been indicated several times, making a model, either as part of a laboratory lesson or as an independent project, can be a meaningful learning experience for the student.

To assist and encourage students to make models, space to make them should be available. In addition, an assortment of inexpensive construction materials and simple tools should be provided. The list below represents a minimum of construction materials and tools that should be kept on hand.

### MATERIALS

| | |
|---|---|
| Adhesive tape | Pins—common |
| Adhesive wax | Plaster of Paris |
| Aluminum foil | Plastic chips |
| Balloons | Plastic foam |
| Balsa | Plastic rings |
| Beads | Plastic sheets —clear and colored |
| Bolts and wing nuts —assorted sizes | Plastic tape —assorted colors |
| Cardboard | Plastic tubes |
| Cellophane tape | Plywood |
| Colored chalk | Pocket mirror |
| Colored corrugated cardboard | Pushpins and map pins |
| Construction paper | Rubber binders |
| Cork panels | Rubber cement |
| Crayons | Sandpaper |
| Drapery cord | Soda straws |
| Drawing paper | Spray paint |
| Elastic thread | Stapler and staples |
| Felt—assorted colors | String |
| Fiberboard | Styrofoam balls and blocks |
| Fishline weights | Tackboard |
| Glue | Thumbtacks |
| Golf tees | Tiles—asphalt, ceramic, plastic |
| Graph paper | Title board |
| Letter stencils | Tongue depressors |
| Letters for mounting | Toothpicks |
| Marbles | Wax paper |
| Masking tape | Wire |
| Modeling clay | Wire mesh |
| Nails, brads, tacks | Wooden dowels |
| Paint | Wooden Venetian-blind slats |
| Paper fasteners | Yarn |
| Pegboard and pegs | |
| Pickup sticks | |
| Pill boxes | |

### TOOLS

| | |
|---|---|
| Brush pens | Mechanical drawing kit |
| Carpenter's rule | Metal shears |
| Carpenter's square | Nylon-point pens |
| Centering square | Paint brushes |
| Compasses | Paper cutter |
| Coping saw | Paper punch |
| Drawing board | Plane |
| Eyelet punch | Pliers |
| Gluing clamps | Protractor |
| Hammer | Razor blades and holder |
| Hand drill and assorted bits | Scissors |
| Hand saw | Screwdriver |
| Knife | Soldering iron |
| | Vise |
| | Wood rasp |

As an example of a student-made model consider a linkage in the form of a parallelogram that is made with four strips of stiff cardboard joined by brass paper fasteners (Figure 8.55). The eyelets make each vertex a movable joint; therefore, the model can be used to represent many different parallelograms. With the aid of this model and a protractor, the student can observe that the sum of the measures of the angles of a parallelogram is 360 degrees, that opposite angles have the same measure, and that consecutive angles are supplementary. Using elastic thread to represent diagonals enables the student to see that either diagonal separates the parallelogram into two congruent triangles, and that the diagonals of the parallelogram bisect each other.

Students can test their observations concerning a parallelogram by using a linkage that has the shape of a quadrilateral that is not a parallelogram (Figure 8.56). The sum of the measures of the angles in the figure represented by the new model is again 360 degrees, but opposite angles do not have the same measure, diagonals do not bisect each other, and neither diagonal separates the figure into two congruent triangles. By stitching elastic thread through the midpoints of consecutive sides of the quadrilateral model, students can see that the elastic thread represents a parallelogram regardless of how the linkage is transformed. If students are asked to compare the lengths of the sides of this parallelogram with

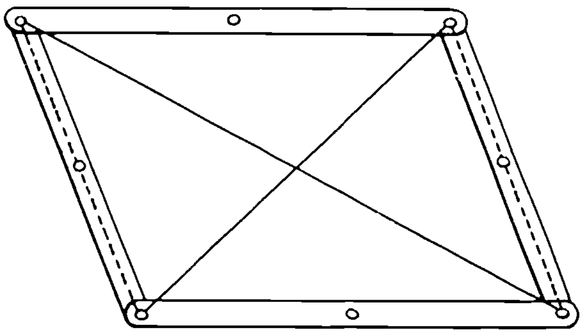


FIGURE 8.55

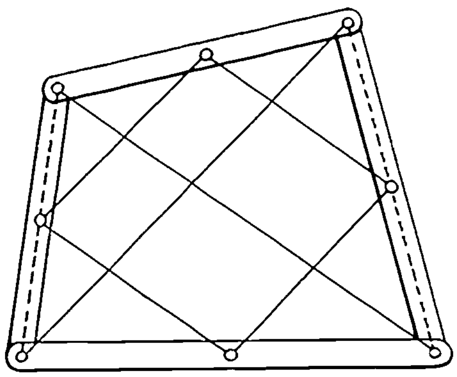


FIGURE 8.56

the diagonals of the quadrilateral, they have an opportunity to discover an important generalization that can be proved as a theorem.

The linkage that has been described is one that every student of geometry should be able to make. Construction of more elaborate models should probably be undertaken as special projects. Such projects can be very rewarding to the student. In designing and constructing a model, the student has an opportunity to add depth to his understanding of ideas with which he is already familiar or to discover ideas that are new to him.

Figure 8.57 pictures a model of the most general kind of parallelepiped. The model in the picture is made of walnut veneer. Each face has the shape of a parallelogram. The student who made this model wanted to be able to show that a plane can be "passed" through a pair of diagonally opposite edges. The model is built so that the representation of the plane through a pair of diagonally opposite edges is removable. The model also has various other mathematically significant features. However, the point we wish to make is this: The design problems encountered by a student who builds a model like this compel him to think about mathematical details that may not have occurred to him before.

The student-made model pictured in Figure 8.58 is made of sheet aluminum. Except for extreme cases, spherical triangles of every shape can be represented with it. The student who constructed this model had only the usual two-dimensional drawings from a geometry textbook to guide him. This model represents a remarkably creative effort. Building it required a great deal of knowledge not only of mathematics but also of metal working.

FIGURE 8.57

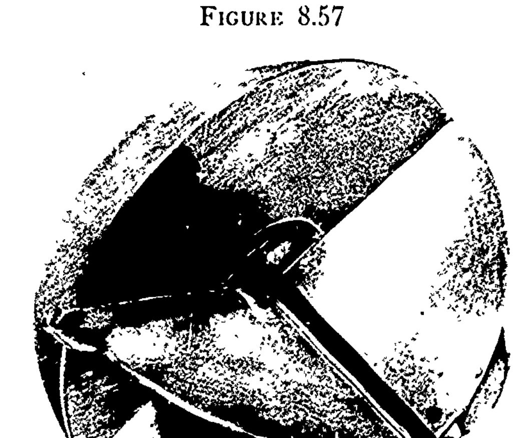


FIGURE 8.58

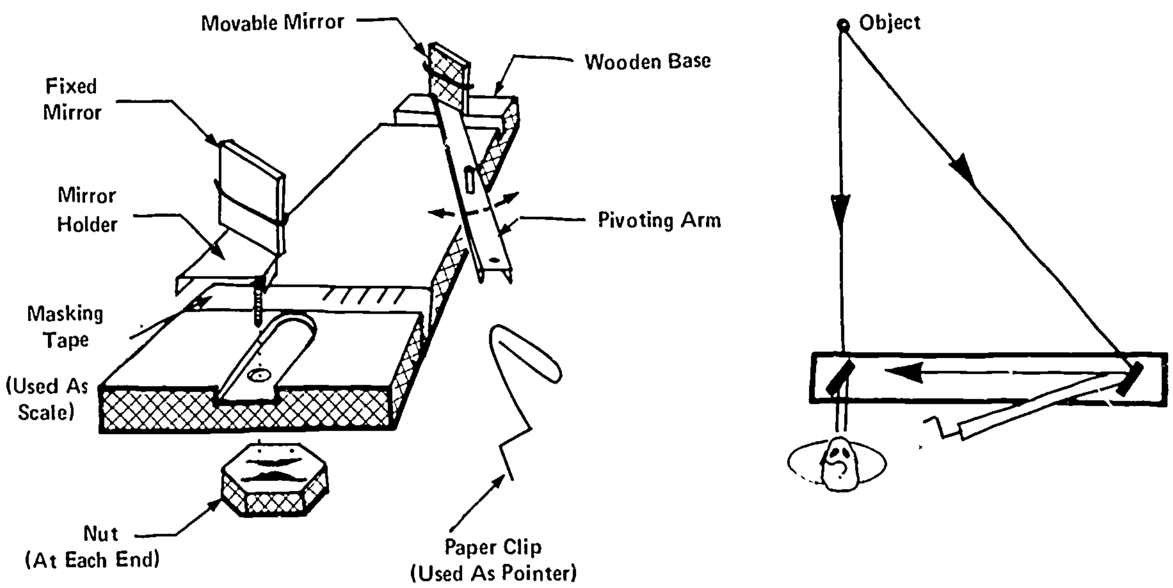

FIGURE 8.59

*Courtesy of LaPine Scientific Company*

The opening sentence of this section (". . . making a model, either as part of a laboratory lesson or as an independent project, can be a meaningful learning experience for the student.") is no empty claim. Building a model *is* a meaningful learning experience for the student. One reason is that students are basically creative. Even when a student lacks the needed skills, he still likes to make things. Fortunately, this characteristic of students has been recognized by producers of instructional aids for mathematics—so there is on the market today a variety of kits containing partially assembled models. Assembling models with these kits gives the student the feeling of accomplishment that comes from creating something. In the process, he can learn important mathematical ideas.

Displayed in Figure 8.59 are some of the contents of the Sage Kit with illustrations showing ways in which the student can use the kit to build models. Accompanying the kit is an instruction booklet that serves a purpose similar to the guide sheet suggested earlier.

Displayed in Figure 8.60 are the contents of a distance-measuring instruments kit. Three devices can be built with this kit—a range finder, a parallax viewer, and an optical micrometer. Figure 8.60 also shows how the range finder can be used to find distances. Building the three devices will give the student the thrill of creating something; using it (especially the range finder) will make him wonder why the device works; and proving why it works will make him think about exciting mathematical ideas.




*Courtesy of Macalester Scientific Company*

FIGURE 8.60. *Distance-measuring instruments kit. In use, an object is viewed over the fixed mirror as shown in the right-hand diagram. The movable mirror is then rotated with the pivoting arm until an image of the object can be seen in the fixed mirror. The image and object are aligned. The instrument is calibrated by marking on a strip of masking tape on the base the positions of the paper-clip pointer for objects of known distances.*

We conclude this section by mentioning one other way in which teachers can encourage students to build models. This is really a variation of the kit idea. Supplying students with a scale drawing of a pattern for a model, or with the pattern itself, will often be sufficient to get them started.

The model that can be made with the patterns displayed in Figure 8.61 is a three-piece dissectible triangular right prism.[3] A suitable construction material is cardboard. After cutting out the patterns, the cardboard should be folded along the dashed lines. Edges may be joined with tape. When the three pieces of the completed model are placed together so that vertices with the same letter are together, the resulting model should look like the one in Figure 8.11.

3. This set of diagrams is an adaptation of those shown in *Mathematics for Elementary School Teachers*, by Helen L. Garstens and Stanley B. Jackson (20, pp. 467–71).

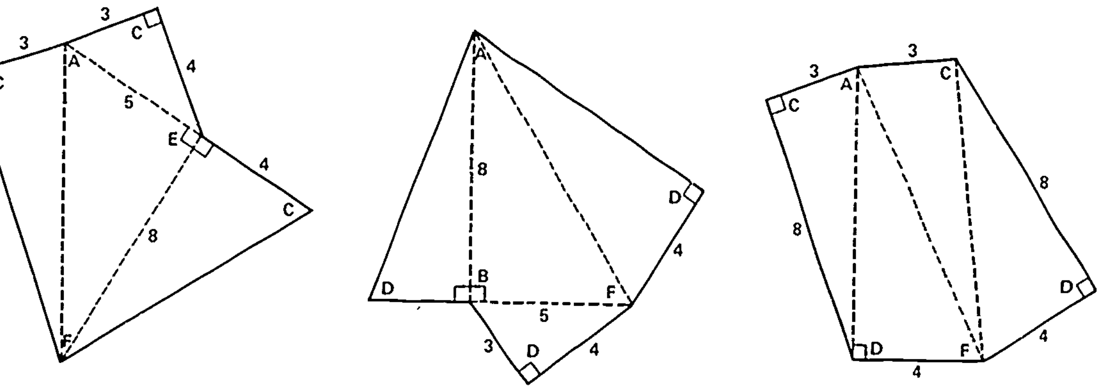


FIGURE 8.61

## BUILDING A MODEL COLLECTION

A model collection is not born. It simply grows. How does a model collection start? It usually starts spontaneously—when the teacher discovers some object that is useful as a model and decides to keep it.

One way of making a model collection grow is nourishing it with useful realia such as exciting-looking boxes, pieces of pipe, lampshades, and so on. Adding student-made models is another way of making a model collection grow. Students are usually pleased to be asked to contribute their models. A word of caution is in order here, however. Having students make models simply to enlarge a school's collection is not recommended as an end in itself. There are several reasons for this. First of all, the purpose of having students make models is not always served by having them concentrate on the sort of craftsmanship looked for in a model that is to be put in a school's collection. Secondly, students cannot be expected to produce the variety of models that a school needs. Finally, students (being students) are ordinarily not able to produce many of the models that they themselves need in order to learn mathematics.

Collecting realia and accumulating student-made models are helpful schemes for building a model collection. However, neither scheme, nor even both together, are adequate means. Therefore, the purchase of commercially produced models should be anticipated as the most reliable way of building a model collection for a school. It is best to begin by acquiring a few basic models that are versatile and sturdy and which serve purposes that cannot be met by using pictures from a book, chalkboard diagrams, objects in the classroom, or student-made models. Once started, the collection can be enlarged year by year.

Before purchasing a model the teacher should ask himse ' the nine major questions listed below. For a penetrating appraisal, the subquestions might be considered.

1. For what purpose can this model be used?
   *a*) Can it be used to motivate learning?
   *b*) Can it be used to awaken student interest?
   *c*) Can it be used to enrich learning?
   *d*) Can it be used to help students discover a concept?
   *e*) Can it be used to help clarify an idea?
   *f*) Can it be used to speed up communication?
   *g*) Can it be used to teach an algorithm?
   *h*) Can it be used to help students solve a problem?
   *i*) Can it be used for practicing a skill?
   *j*) Can it be used to help students grasp a spatial relationship?
   *k*) Can it be used to help students make transitions from the concrete to the abstract?
   *l*) Can it be used for a recreational activity?
   *m*) Can it be used to illustrate application of a mathematical idea?
2. In what kinds of instructional arrangements is this model to be used?
   *a*) As a demonstration aid with large or medium-sized class groups?
   *b*) For laboratory work?
   *c*) For independent investigation by students?
   *d*) For displays or exhibits?
3. With what subjects in mathematics can this model be used?
4. Is the mathematical idea that this model depicts significant?
5. Is the construction of this model faithful to the mathematical idea it is supposed to depict?
6. Is this model large enough so that details pertinent to the idea depicted are clearly visible?
7. Is this model well designed and well constructed?
8. Is this model sturdy enough to be handled by students?
9. Can this model be used to do something that cannot be done as well or better in some other way or with some other instructional aid?

It is unlikely that any model you purchase will have all the qualities you wish it to have. Hence your selection must be tempered by your

judgment of the total contribution it is likely to make to your instruction.

We have left until last the matters of cost and storage. Cost should be commensurate with the use expected of a model. If a model is versatile and likely to be used a great deal, then a sizable expenditure of money would be justifiable. However, if a model is going to be used only once, extravagance should be avoided. Models that depict the same idea are available today at different levels of cost and quality. The expected use of a model should be a determining factor in making a proper choice.

In building a model collection for a school, it should be kept in mind that models need to be stored when they are not being used. In a large school, models can be stored in a mathematics resource center. In a small school, it is likely that the entire model collection of the school will have to be stored in a single room. The roller-mounted combination storage cabinet and demonstration table pictured in Figure 8.62 is one example of a piece of equipment that can be used for this purpose.

## WHAT MODELS ARE AVAILABLE?

To use models effectively, a teacher needs to be aware of what models are available. To keep informed about the availability of new models it is helpful to read professional journals in which articles about models and advertisements for commercial models frequently appear. Visiting exhibits of manufacturers and distributors of models at conferences and maintaining an up-to-date catalog file are also helpful.

To assist teachers in selecting and purchasing models, a compilation of the usual catalog names of models considered useful as instructional aids

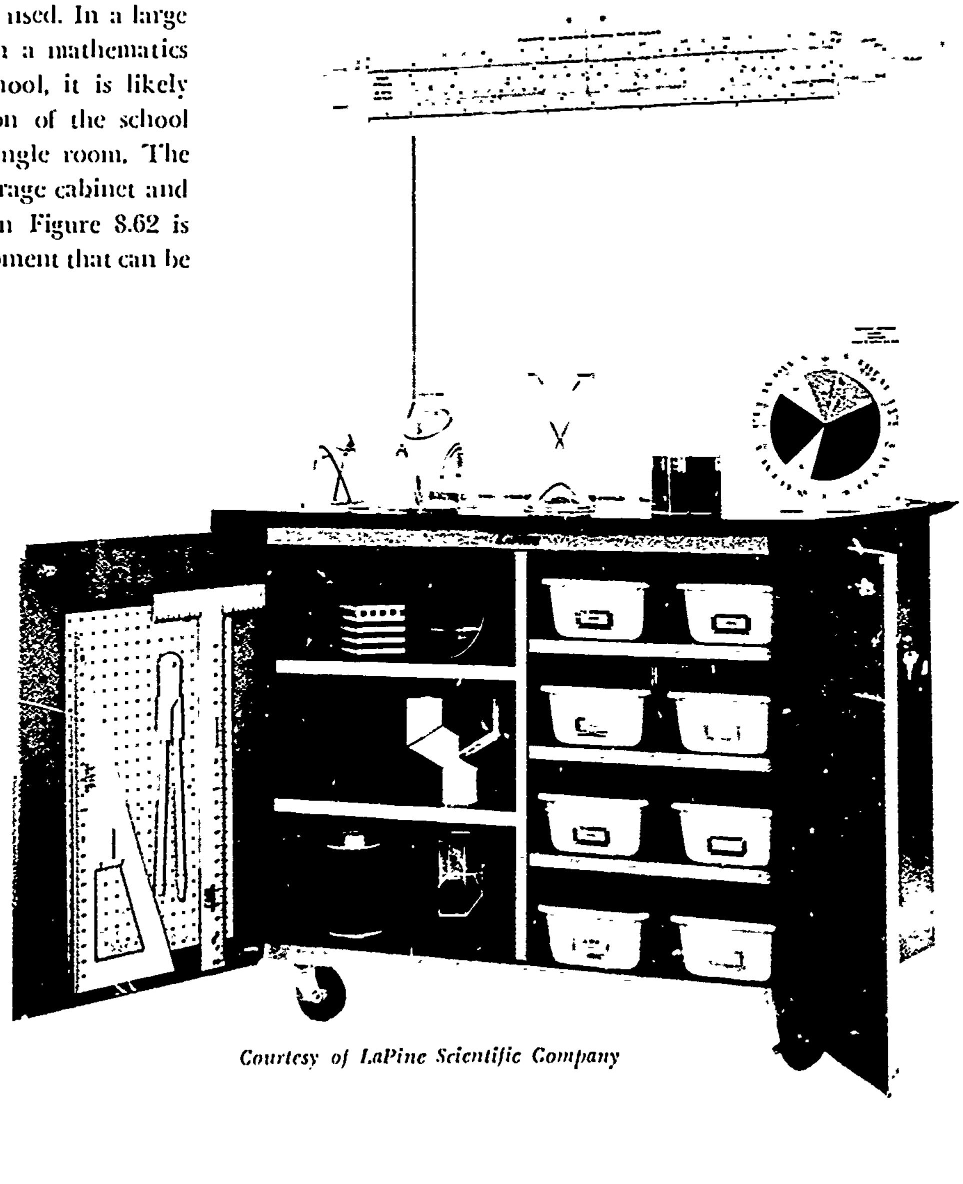


FIGURE 8.62

*Courtesy of LaPine Scientific Company*

in mathematics is presented at the end of this chapter. This compilation is part of a larger list prepared by a class in mathematics education at the University of Minnesota. Members of the class examined more than 1,000 distributors' catalogs in preparing their list.[1]

The compilation that appears at the end of this chapter is organized into twenty categories according to the mathematical topics and concepts shown in the box. A model considered appropriate for use with more than one topic is listed under each topic for which it is suitable. In addition, the compilation classifies each model according to the eight types outlined at the beginning of the chapter (demonstration devices, manipulative devices, and so forth) and lists one or more distributors for each model.

MATHEMATICAL TOPICS AND CONCEPTS

1. Sets
2. Numeration Systems, Number Systems
3. Plane and Solid Geometry
4. Coordinate Geometry
5. Measurement
6. Probability and Statistics
7. Formulas
8. Logic
9. Ratio, Proportion, Percentage
10. Operations and Their Properties
11. Open Sentences
12. Graphs
13. Relations and Functions
14. Logarithms and Exponents
15. Trigonometry
16. Linear Algebra
17. Topology
18. Transformations
19. Calculus
20. Computer Concepts

1. Catalogs used were obtained from the Saint Paul Public School's Mathematics Resource Center, which keeps an up-to-date file of catalogs of distributors. Helen Hughes of Hudson, Wisconsin, is responsible for compiling the list in a format that is accessible for use by teachers.

## WHAT IS THE FUTURE OF MODELS?

The number of commercial models available for teachers of mathematics has increased dramatically in recent years, and indications are that an ever-increasing supply will be available in the future.

The necessity for more students to learn more mathematics more efficiently has been the cause of numerous experimental programs. One of the chief concerns of these programs is individualization of instruction. To this end many of the programs emphasize discovery and laboratory approaches. Models play an important role in such approaches.

Programs for students in inner-city schools, for students with special learning disabilities, and for culturally deprived students stress the importance of providing nonverbal means of communication. Such communication is an important function of models.

Recently mathematics textbook publishers have begun producing materials packages to accompany the textbooks they publish. It seems almost certain that this practice will lead to expanded use of models by teachers of mathematics.

Finally, continued government support for innovation in education will undoubtedly provide continued impetus for the development of more new models and for wider use of models.

## SUMMARY

In this chapter we have described the role of models in teaching and learning mathematics, made various suggestions for using models effectively, and attempted to give some guidance in the selection and procurement of models. If you are not now using models in your teaching, we trust that you will in the future. And if you do, we are confident that you, like thousands of other teachers of mathematics, will rarely want to teach without them.

## MODELS FOR TWENTY MATHEMATICAL TOPICS AND CONCEPTS

Models are classified in two ways in this compilation: (1) under the topics and concepts for which they are considered useful as instructional aids and (2) according to type as identified by the code system below. Listed with each model is the name of one or more distributors. The full names and addresses of distributors are listed in the Appendix.

| | |
|---|---|
| D—Demonstration Devices | K—Kits |
| M—Manipulative Devices | G—Games and Puzzles |
| C—Computational Devices | S—Science Apparatus |
| I— Instruments for Measuring and Drawing | R—Realia |

### 1. SETS

| | | |
|---|---|---|
| Array Board | D M | Science Research Associates |
| Electronic Logic and Sets Demonstrator | D | Lano |
| Hold-on Set Maker | D M | Weber Costello |
| Minne-Bars | M | Judy |
| Modern Math Demonstration Board | D M | Gamco |
| Modern Math Symbols | D M | Gamco, Geyer, NASCO, Science Research Associates |
| New Math Charts | D | Instructor, NASCO, Nystrom, Weber Costello |
| On Sets (game of set theory) | G | WFF 'N PROOF |
| Series of Wall Charts | D | Walker Educational Book |

### 2. NUMERATION SYSTEMS AND NUMBER SYSTEMS

| | | |
|---|---|---|
| Abacus (13, 15, and 23 reeds) | D M C | Tuttle |
| Base Blocks | M | Ideal |
| Base Conversion Chart | D | Ideal |
| Base-Five Place Value Chart | D C | Ideal, NASCO |
| Base-Ten Counting Frame | M | Science Research Associates, Weber Costello |
| Binary Counter | D M C | Sargent-Welch |
| Binary System Fact Cards and Slide Rules | M | Bowmar |
| Chinese Abacus | M C | Sargent-Welch |
| Circles for Calculating Fractions | D M C | Franklin, Freyer |
| Direct Reading Ball Abacus | D M C | TRI-TIX |
| Expanded Notation Cards | D | Ideal, NASCO |
| Fraction Chart | D M | Exton, Ideal |
| Fraction Finder | M | AIM |
| Fraction Inlay Boards (squares, circles) | D M | Judy |
| Fraction Kit (teacher and pupil) | D M | Ideal |
| Fractional Parts (squares, circles) | D M | Bowmar, Ideal, Instructo, Weber Costello |
| Giant Dominoes | M C | NASCO |

2. NUMERATION SYSTEMS AND NUMBER SYSTEMS (Continued)

| | | |
|---|---|---|
| Hundred Chart/Board | D M | Ideal. Instructo, Judy, NASCO |
| Integer Slide Rule | M C | Bowmar |
| Kinesthetic Numeral Cards | M | Instructo |
| Magnetic Numb-R-Ods | D M C | Yoder |
| Magnetic Numerals | D M | Instructo |
| Modern Computing Abacus | D M C | Ideal |
| Multiple Base Abacus | D M C | Weber Costello |
| New Math Charts | D C | Instructo. Instructor, Nystrom, Weber Costello |
| Number Line Rule | M | Sigma |
| Number Lines | D | Bowmar, Houghton Mifflin, Ideal, Instructo, Instructor, Judy. NASCO, Weber Costello |
| Number and Picture Dominoes | M | Weber Costello |
| Number Relation Blocks | M | NASCO, Weber Costello |
| Open End Abacus Kit | D M C K | Houghton Mifflin |
| Place Value Board | D M | Ideal, Instructo, NASCO |
| Place Value Charts (student and teacher) | D M C | Ideal, Instructo, NASCO. Weber Costello |
| Place Value Indicator | D M | Bradley |
| Place Value Tab Rack | D M | Judy |
| Projectable Abacus | D C | Tweedy |
| Rational Number Line (printed chart and pegboard) | D | Ideal |
| Real Number Game | G | WFF 'N PROOF |
| Square-Root Chart | D | Geyer |

3. PLANE AND SOLID GEOMETRY

| | | |
|---|---|---|
| Angle Prism | I S | Cenco |
| Applied Geometry Experiments Kit | M K | LaPine |
| Archimedes' Theorem Model (cylinder, hemisphere, and cone for filling) | D | Viking |
| Beam compasses | M I | Charvoz, Lietz, Math Master, Post |
| Burns Boards | D M | Ideal, NASCO |
| Cavalieri's Theorem (cylinder and oblique prism set. cone-cylinder-hemisphere set, pyramids set) | D | LaPine |
| Chalkboard Beam Compasses | D I | Sargent-Welch |
| Chalkboard Drawing Instruments (T-square, compasses, ruler, protractor) | D M I | Bowmar, Geyer, Ideal, Lano, LaPine, Lietz, Math Master, Math-U-Matic, NASCO, Post. Sargent-Welch, Weber Costello, Yoder |
| Chalkboard Drawing Instruments and Templates | D I | LaPine. Post, Yoder |
| Chalkboard Graph Charts (rectangular and polar) | D | Geyer, LaPine, Math Master, Math-U-Matic, Sargent-Welch, Weber Costello |

| | | |
|---|---|---|
| Chalkboard Graph Panels | D | Claridge, LaPine, Math Master, Math-U-Matic, Weber Costello |
| Chalkboard Graph Stencils (rectangular and polar) | D | Lano, LaPine, Math Master, Sargent-Welch |
| Clear Transparent Markable Globes | D | Farquhar |
| Coaxial Cylinders Intersected by an Axial Plane | D | Sargent-Welch |
| Cone with Circular Section | D | Sargent-Welch |
| Cones and Conic Sections | D M | Lano, LaPine, Sargent-Welch |
| Cones and Cylinders | D M | LaPine, Math Master |
| Cone with Parabolic Section and Dandelin's Sphere | D | LaPine, Viking |
| Cone of Two Nappes with Hyperbolic Section | D | LaPine, Sargent-Welch, Viking |
| Cone of Two Nappes with Hyperbolic Section and Two Spheres of Contact | D | LaPine, Viking |
| Configurations | G | WFF 'N PROOF |
| Crystal Models | D M | Sargent-Welch, Science Related Materials, Viking |
| Cube and Pyramid | D | Sargent-Welch |
| Dandelin's Cone | D | LaPine, Sargent-Welch, Viking |
| Demonstration Conic Section Curves Scaled for Use with a Pegboard | D | Lano |
| Dissectible Binomial Cube | D | Lano, LaPine, Sargent-Welch, Science Related Materials |
| Dissectible Prism Set | D M | LaPine |
| Dissectible Wood Cone | D M | Sargent-Welch |
| Eight-Octants Model | D | Viking |
| Ellipsoid | D | Viking |
| Fillable Hollow Plastic Geometrical Bodies (cube, sphere, cones, cylinders, prisms, pyramids) | D | Viking |
| Flexible and Adjustable Polygons | D M I | Geyer, LaPine, Sargent-Welch, Vis-X |
| Flexible Plastic Templates Showing Areas of Circular Cylinder, Cone, Frustum of Cone | D | LaPine |
| Fold-Out Lateral Area Set | D M | LaPine, Math Master |
| Fundamental Locus Demonstration Kit | D M K | Viking |
| Geoboard (square and circle) | M | Cuisenaire, Sigma |
| Geodestix Kit | K | Geodestix |
| Geometric Models Construction Kit | K | Bradley |
| Geometric Solids and Surfaces (39-piece set, wood) | D M | Yoder |
| Geometric Solids (large plastic) | D M | Ideal |
| Geometric Transformation Animation Boards | D | LaPine |
| Geometry Charts | D | Ideal, Instructor, Judy, Viking |
| Geometry Charts (elementary) | D | Ideal, Walker Educational Book |

3. PLANE AND SOLID GEOMETRY (Continued)

| | | |
|---|---|---|
| Geometry of the Circle Devices (measurement of angles formed by chords, tangents, secants; ratio of segments of chords, secants, etc.) | D M I | LaPine, Sargent-Welch |
| Hemisphere, Cylinder and Cone | D | LaPine |
| Hemisphere with Spherical Pyramid | D | Sargent-Welch |
| Hemisphere with Spherical Sector | D | Sargent-Welch |
| Hinged Layout Model of Cube and Pyramid | D | LaPine |
| Hyperboloid | D | Viking |
| Instruments for Determining $\pi$ | D M I | Geyer, LaPine, Sargent-Welch, Bowmar |
| Inverse Square Law Demonstrator | D | Sargent-Welch |
| Jigged Geometric Shapes | D M | Weber Costello |
| Locus Construction Devices (hyperbola, parabola, ellipse) | D M | Geyer, Lano, LaPine, Math Master, Yoder |
| Locus Kit | D | STAS |
| Mathematics Design Maker | G I | NASCO |
| Miniature Geometric Solids (27-piece set, solid plastic) | M | Edmund |
| Moby Lynx Construction Kit | K | Geodestix, Kendrey |
| Models of Polyhedra | D | Cenco, LaPine, Math Master, Sargent-Welch, Science Related Materials, Viking, Vis-X |
| Multi-Model Geometric Construction Kit | D K | Yoder |
| Multi-Model Geometric Construction Set | D M | Yoder |
| Multi-Purpose Pegboard | D | Bradley, Ideal, Lano, Yoder |
| Oblique Pyramids | D M | LaPine, Math Master |
| Pantograph | D M I | Charvoz, Geyer, Lano, Lietz, Math Master, Sargent-Welch, Yoder |
| Paraboloid | D | Viking |
| Parallel Rulers | D M I | Charvoz, Geyer, Math Master, Sargent-Welch, Yoder |
| Perimeter and Area Demonstration Device | D M | Ideal, NASCO |
| Perspectograph | M I | Yoder |
| Piometer | I | Geyer |
| Plane Figures Set | D | Bowmar, Lano, Math Master, Vis-X |
| Plane and Solid Geometry Charts | D | Viking |
| Poleidoblocs | M | Math Media |
| Polyhedra Construction Kit | D M K | Geodestix, Hartley, Lano, Math Master, Science Related Materials |
| Polyhedron Construction Kit | D M K | Lano, Math Master |
| Prism and Pyramid Sets | D | Lano, LaPine |
| Pythagorean Theorem Demonstration Device | D M | Bowmar, Cenco, Lano, LaPine, Math Master, Sargent-Welch, STAS, Viking |
| Regular Polygons Set | D | Math Master |
| Rotation of Plane Figures Devices | D | LaPine |
| Sectioned Circular Cylinder | D | Sargent-Welch |

| | | |
|---|---|---|
| Sectioned Prisms (four- and six-sided) | D | Sargent-Welch |
| Segmented Sphere | D M | Sargent-Welch |
| Single Octant Model | D | Viking |
| Slated Globe | D M | Geyer, LaPine, Math Master, NASCO, Nystrom, Sargent-Welch, Yoder |
| Solid Geometrical Figures (wood) | D M | Ideal, LaPine, Sargent-Welch, Viking |
| Solids-In-Solids | D | LaPine |
| Solid Shapes Lab | K | NASCO |
| Soma Cubes | G | Parker |
| Space Geometry Lab Kit | K | STAS |
| Space Spider Kit | D K | Cooper, Math Master, NASCO |
| Speed-Up Geometry Blackboard Stencils | D I | Speed-Up, Yoder |
| Sphere, Cone, and Cylinder Set (metal) | D | Sargent-Welch |
| Sphere with Great Circle Sections Showing L'Huilier's Triangle | D | Lano, LaPine, Sargent-Welch, Viking |
| Spheres with Removable Lunes, Segments, Spherical Cones | D M | LaPine |
| Spherical Triangle | D M | LaPine, Sargent-Welch |
| Spirograph | I K | Kenner |
| Student Drawing Instruments | M I | Charvoz, C-Thru, Geyer, Lietz, Math Master, NASCO, Post, Sargent-Welch, Sterling, Yoder |
| Student Geometry Drawing Template | M I | Academic Industries, Speed-Up, Yoder |
| Three-Dimension Graphing Device | D | Science Related Materials, Viking |
| Transparent Cones and Conic Sections | D | Lano, LaPine |
| Transparent Models of Geometric Solids with Sections | D M | LaPine |
| Trisectible Triangular Prism | D | Sargent-Welch |
| Twixt | G | Minnesota Mining & Mfg. Co. (3M) |
| Variable Cylinder-Cone String Device | D M | LaPine |
| Volume Demonstration Set | D M I | Lano. NASCO, Viking |

4. COORDINATE GEOMETRY

| | | |
|---|---|---|
| Chalkboard Drawing Aids | D M I | Lano, LaPine, Lietz, Math Master, NASCO, Post, Sargent-Welch, Yoder |
| Chalkboard Graph Chart Stencils (rectangular and polar coordinates) | D | Cram, Daintee, Math Master, Sargent-Welch |
| Chalkboard Graph Charts (rectangular and polar coordinates) | D | Claridge, Cram, Denoyer-Geppert, Geyer, Lano, LaPine, Math Master, Math-U-Matic, Nystrom, Sargent-Welch, Weber Costello, Yoder |
| Chalkboard Liner | D I | Math Master, Viking |
| Dissectible Right Circular Cone (sections—circle, ellipse, parabola, and hyperbola) | D M | Lano, Sargent-Welch, Viking |
| Eight-Octants Graphing Model | D M | Viking |
| Ellipsograph | D I | LaPine |
| Geoboard (square and circle) | D M | Cuisenaire |
| Geometric Transformation Boards | D | Günter Herrmann, LaPine |
| Graph Stamps (rubber) | I M | Geyer, LaPine, Math Master |

4. COORDINATE GEOMETRY (Continued)

| | | |
|---|---|---|
| Hyperbolic Paraboloid (string model) | D M | Math Master |
| Hypocycloid and Cycloid Models | D | Geyer, LaPine |
| Logarithmic Graph Chart | D | Denoyer-Geppert |
| Map Projection Models | D | Farquhar, Math Master |
| Moto Math Demonstration Device | D I | Yoder |
| Movable Vector Adder | D | Lano |
| Multi-Purpose Graph Board | D | Lano. Yoder |
| Pantograph | D M I | Charvoz, Lano, Math Master, Sargent-Welch, Yoder |
| Parallel Rulers | D M I | Charvoz, Math Master, Sargent-Welch, Weems and Plath, Yoder |
| Pegboard Grid | D M | Math Master, Weber Costello, Yoder |
| Radian Protractor | M I | Math Master |
| Slated Globe | D M | Cram, Denoyer-Geppert, Math Master, NASCO. Sargent-Welch, Yoder |
| Solids of Revolution (ellipsoids, hyperboloids, paraboloids) | D M | LaPine, Viking |
| Templates for Drawing Conic Sections | D I | Charvoz, Lano |
| Templates for Drawing Ellipses | I | Charvoz, Post, Yoder |
| Three-Dimension Graphing Device | D M | Science Related Materials, Viking |
| Variable Cylinder-Cone String Device | D M | LaPine |

5. MEASUREMENT (DISTANCE, AREA, VOLUME, ANGLE MEASURE, ETC.)

| | | |
|---|---|---|
| Arc Measurement Device | D M | Sargent-Welch, Vis-X |
| Area Kit | K | School Service |
| Assorted Equal Volume Solids (1 cubic inch each) | D M | Gamco, Houghton Mifflin |
| Blocks and Trays | M | Bradley |
| Board Foot | D | Vis-X |
| Bubble Sextant | I | Geyer, LaPine, Sargent-Welch, Weems and Plath, Yoder |
| Calipers (vernier, micrometer, regular) | D M I | LaPine, Sargent-Welch, Yoder |
| Centimeter Grid Paper | C | Houghton Mifflin |
| Centimeter Number Line | I | Houghton Mifflin |
| Chain Tape | I | Post, Yoder |
| Chalkboard Beam Compasses | I | Sargent-Welch, Speed-Up, Yoder |
| Chalkboard Drawing Instruments | D M I | Bowmar, Ideal, Lano, LaPine, NASCO, Post, Sargent-Welch, Yoder |
| Circle (30-inch circumference) | D | Vis-X |
| Cone, Sphere, Cylinder Set (metal) | D | Sargent-Welch |
| Cube and Pyramid Volume Set | D | NASCO, Sargent-Welch |
| Cubic Foot | D | NASCO, Sargent-Welch, Science Related Materials, Vis-X |
| Cylindrical Cup | I | Sargent-Welch |
| Demonstration Micrometer Caliper | D | Sargent-Welch |
| Demonstration Vernier Caliper | D | Sargent-Welch |
| Dissectible Cubic Foot | D M | Ideal |

| | | |
|---|---|---|
| Dissectible Liter Block | D M S | Cenco, Lano, LaPine, NASCO, Sargent-Welch |
| Dry Measure Set (pint, quart, peck, bushel) | D | Ideal, NASCO |
| Fillable Hemisphere, Cylinder and Cone Set | D M | Cenco, LaPine |
| Geometry Demonstration Set for Calculating Area and Volume | M K C | Viking |
| Gravity Protractor | D I | Geyer, Lietz, Sargent-Welch, Yoder |
| Inscribed Angle Measurement Device | D M | Cenco, LaPine, Sargent-Welch |
| Isometric Protractor | I | GraphiCraft |
| Leveling Rod with Target | I | Berger, Cenco, Geyer, Post, Sargent-Welch, Yoder |
| Liquid Measure Set (gallon, quart, pint, gill) | D | Ideal |
| Map Measures | I | Lietz |
| Measuring Disk (for determining $\pi$) | M I | Sargent-Welch |
| Measuring Tape | I | Post, Sargent-Welch, Yoder |
| Meter-Yard Comparison Chart | D | Bradley |
| Metric System Chart | D | Sargent-Welch |
| Model for Demonstrating $A = \pi ab$ (area of ellipse) | D | Viking |
| Number Line Ruler | M | Sigma |
| One-Inch Cubes and Half-Inch Cubes | D M | Vis-X |
| One Square Yard | D | Ideal, NASCO |
| Pattern Dial for Drawing Regular Polygons | M I | Sargent-Welch |
| Perimeter and Area Board | D M | Ideal, NASCO |
| Piometer | I | Geyer |
| Planimeter | I | Charvoz, Sargent-Welch, Yoder |
| Pocket Transit | M I S | Post, Sargent-Welch |
| Projection Vernier | D M | Sargent-Welch |
| Proportional Dividers | M I | Post, Sargent-Welch, Yoder |
| Protractor (radian measure) | M I | Math Master |
| Radian and Circle Demonstrator | D M | Sargent-Welch |
| Range Finder, Parallax Viewer and Optical Micrometer | M I | Sargent-Welch |
| Rectangular Geoboard | D M | Freyer |
| Rolatape | I | Post, Rolatape |
| Semicircular Blackboard Protractor | D I | Geyer, Post, Speed-Up |
| Set of Boxes with Different Volumes | D | Ideal, NASCO, Science Related Materials |
| Sextant (student) | I | Cenco, Sargent-Welch, Yoder |
| Spherometer | I | Sargent-Welch |
| Spring Scale | D I | NASCO, Sargent-Welch, STAS |
| Thermometer (centigrade or Fahrenheit) | I S | NASCO, Sargent-Welch, STAS |
| Transit Level with Tripod | I | Berger, Cenco, Geyer, LaPine, Post, Sargent-Welch, Yoder |
| Trip Balance Scale | S | Sargent-Welch, STAS |
| Trisectible Triangular Prism | D | Cenco, LaPine, Sargent-Welch |

6. PROBABILITY AND STATISTICS

| | | |
|---|---|---|
| Coin Tossing Machine | D M | LaPine, Math Master |
| Dice | M | Lano |
| Dissectible Binomial Cube | D M | Cenco, Lano, LaPine, Math Master, NASCO, Sargent-Welch, Science Related Materials |
| Percentage Protractor | M C | Math Master |
| Probability Demonstrator | D M | Cenco, Geyer, Harcourt Brace Jovanovich, Lano, LaPine, Math Master, NASCO |
| Probability Experiment Kit | K | Lano |
| Probability Kit | K | Edmund |
| Probability Slide Rule | D M C | Rolatape |
| Probability and Statistics Lab | D M K | LaPine, STAS |
| Probability and Statistics Lab Unit | D M | Math Master |
| Random Motion Graph Paper | D M | Science Related Materials |
| Roulette Wheel (regulation) | D M | Lano |
| Student Slide Rules | D M C | Compass, C-Thru, Geyer, LaPine, Math Master, Post, Sargent-Welch, Sterling, Yoder |
| Topologic Nomograph for Arithmetic Mean or Average | C | Schur |

7. FORMULAS

| | | |
|---|---|---|
| Burns Boards (pupil and teacher) | K | Ideal, NASCO |
| Circle (30-inch circumference) | D | Vis-X |
| Circle Area Device (sand filling) | D M | Lano, LaPine, Viking |
| Circular Disk Divided into Sectors | M | Yoder |
| Circumference Demonstration Models | D M | STAS |
| Cone, Sphere, and Cylinder Set | D | Math Master, NASCO, Sargent-Welch |
| Cube and Pyramid Set | D | NASCO, Sargent-Welch |
| Demonstration Conics | D | Lano |
| Dissectible Binomial Cube | D M | Cenco, Lano, LaPine, Math Master, NASCO, Sargent-Welch, STAS |
| Dissectible Binomial Square | D M | Bowmar, LaPine, Math Master |
| Dissectible Dimension and Capacity Models | M | Viking |
| Dissectible Pythagorean Theorem Demonstration Model | D | Bowmar, LaPine, Sargent-Welch |
| Dissectible Trinomial Square | D M | LaPine |
| Equal Arms Balance | M | Harcourt Brace Jovanovich, Math Media |
| Flexible Lateral Area Templates (cube, pyramid, cylinder, cone, frustum of a cone) | D M | LaPine |
| Geometry Dial (formulas) | C | Viking |
| Geometry Dial (33 formulas) | C | STAS |
| Interest Computation Board | M C | LaPine |
| Indirect Variation Board ($xy=k$) | M C | LaPine |
| Percentage Board ($P = R \cdot B$) | M C | LaPine |
| Percentage Protractor | M C | Math Master |

| | | |
|---|---|---|
| Perimeter Area Board | M | Ideal, NASCO |
| Rack and Pinion for Determining $\pi$ | D M | LaPine |
| Right Triangle Models (30-60-90 and 45-45-90) | D M | Lano, Math Master, Post |
| Transmobiles (principles of levers, pulleys, gears) | S | STAS |
| Trisectible Triangular Prism | D | LaPine, Sargent-Welch |

8. LOGIC

| | | |
|---|---|---|
| Brainiac Kit | K | Berkeley, Lano |
| Dr. Nim | G | Edmund |
| Electronic Logic and Sets Demonstrator | D | Lano |
| Electronic Logic Trainer | D M | LaPine |
| Hypothesis Machine | D M | NASCO |
| Kalah Game | G | Kalah |
| Logical Blocks | M | Herder |
| Tac-Tickle | G | WFF 'N PROOF |
| Tic-Tac-Toe Game (3-D—also called "Quad") | G | Gangler-Gentry |
| WFF: The Beginners Game of Modern Logic | G | WFF 'N PROOF |
| WFF 'N Proof: The Game of Modern Logic | G | WFF 'N PROOF |

9. RATIO, PROPORTION, PERCENTAGE

| | | |
|---|---|---|
| Circle for Calculating Fractions and Percentages | D M | Freyer |
| Decimal and Percentage Board | D M | Ideal |
| Decimal Place Value Cards | D M | Ideal, NASCO |
| Fraction Discs | M | Bradley, LaPine |
| Fraction Disc Calculator | D M C | LaPine |
| Fraction Equivalents Board | M | Creative Playthings |
| Fraction Finder | D M C | AIM |
| Fraction Inlay Board (squares, circles) | D M | Judy, NASCO |
| Fraction Pies, Cubes, Blocks, etc. | M | Creative Playthings |
| Fractions Game | G | Bradley |
| Indirect Variation Board | D M | LaPine |
| Inverse-Square-Law Demonstration Device | D | Sargent-Welch |
| Magnetic Fraction Circles | D | Instructo, NASCO |
| Markable Plastic Demonstration Boards for Ratios and Fractions | D M | LaPine |
| Math Wheel | D | Bradley, NASCO |
| Pantograph | M I | Geyer, Lano, Math Master, Sargent-Welch, Yoder |
| Parts Imparter | D M | Exton |
| Proportional Dividers | I | Compass, Post, Sargent-Welch, Yoder |
| Rubber Inset Fraction and Area Boards | D M | Physics |

## 10. OPERATIONS AND THEIR PROPERTIES

| | | |
|---|---|---|
| Addition Cubes | M | Creative Playthings |
| Addition-Subtraction Slide Rule | M | Judy |
| Array Board | D M | Science Research Associates |
| Chinese Abacus | D M C | Sargent-Welch |
| Circles for Calculating Fractions | D | Freyer |
| Combinations Card Game | G | Creative Playthings |
| Computing Abacus | M C | Ideal, NASCO |
| Cross Number Puzzles | G C | Ideal |
| Cuisenaire Rods | M | Cuisenaire |
| Direct Reading Abacus | D | Edmund, NASCO, TRI-TIX |
| Electric Math Quiz Game (elementary) | G | Edmund |
| I-Win (card game) | G | Exclusive |
| Magnetic Numb-R-Ods | D M | Yoder |
| Matrix Cards for Building Number Patterns | M C | Ideal |
| Modern Math Demonstration Board | D | Math Master |
| Moto-Math Abacus Parts | D M | Yoder |
| Multiple Base Abacus | M C | Weber Costello |
| Napier's Rods | D | Bowmar, Ideal |
| New Math Charts | D | Nystrom, Walker, Weber Costello |
| New Math Number Frame | D | Ideal, NASCO |
| Number Facts Matrix | D | Ideal |
| Number Frame | D M C | Creative Playthings, NASCO, Sargent-Welch |
| Number Frame to Demonstrate Distributive Property | D | Ideal, LaPine |
| Number Line Ruler | M | Gamco, LaPine, NASCO |
| Number Lines | D M | Bowmar, Gamco, Ideal, LaPine, NASCO |
| Number Relation Blocks | M C | Weber Costello |
| Pattern Boards | M | Creative Playthings |
| Projectable Abacus | D M | Weber Costello |
| Relationship Cards | D C | Bradley, Ideal, NASCO |
| Spinno | D M | Weber Costello |
| Symbolic Bead Material for Multiplication and Division | M | Creative Playthings |
| TUF (mathematical sentence-game) | G | Avalon Hill |
| Winning Touch | G | Ideal |

## 11. OPEN SENTENCES

| | | |
|---|---|---|
| Basic Facts Cards | D | Scott Foresman |
| Equal Arms Balance | D M | Harcourt Brace Jovanovich, NASCO |
| Equation Cards | D M | Houghton Mifflin |
| Equations Game | G | WFF 'N PROOF |
| Flip-Flash Combinations | M | Bowmar |
| Greater Than Less Than Chart | D M | Instructo |
| Mathematics Equalizer Balance | D M | Math Media |

| | | |
|---|---|---|
| Moto-Math Demonstration Board and Accessories for Graphing | D | Yoder |
| Multi-Purpose Graph Board | D | Lano |
| New Math Relationship Cards | D | Ideal, NASCO |
| Pan Balance and Weights | D M | Math Media |
| TUF (mathematical sentence game) | G | Avalon Hill |
| WFF 'N PROOF | G | WFF 'N PROOF |
| "What's the Number?" Cards | D | Franklin |

## 12. GRAPHS

| | | |
|---|---|---|
| Centimeter Grid Paper | D C | Houghton Mifflin |
| Chalkboard Graph Charts (rectangular and polar) | D | Claridge, Cram, Denoyer-Geppert, Geyer, Lano, Math Master, Math-U-Matic, Nystrom, Sargent-Welch, Weber Costello, Yoder |
| Chalkboard Graph Chart Stencils (rectangular and polar) | D | LaPine, Math Master, Sargent-Welch |
| Chalkboard Graph Panel for Plotting Trigonometric Curves | D | Math Master |
| Demonstration Conics | D | Lano |
| Drafting Machine with Drawing Board | M I | Charvoz, Post, Sargent-Welch |
| Eight-Octants Graphing Device | D M | Viking |
| Flannel Board Graph Set | D M | Instructo |
| Logarithmic and Semi-Logarithmic Graph Charts | D | Math Master |
| Movable Vector Adder | D | Lano |
| Number Line Paper | D | Educational Supply, Ideal, NASCO |
| Pegboard Grid System | D | Lano, Math Master |
| Rubber Stamps for Small Graph Charts | I | Edmund, Geyer |
| Space Spider | K | Edmund, Geyer, LaPine, Math Master, NASCO |
| Three-Dimension Graphing Device | D M | Science Related Materials, Viking |

## 13. RELATIONS AND FUNCTIONS

| | | |
|---|---|---|
| Chalkboard Graph Charts (polar and rectangular) | D | Lano, LaPine, Math Master, Sargent-Welch, Weber Costello, Yoder |
| Inverse-Square Law Demonstration | D | Sargent-Welch |
| Logarithmic and Semi-Logarithmic Graph Charts | D | Math Master |
| Logarithmic and Trigonometric Functions Charts | D | LaPine, Math Master, Sargent-Welch |
| Moto-Math (demonstration board and accessory materials for graphing) | D | Yoder |
| Multi-Purpose Graph Board | D | Lano |

## 14. LOGARITHMS AND EXPONENTS

| | | |
|---|---|---|
| Analog Computer Kit | D K | Edmund, Lano |
| Circular Slide Rule | M C | Charvoz, Compass, C-Thru, Post, Sargent-Welch, Yoder |
| Demonstration Analog Computer | D | Sargent-Welch |
| Demonstration Slide Rules | D | Claridge, Keuffel and Esser, Lano, LaPine, Math Master, Pickett, Post, Sargent-Welch |
| Log and Trig Functions Charts | D | LaPine, Math Master, Sargent-Welch |
| Logarithmic and Semi-Logarithmic Graph Charts | D | Math Master |
| Nomographs—Getting a Line on Mathematics | C | Cuisenaire |
| Precision Slide Rule | M C | Charvoz, Compass, Crawford, Keuffel and Esser, Post, Sargent-Welch, Sterling |
| Projection Slide Rule | D | Charvoz, Math Master, Pickett, Post, Sargent-Welch, Tweedy, Yoder |
| Student Slide Rules | M C | Charvoz, C-Thru, Geyer, Lano, LaPine, Math Master, Post, Sargent-Welch, Sterling, Yoder |
| Tyler Slide Rule | M C | Weems and Plath |

## 15. TRIGONOMETRY

| | | |
|---|---|---|
| Ambiguous Case Demonstration | D | Math Master |
| Angle Mirror | D M I | Yoder |
| Bubble Sextant | S I | Weems and Plath, Yoder |
| Chalkboard Graph Chart for Plotting Trig Curves | D | Math Master |
| Chalkboard Graph Charts (polar and rectangular) | D | Cram, Geyer, Lano, LaPine, Math Master, Math-U-Matic, Sargent-Welch, Weber Costello, Yoder |
| Clear Markable Globe | D | Farquhar |
| Demonstration Slide Rules | D | Claridge, Keuffel and Esser, Math Master, Pickett, Post, Sargent-Welch |
| Dial-A-Function | M C | Math-Aide |
| Dial-A-Trig | M C | LaPine, Viking |
| Drafting Machine with Drawing Board | D M I | Charvoz, Post, Sargent-Welch |
| Full-Circle Protractors | M I | Cenco, LaPine, Post, Sargent-Welch, Wabash, Yoder |
| Hypsometer with Clinometer and Graphic Sine, Cosine, and Tangent Table | D M I | Yoder |
| Leveling Rod and Target | D S I | Cenco, Geyer, Math Master, Post, Sargent-Welch, Yoder |
| Level-Transit with Tripod | D M S I | Cenco, Geyer, Math Master, Post, Sargent-Welch, Yoder |
| Log and Trig Functions Charts | D C | LaPine, Math Master, Sargent-Welch |

| | | |
|---|---|---|
| Map Projection Unit | D | Farquhar, Sargent-Welch |
| Plane Table, Tripod, and Alidade | M I | Cenco, Yoder |
| Precision Slide Rules | M C | Charvoz, Compass, Crawford, Geyer, Keuffel and Esser, Pickett, Post, Sargent-Welch, Sterling |
| Projection Slide Rule with Trig Scales | D | Charvoz, LaPine, Pickett, Post |
| Protractor Triangle | I | Post, Weems and Plath |
| Radian and Circle Demonstrator | D | Sargent-Welch |
| Radian Protractor | I | Math Master |
| Range Finder, Parallax Viewer, Optical Micrometer | D M I | Sargent-Welch |
| Ranging Poles | M I | Cenco, Post, Yoder |
| Schacht Dynamic Adjustable Triangle | D M | Sargent-Welch |
| Steel Measuring Tape | I | Cenco, Geyer, LaPine, Post, Sargent-Welch, Yoder |
| Student Slide Rules | M C | Charvoz, Compass, C-Thru, Geyer, Keuffel and Esser, Lano, LaPine, Math Master, Pickett, Post, Sargent-Welch, Sterling |
| Transparent Grid Globe | D | Farquhar |
| Trig-Aide (student and teacher) | D M | Brooks |
| Trig-Check | M C | Math-Aide |
| Trig Circular Slide Rule | M C | Fullerton, Post |
| Trig Demonstrator, Unit Circle | D | LaPine |
| Trigonometric Functions Device | C | LaPine, Math Master |
| Trig Tracer (student and teacher) | D M | Math Master |
| Trigtracker (student and teacher) | D M | Lano |
| Tyler Slide Rule | M C | Weems and Plath |
| Unit Circle Device for Determining Line Values | D M | LaPine, Vis-X |
| Viza-Trig (student and teacher) | D M | Math-U-Matic |

## 16. LINEAR ALGEBRA

| | | |
|---|---|---|
| Moto-Math (demonstration board and accessory materials for graphing) | D K | Yoder |
| Movable Vector Adder | D | Lano |
| Multi-Model Geometric Construction Set | D M K | Yoder |
| Multi-Purpose Graph Board | D | Lano |

## 17. TOPOLOGY

| | | |
|---|---|---|
| Clear Markable Globe | D M | Viking |
| Dissectible Sphere | D M | LaPine |

17. TOPOLOGY (Continued)

| | | |
|---|---|---|
| Five Regular Solids | D | Lano, LaPine, Math Master, Sargent-Welch, Science Related Materials |
| Geometrical Solids (dissectible or flattened) | D | LaPine, Viking |
| Map Projection Models | D M | Sargent-Welch, Viking |
| Slated Globe | D | Cram |
| Sphere with Great Circle Sections Showing Euler's Triangle | D | LaPine, Sargent-Welch, Viking |
| 18. TRANSFORMATIONS | | |
| Area Transformation Boards | D | Günter Herrmann, LaPine |
| Moto-Math (demonstration board and accessory material for graphing) | D K | Yoder |
| Multi-Purpose Graph Board | D | Lano |
| Pantograph | D I | Charvoz, Geyer, Lano, LaPine, Sargent-Welch |
| Reflection, Rotation, Half-Turn Translation Boards | D | Günter Herrmann, LaPine |
| 19. CALCULUS | | |
| Coaxial Cylinders Intersected by a Plane | D M | Sargent-Welch, Viking |
| Differential of a Function of Two Variables Model | D | Viking |
| Differential of Volume in Spherical Coordinates Model | D | Viking |
| Eight-Octants Model | D | Viking |
| Single-Octant Model | D | Viking |
| Solids of Revolution Models (ellipsoids, hyperboloids, paraboloids) | D M | Math Master |
| Spherical Shell Model | D | Viking |
| Volume of Revolution Model | D | Viking |
| 20. COMPUTER CONCEPTS | | |
| Analog Computer Kit | D K | Edmund |
| Binary Computer Cards and Kit | M | Science Related Materials |
| Binary Counter | D M | Sargent-Welch |
| Binary Teacher | M | Math-Media |
| Brainiac | D K | Berkeley |
| Brain Teasing Flip-Flop Toy | G | Edmund |
| Computer Simulator | D | Cenco |
| Computer Trainer | D | Arkay, LaPine, Math Master |
| Demonstration Analog Computer | D | Sargent-Welch |
| Digi-Comp | D M | Edmund, Sargent-Welch |
| Digi-Comp Game | G | Edmund |
| Dr. Nim (digital computer game) | G | Edmund |
| Model Digital Computer | D C | Sargent-Welch |
| Modern Abacus | D | Edmund, LaPine |

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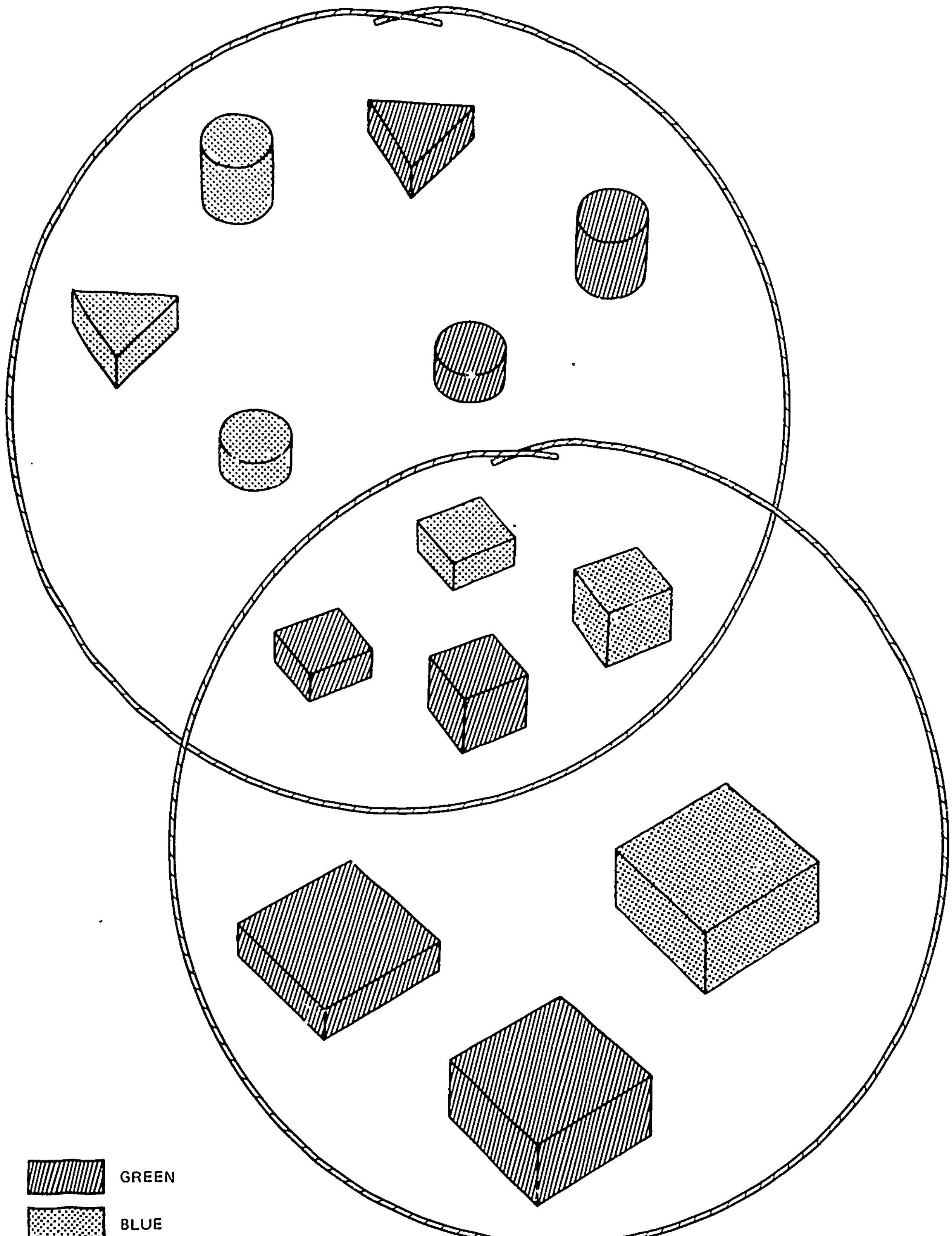
# 9. MANIPULATIVE DEVICES IN ELEMENTARY SCHOOL MATHEMATICS




BLOCKS CAN BE USED TO ILLUSTRATE SET OPERATIONS
SUCH AS UNION AND INTERSECTION

CHAPTER 9

# MANIPULATIVE DEVICES IN ELEMENTARY SCHOOL MATHEMATICS

*by*

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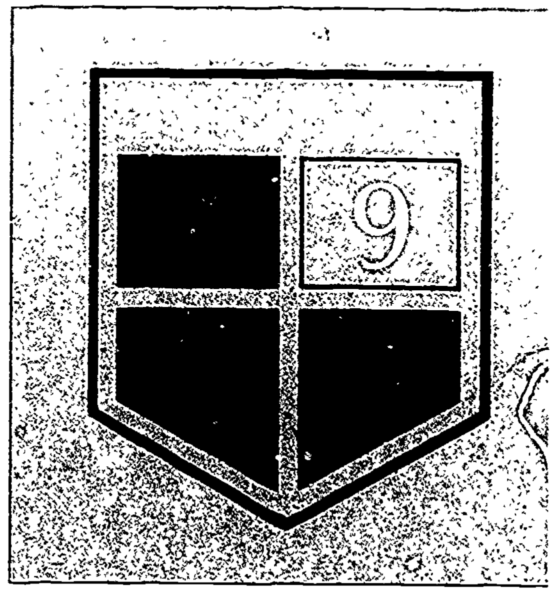



Chapter 9 explores the types and uses of manipulative devices that are appropriate to the teaching and learning of elementary school mathematics. A classification scheme is developed so that types of items, as well as individual items, can be examined. Suggestions for use of devices are presented, related research is examined, and evaluation techniques are detailed. An extensive listing is given of sources of different kinds of devices. The chapter concludes with a bibliography grouped into general references and nine special classifications.

# 9. MANIPULATIVE DEVICES IN ELEMENTARY SCHOOL MATHEMATICS

The impact of technology on the educational process and, in particular, on the learning of mathematics has been amplified by new knowledge about the nature of the learning process, the nature of children, and the structure of mathematics itself, as well as by new mathematical content. We have learned more about how children learn, think, and behave, and at the same time have been able to integrate this knowledge into classroom activities and curriculum planning. We have learned more about mathematical structure and have discovered ways in which this structure can be made understandable to children. We have introduced new mathematical content and eliminated outdated content so that objectives of mathematics instruction can be more readily attained. These changes combined with technological growth point to the fact that now, more than ever before, educators recognize the importance of the child's active participation in the learning process. Furthermore, these changes have given rise to new educational tools that enable us to attain goals more readily than ever before.

The educational tools with which we are concerned in this chapter are commonly called *manipulative devices*. It is the purpose of this chapter to examine the nature and role of manipulative devices in the teaching of elementary school mathematics.

In order to understand the role of manipulative devices in the learning of mathematics, it is necessary to keep in mind the main goals of mathematics instruction and to examine how such devices contribute to the attainment of these goals. Sobel and Johnson (29) describe the purposes of mathematics instruction as follows:

1. Formation and understanding of mathematical concepts, facts, and processes
2. Development of skills in computing with accuracy and efficiency
3. Development of a general problem solving technique
4. Formation of positive attitudes toward mathematics
5. Understanding of the logical structure of mathematics
6. Utilization of mathematical concepts to discover new generalizations
7. Recognition and appreciation of the role of mathematics in society
8. Development of study habits essential for independent progress
9. Demonstration of mental traits such as creativity, imagination, curiosity, and visualization.

A manipulative device is considered to be one that utilizes primarily the senses of sight and touch. Implicit in this statement is the understanding that sensory contact with objects in the real world is something more than a casual "look at" or "feel" interpretation by the senses; for to manipulate means to handle or to use. Thus, the term *manipulative devices* refers to those concrete objects which, when handled or operated by the pupil and the teacher, provide the pupil with an opportunity to attain specific objectives identified under the broad goals listed above.

There seems to be wide agreement among mathematics educators and psychologists that sensory learning is fundamental, or at least basic, to all other learning. There is no question that in the early stages of learning, modification of behavior in a young child is largely dependent on his ability to perceive his environment, and perceptions formed by young children arise from the use of their senses. Since children can only sense that which exists in the real world, it is important that they have available concrete objects with which to carry out explorations. Perhaps the very young child who in his quest for learning picks up small objects to eat them is not unlike the nine-year-old boy who takes an electric clock apart to see how it works. Although far different in maturation and in established knowledge systems, the two are learning by manipulation. Syer's scholarly discussion

of sensory learning points out that motivation, memorization, mathematical understandings, problem solving skills, and attitudes—all important facets of the learning process and overall goals of mathematics instruction—arise from sensory learning (35). Van Engen emphasizes the role of sensory learning in concept formation when he states:

> Reactions to the world of concrete objects are the foundation from which the structure of abstract ideas arises. These reactions are refined, reorganized, and integrated so that they become even more useful and even more powerful than the original response. [37]

Dienes's research with structural materials has led him to conclude:

> The organism seems to wish to explore and manipulate the environment. It does this, presumably, for the purpose of being able to predict how the environment is going to respond. Mathematics learning would probably be no exception to this, but a preliminary groping period is notably lacking in most mathematics lessons . . . . So play, it seems, should be regarded as an integral part of any learning cycle. Mathematical play can be generated simply by providing children with a large variety of constructed mathematical materials. [6]

In examining Piaget's theory of concept formation for implications for teaching, Adler comes to this conclusion:

> Physical action is one of the bases of learning. To learn effectively the child must be a participant in events, not merely a spectator. To develop his concepts of number and space, it is not enough that he look at things. He must also touch things, move them, turn them, put them together, and take them apart. For every new concept that we want the child to acquire, we should start with some relevant action that he can perform. [2]

Gagné points out:

> Instruction needs to be fundamentally based on the stimulation provided by objects and events. . . . Objects and events are the stimuli from which concepts are derived. [10]

The sometimes controversial Cambridge report makes a strong case for manipulative devices when it states:

> The conclusion is inescapable that children can study mathematics more satisfactorily when each child has abundant opportunity to manipulate suitable physical objects. [12]

The learning theorists Piaget, Bruner, and Flavell agree that an operation is something that may be performed either internally with symbols representing materials or externally with the actual materials. Further, the probability of performing internal operations accurately and consistently is increased by repeated experience with concrete materials. Thus there appears to be strong support among educators and psychologists for the position that the manipulation of a device or devices should precede the requirement of abstracting an idea or model. In short, it would appear that manipulative devices are essential in the instructional program for elementary mathematics.

## RESEARCH ON MANIPULATIVE DEVICES IN ELEMENTARY SCHOOL MATHEMATICS INSTRUCTION

Some support and direction for the use of manipulative devices can be found in various research projects. However, much of the research evidence is inconclusive. Sole found that using a variety of materials does not produce better results than using a single device if both procedures are used for the same amount of time (30). If more time is spent, achievement improves, regardless of what or how many materials are used.

In attempting to determine the effectiveness of devices on the achievement of first-grade pupils, Harshman found very little difference in achievement between groups using expensive commercial materials, inexpensive commercial materials, and teacher-made materials (16). When small differences in achievement were found, they were in favor of the group using teacher-made materials.

Regarding the use of materials by teachers, Harvin found that teachers with mathematics methods courses use more manipulative devices than teachers without such courses; she further found that pictures and symbolic materials are

used more frequently than actual manipulative devices (17). She also established through an extensive questionnaire that frequency of use of manipulative materials appears to be a contributing factor in achievement in elementary mathematics.

In defining the role of instructional aids, Eidson reported that it is generally accepted that proper use of aids is based on the same framework of principles as all good teaching procedures (7). His study showed that most effective results are obtained by placing major emphasis on the use of devices to obtain data rather than for supplementary activities. He cautions users to remember that aids never teach mathematics and that the role of the teacher is paramount. Another warning is given to teachers using manipulative devices by Sole when he states that manipulation of any device must not be confused with the learning of mathematics (30). Mathematics is made up of ideas—not materials; it can be illustrated with devices and has application to many concrete situations, but it must be understood as a system of ideas.

The findings of two researchers, Swick and Spross, give strong support to the desirability of using multisensory aids in teaching both computation and reasoning (36; 32). In addition, the use of aids and devices was found to improve the attitude of second- and third-graders toward mathematics.

Adkins and Suddeth concluded that there seems to be a tendency to use more instructional materials in the primary grades for the purpose of discovering relationships, for motivation, and for influencing attitude (1; 33).

This research gives credence to the hypothesis that manipulative devices are useful in the attainment of the goals of mathematics instruction.

## SOME CHARACTERISTICS OF GOOD MANIPULATIVE DEVICES

Many attempts have been made to prepare lists of characteristics that good manipulative devices should possess. Most authors agree that a good manipulative device should—

1. be relevant to the mathematical content with a desirable outcome in mind;
2. exploit as many senses as possible;
3. be durable, its durability being commensurate with its cost and anticipated usage;
4. be constructed so that its details are accurate;
5. have high standards of craftsmanship so that parts are not easily broken or lost;
6. be attractive in appearance;
7. be maintained easily and at a reasonable cost;
8. be adaptable to the school facilities (considering mobility and convenience of storage);
9. be simple to assemble;
10. be flexible and have a variety of uses;
11. be simple to operate;
12. be large enough to be easily visible to all pupils, if used for demonstrations;
13. either involve a moving part or parts or be something that is moved in the process of illustrating the mathematical principle involved.

In the final analysis, the devices that are effective and efficient in helping the child understand the mathematical concept to be learned are the devices that should be used.

## SOME GENERAL GUIDELINES FOR THE USE OF MANIPULATIVE DEVICES

Each manipulative device has its own particular set of instructions that should be followed if one is to achieve maximum effectiveness from its use. There are, however, a number of general guidelines that can be helpful to the teacher in the selection and use of manipulative devices. The following are suggestive:

1. Choose the device that best suits the purpose of the lesson and will help attain the objective.
2. Become familiar with the device and the techniques for making effective use of it.

3. Correlate the operations depicted by the device with those to be done either with pencil and paper or on the chalkboard.
4. If possible, arrange for each child to have his own device. If this is not possible, make certain that each child has the opportunity to use the device many times.
5. Try to create an atmosphere that encourages rather than forces the use of devices.
6. Create opportunites, as soon as possible, for each child to become progressively less dependent on manipulative devices and more dependent on using symbolism and abstractions.
7. Have the child stop using a device when he is ready for a higher, more abstract level so that the device does not become a crutch.
8. Use a "math caddy" or movable storage unit designed to hold and properly display devices in order to increase their usefulness.

Manipulative devices will not teach mathematics. It is their use by the teacher and his guidance of pupils' use that determines the usefulness of devices in learning.

## CLASSIFICATION OF MANIPULATIVE DEVICES

The presently vast and ever-growing supply of manipulative devices for use with elementary children rules out the possibility of presenting a detailed discussion of all available devices. Since most of the devices currently available are mutipurpose devices and can be used to objectify many mathematical concepts, it is necessary to select a classification scheme that will keep overlapping of discussion at a minimum. For this reason we have chosen to bypass a classification scheme built entirely around the mathematical concepts to be taught to pupils. For the purposes of discussion in this chapter, we shall consider specific manipulative devices under the following categories:

1. Demonstration boards and devices
2. Place value devices
3. Colored beads, blocks, rods, and discs
4. Number boards
5. Cards and charts
6. Measurement devices
7. Models of geometric relationships
8. Games and puzzles
9. Special computational devices

## DEMONSTRATION BOARDS AND DEVICES

There are many desirable teaching-learning experiences in an elementary mathematics program which, for one reason or another, cannot be provided effectively by having each child work individually with his own set of materials. Small group and class instruction can make an important contribution to the total growth pattern of children. To accomplish these ends and still provide children with real-world referents on which to build abstractions demonstration boards must be available to the teacher.

The demonstration boards most familiar to elementary teachers are the flannel board and the magnetic board. While both types of demonstration boards can serve the same function, the flannel board is perhaps the more widely used because it can be readily constructed by the teacher. Advocates of the magnetic board prefer the durability and workmanship qualities of a commercially built device. Sueltz describes specifications and construction procedures for both flannel boards and magnetic boards (34).

The great strength of these devices is their versatility. Most of the concepts taught in elementary school mathematics can be effectively presented on these boards.

Generally, the flannel board is a 2 by $2\frac{1}{2}$ foot piece of hardwood covered with flannel or felt. If a piece of sheet metal is inserted between the board and the flannel, it can also serve as a magnetic board. It is propped on an artist's easel or placed on a chalkboard tray so that it gives the appearance of a blank picture. Accompanying the board are flannel or felt cutouts that are held in position by the cohesiveness of the materials.

One use of demonstration boards can be illustrated by the following classroom situation in which kindergarten or first-grade children are seated in a semicircle about the teacher, who is seated near a flannel board in the center.

> Today, I am going to tell you an interesting story. This story is about something that really happened.
>
> Behind my house, I planted a little vegetable garden. I planted tiny seeds of radishes, lettuce, carrots, and several other vegetables. One summer morning I awoke; and, while eating my breakfast, I looked out the window at my garden and saw something move. What do you think it was? [Children make guesses.] Let's see what it was. [Place a rabbit cutout on the flannel board.] A rabbit! And what do you suppose he was doing in my garden? Eating vegetables! He was so cute I thought I would watch him for awhile. Soon I saw something else moving! It was ________. [Place another rabbit on the flannel board.] Do you suppose he was eating vegetables, too? [Response: "Yes!"]
>
> How many rabbits were eating in my garden? [Response: "Two."] Yes, there were two rabbits. But as I watched, I saw something else moving just outside my garden fence. What do you suppose that was? [Response: "Another rabbit!"] Yes, but not just one rabbit. There were one, two, three rabbits. [Place three more rabbits on the flannel board (Figure 9.1).]
>
> They were so hungry they wanted to ________. [Response: "Eat all the vegetables in your garden!"] That's right! I had two rabbits eating and three rabbits watching. But soon the three rabbits watching found a hole in my garden fence and hopped into the garden to eat. How many rabbits are now eating? [Response: "Five."] How do you know? [Response: "I counted them."] Right! First there were two rabbits eating, and then three more rabbits joined them so that there were five rabbits eating.

This little story and similar ones help provide children with experiences in assigning numbers to sets and to the union of disjoint sets in a concrete setting. This helps prepare them for work in addition. Such stories provide opportunities for the development of mathematical sentences such as the one that is illustrated in Figure 9.2, where the operation symbol indicates the main action of the story.

Once a mathematical sentence is developed, children are able to make up other stories for it. In this way they develop an intuitive feeling for the generalization represented by the sentence. Similarly, related subtraction facts can be developed by an undoing of additon or by a separating rather than a joining action. The children should have extensive opportunities to use the demonstration board in forming their own stories and in suggesting mathematical sentences for others in the group.

At a little higher level, Perkins and Hanson illustrate the formation of Cartesian products by forming arrays on the flannel board (40).

FIGURE 9.1

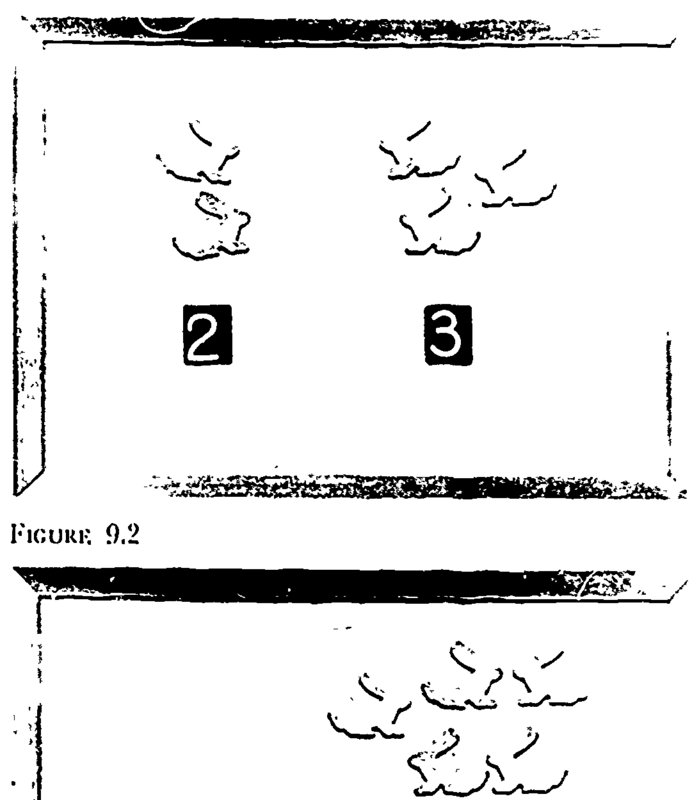
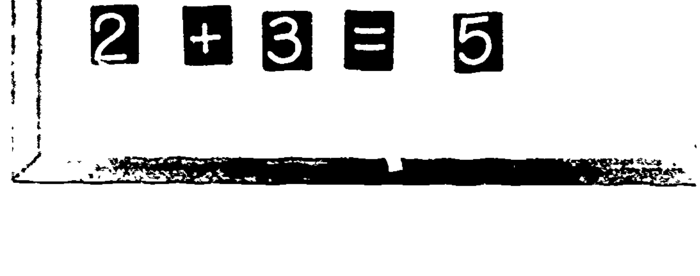



FIGURE 9.2



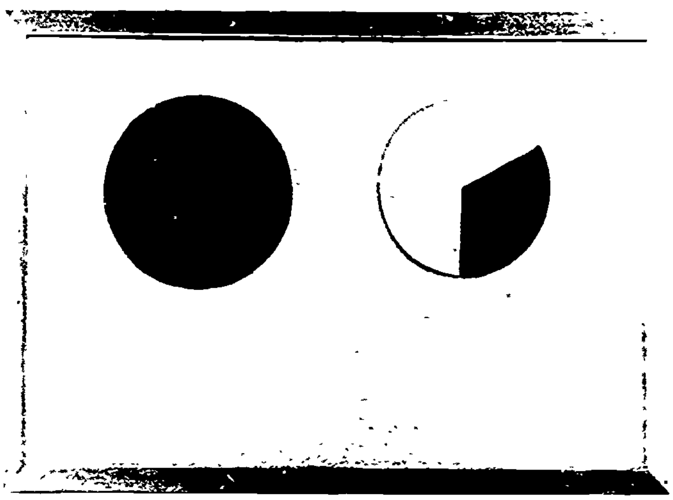


FIGURE 9.3

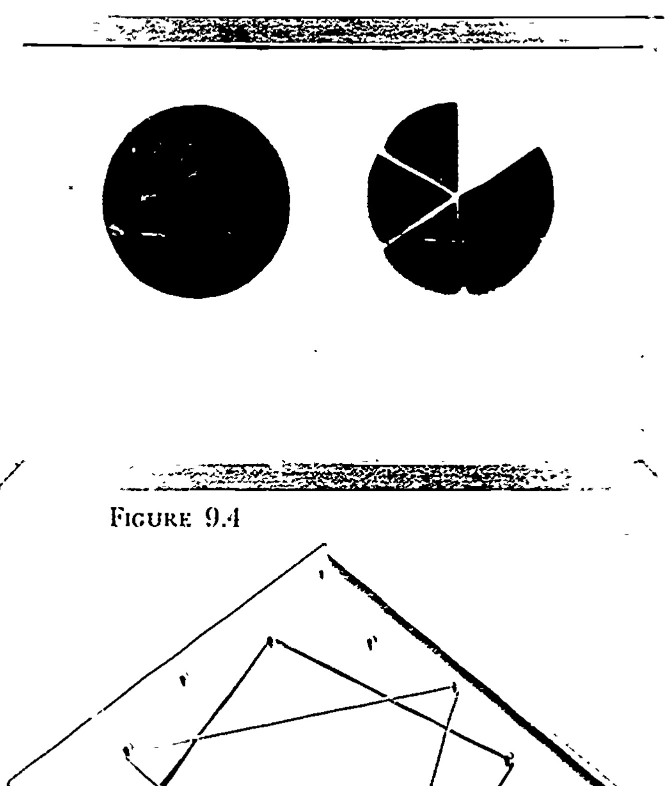


FIGURE 9.4

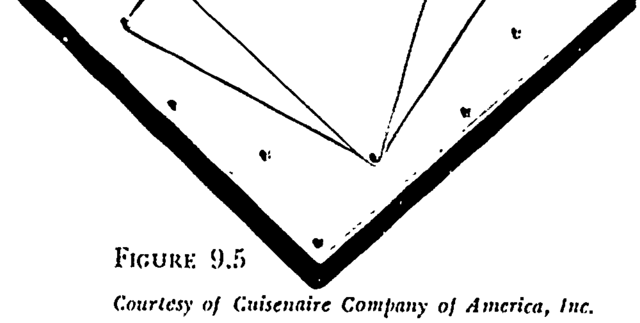


FIGURE 9.5

*Courtesy of Cuisenaire Company of America, Inc.*

The same strategy used with the addition of whole numbers can be used to illustrate the addition of rational numbers. If the circular regions shown at the left in Figures 9.3 and 9.4 represent a standard unit, then the addition of $\frac{1}{2}$ and $\frac{1}{3}$ can be represented by the diagram at the right in either figure. Although the addition is complete, it is necessary to find an ordered pair of numbers which can be used to represent the sum. The appropriate ordered pair to use in this case is (5, 6).

Peskin suggests the use of a flannel board at an even more advanced level, in relating figures geometrically and algebraically so that the multiplication of a binomial by a binomial can be visualized (41).

Other items in the category of demonstration boards which are rapidly gaining in popularity are pegboards and geoboards. Many pegboards are so constructed that they can serve as hundred boards. Geoboards (Figure 9.5) are particularly helpful when developing the concepts of perimeter and area. Squares, triangles, rectangles, quadrilaterals, pentagons, and other polygons can all be easily shown on a geoboard using elastic.

Cunningham and Raskin, in support of the use of physical objects as models for concepts, suggest the use of a pegboard as a model for developing an understanding of rational numbers (39). Kennedy suggests additional uses for this class of materials (21).

As teachers begin to use demonstration devices with children, a new universe of creative ideas will open up. These will result in stimulating learning experiences for both children and teachers.

## PLACE VALUE DEVICES

Two features of a place value system of numeration are base and position, or place value. Before a child can comprehend place value, he needs to have an understanding of a single or ones unit and also an understanding of base number. To help him gain this understanding, manipula-

tive devices that provide visualization are most helpful.

Manipulative devices that can be used to teach place value are numerous; they include different types of abacuses, place value boards, pocket charts, and sticks or counters which can be sorted and grouped.

A first experience for children may be grouping counters or bundling sticks to exhibit 1 ten. Later, 10 tens can be bundled to show a hundred. If a board similar to the one shown in Figure 9.6 is available, children can be asked to represent different numbers. In the illustration, the number 133 is represented at lower right.

The pocket chart shown in the background can be used in the same way, as well as to help explain renaming in addition and subtraction. Consider the problem that is illustrated: 14 + 19 + 31. It can be seen that the sum is represented by the 5 tens and 14 ones. But 14 ones equals 1 ten and 4 ones, and 10 ones can be grouped together and placed with the tens, showing a sum of 64. Such an example makes meaningful the addition of 1 ten when the child shows his work like this:

$$\begin{array}{r} 1 \\ 14 \\ 19 \\ \underline{31} \\ 64 \end{array}$$

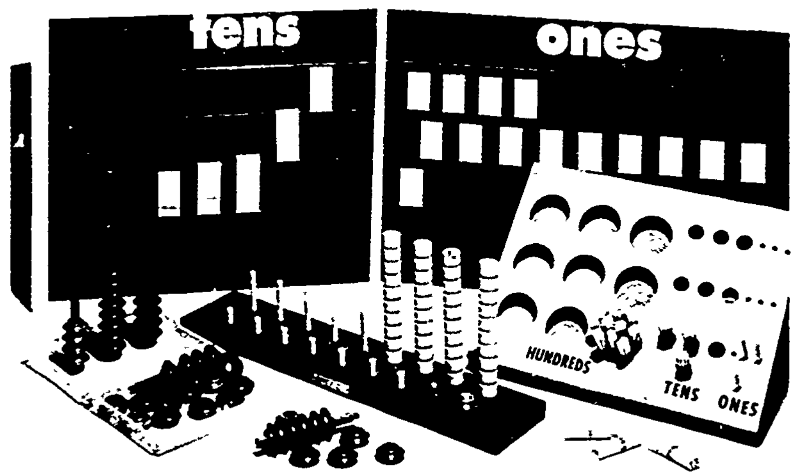


FIGURE 9.6

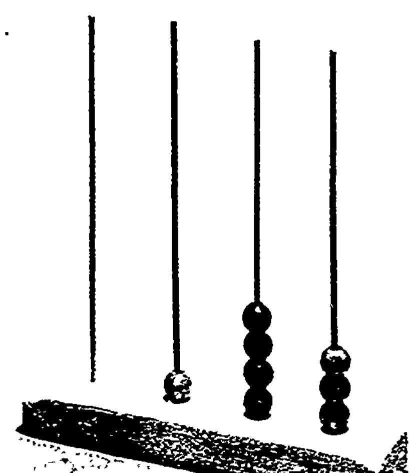
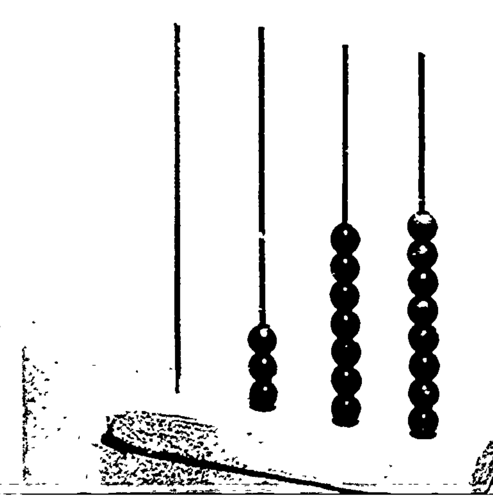


FIGURE 9.7

Vertical abacuses, both the open-end and the closed types, are also adaptable to exercises involving place value, numeration, addition, and subtraction. The abacus shown in Figure 9.7 shows the number 143. The sum of 235 and 143 is illustrated by the abacus shown in Figure 9.8. The necessity for regrouping the beads when the number of ones exceeds 10, the number of tens exceeds 10, and so on, becomes readily apparent to children.

FIGURE 9.8 ▶



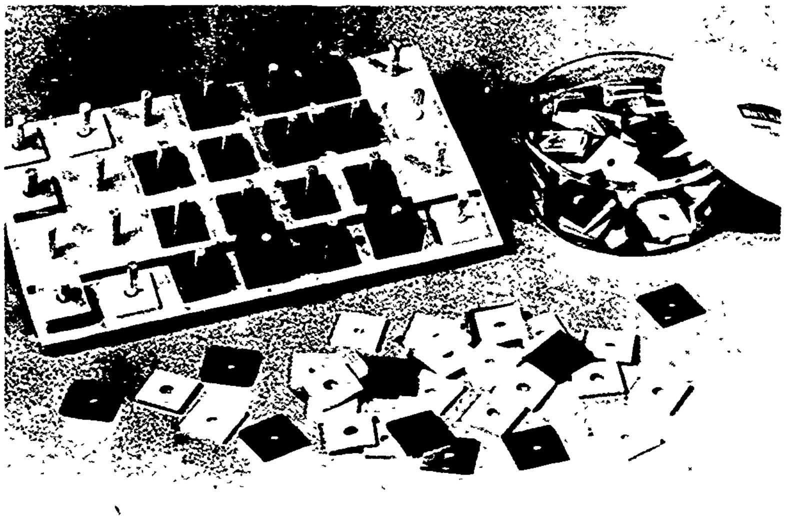


FIGURE 9.9

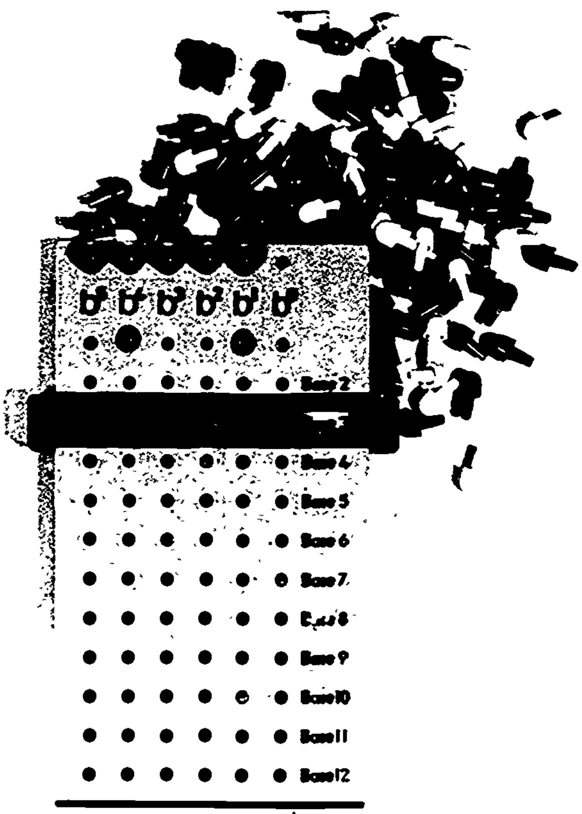



FIGURE 9.10

*Photos on this and the following page, courtesy of Responsive Environments Corporation*

Many boards are adaptable for work with decimals. The board shown in Figure 9.9 uses small squares that children can place on the proper pegs. Thus in the top row the number illustrated is 1102.510. The number illustrated in the second row from the top is 2011.111.

Most vertical abacuses can be used when working with bases other than ten. Multibase converter boards like the one shown in Figure 9.10 are also available. In the illustration, the movable converter is set to show base-three numerals. The base-three numeral illustrated is $10010_{three}$.

The use of the abacus for teaching place value has been the subject of many reports dealing with this class of manipulative devices. Hamilton suggests that the form of the abacus used with children should parallel the mental structure it is hoped to create (14). Thus those abacuses which consist of beads representing fives as well as ones and any abacuses with bead rows placed horizontally would be inappropriate. He suggests using the vertical abacus in one of several forms: string or wire frame, open spike, or with markers moved behind a shield. Clary prefers two different forms—a shoebox abacus that has string sewn through the box and uses colored buttons for counters, and a spool abacus similar to the spike type with spools placed over dowel pins mounted on a board (47).

Cunningham provides detailed instructions for the construction of a "traditional" abacus and a simplified open-end version, both using wooden frames and coat hangers (48). One suggested innovation for vertical wire-frame abacuses using large colored beads is to make the tenth bead in each column the same color as the bottom nine beads in the column to the left.

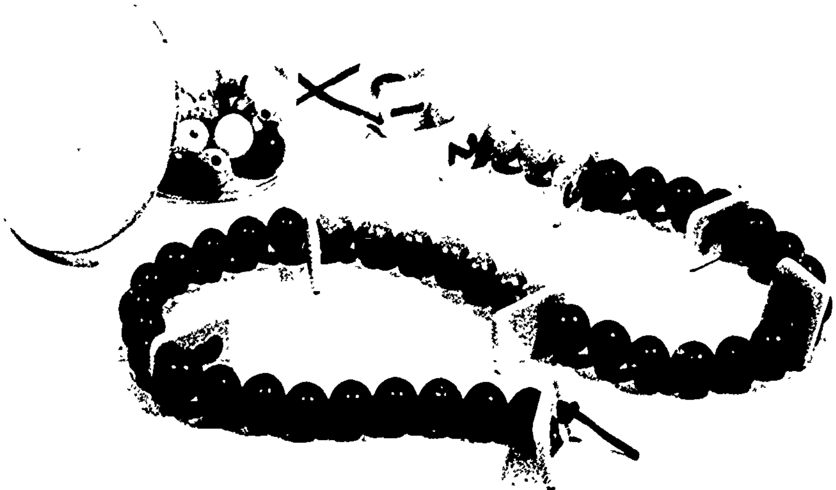
Other suggestions for the construction of abacuses are made by Smith, Peterson, and Rahmlow (62; 59; 61). Nechin and Brower, Spross, and Sueltz recommend the open-end or spike abacus as the most useful type (58; 31; 34). Not only were children motivated to improve computational skills in their experiments, but pupils could see the decimal system of numeration as a system based on logic and order.

A hand-tally counter is suggested by Hooper for teaching place value (54). This counter is similar to the devices used by adults for counting such things as the number of customers at a store and to the devices used on elevators to indicate floors. An advantage noted, particularly for pupils in grade 1, was the improvement of audiovisual and kinesthetic responses.

Bridgers describes the use of a block box for place value work—an open-top box with three layers of blocks inside—the first layer containing 100 cubes, the second layer 10 blocks, each equal in volume to 10 cubes of the first layer, and the last layer consisting of 1 block equal in volume to 10 blocks of the second layer (46). He feels that the uniformity of size of the cubes and the lack of differentiation between colors are strong advantages.

Other suggested devices for teaching place value are popbead necklaces, by Swenson; popsicle sticks, by Weyer; place value charts, by Peterson; and pupil-designed computers, by Rabinowitz (63; 66; 59; 60).

FIGURE 9.11

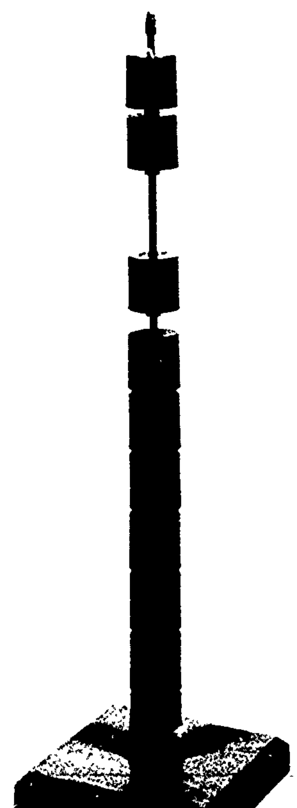


## COLORED BEADS, BLOCKS, RODS, AND DISCS

The many different types of colored beads, blocks, rods, and discs that are now available can be used in a variety of ways to help develop concepts in numeration, logic, and geometry.

For years, colored beads have been used in the primary grades to help children learn about number. Children have strung beads to illustrate a number, stringing five beads on a cord, for example, to illustrate the number five. The little plastic markers shown in Figure 9.11 and the counting pole shown in Figure 9.12 are typical of some of the "extras" that are now part of bead sets.

FIGURE 9.12



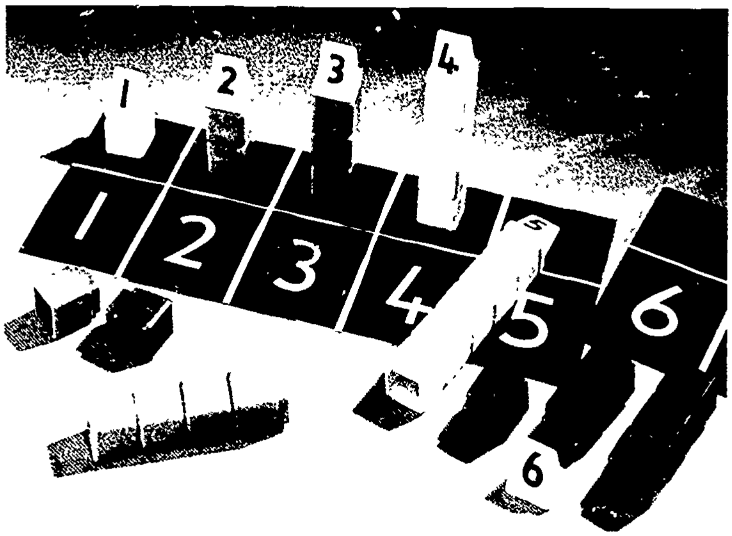


FIGURE 9.13

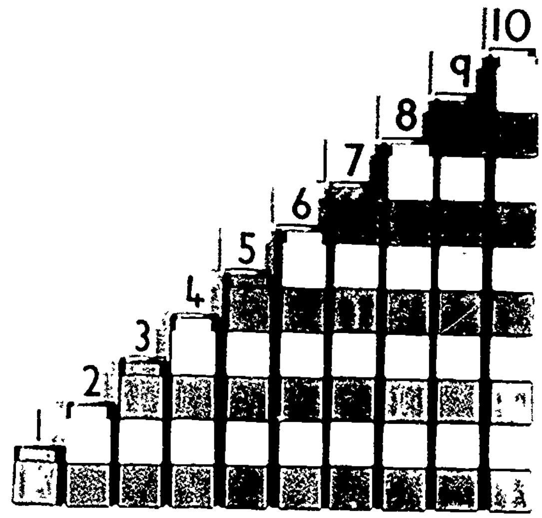


FIGURE 9.14

Blocks of all sorts and descriptions are presently available. These can be used for counting purposes and for showing the one-more relationship as in Figures 9.13 and 9.14.

Counting ladders are helpful in teaching addition and subtraction facts. If a child is learning that 7 + 4 = 11, he may first arrange his blocks as shown at the right in Figure 9.15 and then remove 4 blocks and place them with the 7 blocks. He can then read "11" on his numerical ladder. Similarly, if he wishes to show the subtraction 11 − 4, he will first place his blocks as shown at the left in the illustration and, after removing 4 blocks, read "7" on the ladder. While children can do and have done this type of activity work without the aid of devices, the novelty of the ladder increases interest and prevents counting errors that often occur because a child is careless or because he is unable to count correctly.

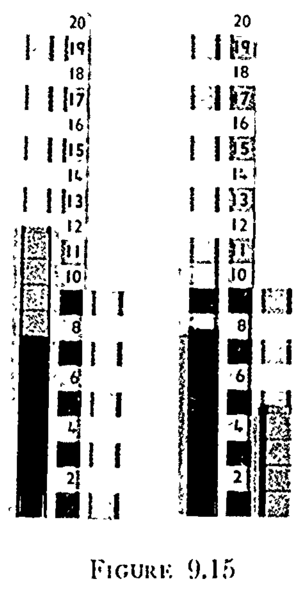
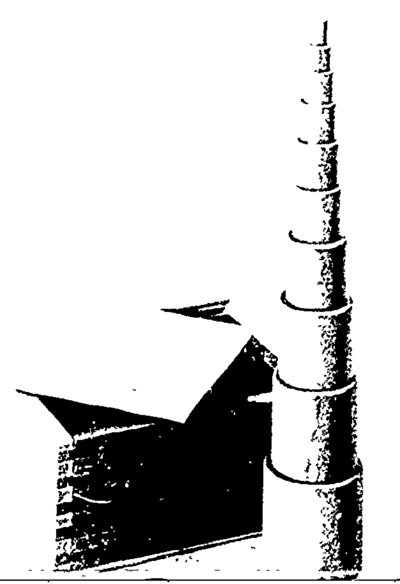


FIGURE 9.15

It is generally agreed that blocks are important in the play stage. Many of the colored block sets now available are being used at the primary level for sorting blocks into classes according to shape, size, and color. Some sets are merely simple nesting sets (Figure 9.16), and children build according to size; but much work can be

FIGURE 9.16 ▶



*Photos on this page, courtesy of Responsive Environments Corporation*

done with seriation (placing blocks or other objects in order determined by some property or characteristic such as size or shape).

Consider a problem in which a child is given objects like the ones shown at the top of Figure 9.17 and asked to arrange them in order from smallest to largest, as in the bottom of the illustration. Or he might be asked to rank them in order of size, as in Figure 9.18. Later he may be asked to pair two sets of objects matching them by size, the smallest with the smallest, and so forth, as is done in Figure 9.19.

FIGURE 9.17

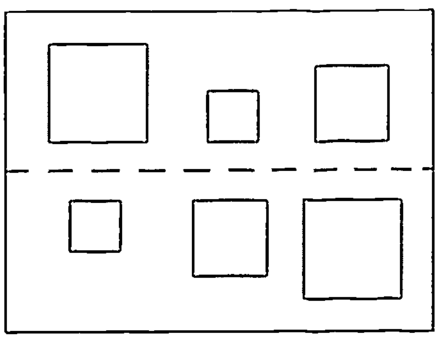


FIGURE 9.18

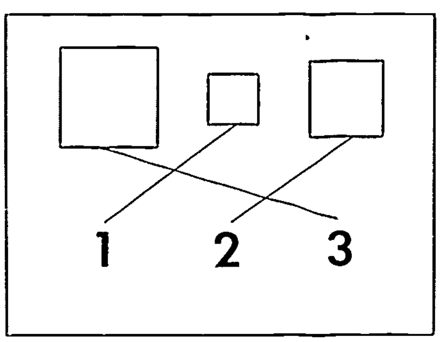
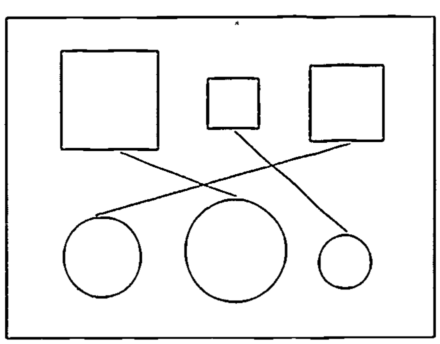



FIGURE 9.19



At the intermediate level, a child may be given 16 objects with 4 different shapes and 4 different colors, as in Figure 9.20, and asked to arrange these objects so that no two having the same shape or color are in the same row or column (Figure 9.21). If 16 objects pose too much of a problem, let the child begin first with 9 objects, using only 3 different shapes and 3 different colors.

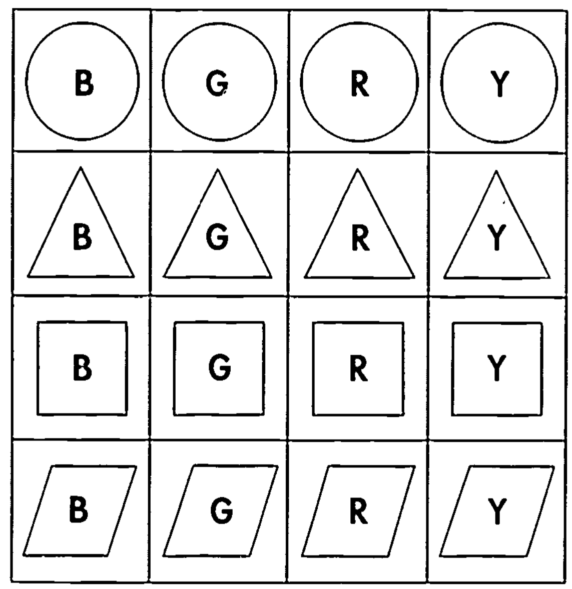
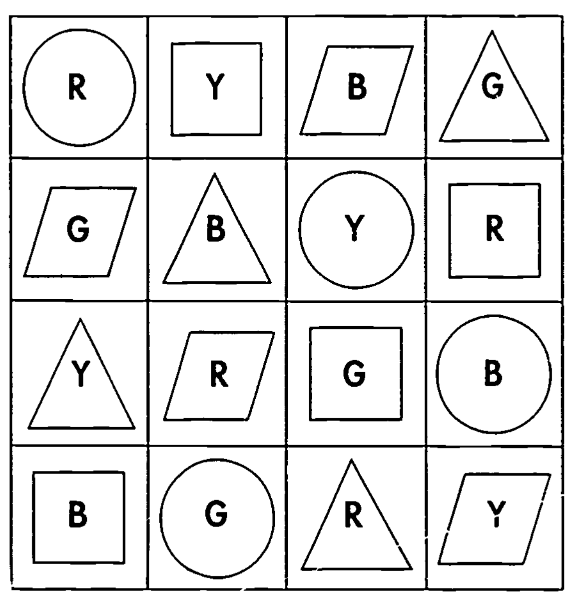



FIGURE 9.20

FIGURE 9.21



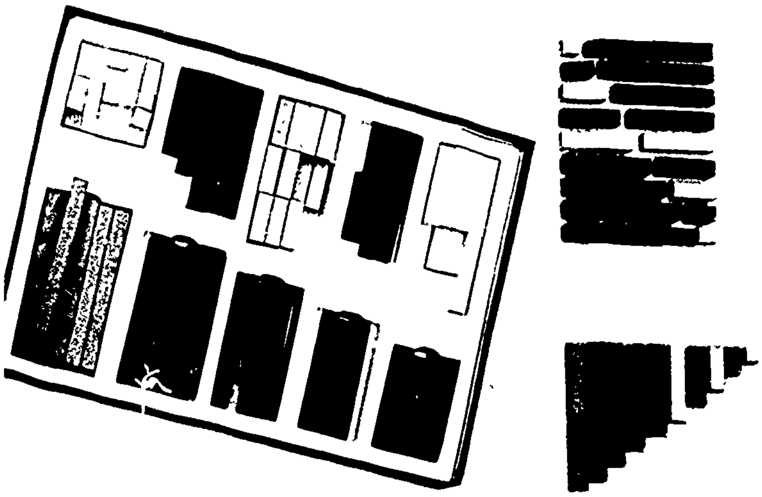


FIGURE 9.22 *Courtesy of Cuisenaire Company of America, Inc.*

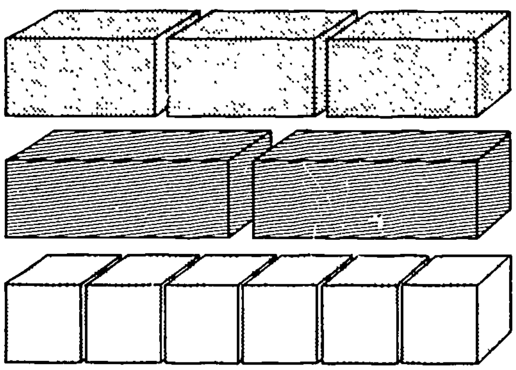
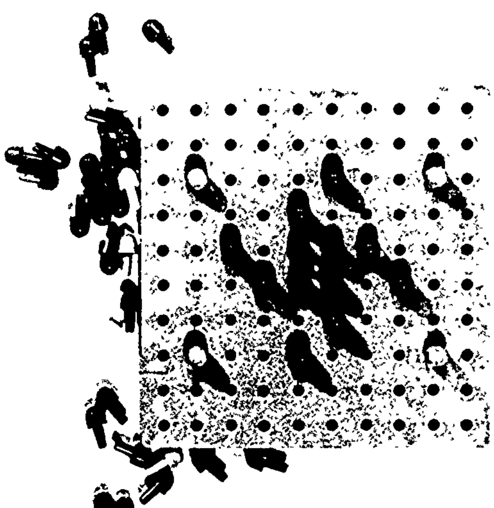


FIGURE 9.23

FIGURE 9.24



*Courtesy of Responsive Environments Corporation*

Typical of the many interesting games that develop logical thinking is the following, which emphasizes the meaning of *not*.

A child is selected to be "it." He then chooses a block from a set of blocks and hides it behind his back in one hand. Suppose, for example, he hides a green, circular, flat block. He then must give the class three clues using the word *not*. He may, for example, give these clues:

"My block is not red."

"It is not square."

"It is not yellow."

He then asks a child to try to guess which block he has hidden. If the second child guesses incorrectly, the first child must give another clue about the block and ask another child to guess which block he has hidden. After each incorrect guess, the first child must give another clue. After a correct response, the child naming the block becomes "it" and gets to select a block to hide.

This also makes an interesting team game if one point is counted against a team for each clue that has to be given.

Rods are particularly useful in building trains to illustrate ideas such as the commutative property of multiplication of whole numbers and to develop multiplication facts (Figure 9.22). In the primary grades, a child might build a train showing that $2 \times 3 = 3 \times 2$. Unit blocks can be used to show that $2 \times 3 = 3 \times 2 = 6$. (See Figure 9.23.)

The uses of beads, blocks, rods, and similar objects are almost limitless. Teachers should not hesitate to use them in situations to develop logical concepts as well as to develop notions of counting. Often children develop new uses for them. One ingenious student used pegs and a pegboard as a hundred board to illustrate two types of symmetry (Figure 9.24). Designs showing line symmetry are, of course, easily constructed and offer other possibilities.

The influx of colored rods during the past few years and the development of mathematics programs built completely around such rods have stimulated numerous research studies to investigate their effectiveness. Most of these studies have involved the Cuisenaire rods and the associated program. Fedon used Cuisenaire

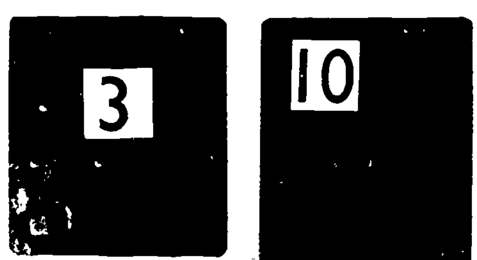
rods with one group of pupils in grade 1 and an eclectic approach with another group (70). From test results he found an apparent trend in achievement that favored the group using the eclectic approach. He concluded that the introduction of color notation as a basis for the symbolization process delays the successful transition from mathematical structure to corresponding mathematical symbolism. Passy's findings were similar (76; 77). He made a study of the use of rods with third-grade pupils. Nasca reports that the rods as a variable had no effect on the achievement of second-grade pupils as measured by a traditional standardized test (75). In a study of two groups of third-grade pupils, one having used the rod approach during both first and second grade and the other following the regular curriculum, Lucow concluded that the rod approach is effective but that there is some doubt as to its general superiority over other methods of instruction in current use (74). In contrasting the rods with the traditional program in England and Scotland, Brownell found no striking differences in the pupils' progress toward maturity of mathematical concepts and understanding (68). An interview technique employed by Howard showed that teacher reaction to rods was favorable and that rods hold promise as a supplement to current methods (72). Hollis, working with first-grade pupils, concluded that as much traditional subject matter was learned by using rods as by traditional teaching techniques. Studies seem to indicate that high-ability pupils appear to benefit more from using rods than low-ability pupils (71; 72; 74), although Callahan and Jacobson, in an experiment with retarded children, indicate that results achieved by pupils using rods were definitely better than might have been achieved in the same time without rods (69). The evidence, although inconclusive, seems to indicate that children do need the help of structured manipulative materials in gaining abstract number ideas but that flexibility of approach rather than one particular physical structure should be employed.

In addition to colored rods, many types of blocks are advocated as useful in teaching abstract concepts. Lucas used attribute blocks with first-grade pupils in an attempt to show that the teaching of classification, seriation, operations, and comparison with colored blocks is analogous to teaching the same things through set abstraction (73). His conclusions indicate that children trained in an attribute-block program conserve cardinality better than traditionally-taught children and show a superior ability to conceptualize addition-subtraction relations, but do not learn computational procedures to the same extent. Additional uses of this class of materials are described by Stern and Bridgers (79; 46).

## NUMBER BOARDS

Teachers have found various forms of number boards effective for representation of abstract mathematical ideas. The boards shown in Figure 9.25 allow young children to represent the concept of number in several ways. Each board has a raised numeral at the top, over which children can place the appropriate notation card. There is also a vertical channel into which the appropriate number of plastic cubes can be placed, and the raised outlines below

FIGURE 9.25

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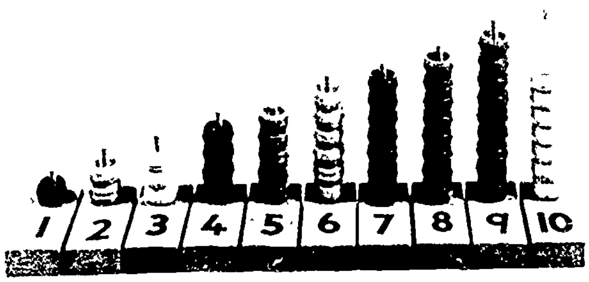



FIGURE 9.26

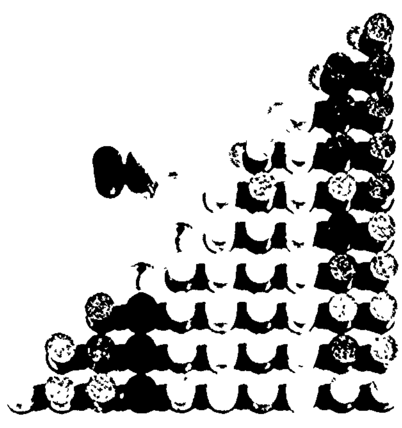


FIGURE 9.27

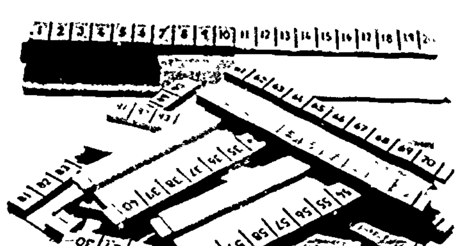


FIGURE 9.28

the numeral provide children with an opportunity to place cubes in a pattern. The raised outlines on the boards suggest an "odd and even" grouping. Boards of this nature give children experiences in associating objects, numerals, and numbers. In addition, the number property of equivalent sets and pattern relationships can be observed. Other boards of a similar nature have raised outlines in "domino" patterns for easy recognition.

Another important concept for primary children involves the ordinal property of number. Understanding of this property is enhanced by having children build and order sets. The apparatus shown in Figure 9.26 is a typical device that can be used for this purpose. Sets with from one to ten members can be built. The illustration suggests that there are ten separate baseboards, but there is really only one. The baseboard shown in the picture has been grooved to give the impression that there are ten little baseboards. Since there is only one baseboard, it serves as a control in the ordering process. A variation of this device that uses multicolored cylinders and a wooden frame serves the same function (Figure 9.27).

Activities with number trays can be effective substitutes for number line activities, especially in the early stages of learning when manipulation is desirable. Placement of cubes in interlocking trays can be used to illustrate counting, adding, subtracting, multiplying, and dividing numbers. The cubes in the tray in Figure 9.28 show an alternating pattern of odd and even numbers.

Perhaps the number board in most common use is the hundred board. Its flexibility permits concrete representation of counting, the four fundamental operations, the meaning of place value notation in the decimal system, and pattern analysis. Although available commercially in a variety of forms, number boards can be easi-

*Photos on this page, courtesy of Responsive Environments Corporation*

ly constructed by pupils or teachers. Two commercial variations are illustrated in Figures 9.29 and 9.30.

Volpel suggests that the simplest homemade board is one made of plywood with nails or pegs in the center of each square (83). Variations can be effected by using spools, curtain rings, rubber washers, or number cards with the board. Osborn's variation of the hundred board is a cardboard chart on which each numeral has a circle or square above it (81). Circles represent odd numbers; squares represent even numbers; black circles and squares represent prime numbers; and white circles and squares represent composite numbers. Circles or squares that represent products have lines across them—one line means one distinct pair of factors; two lines mean the number has two distinct pairs of factors.

A somewhat unusual use of the hundred board in the intermediate and upper grades is shown in Figure 9.31. The black beads are supposed to represent prime numbers; but one mistake has been made. If the board is used as shown in the display, children can be asked to correct the placement of the black bead so that all prime numbers are represented by black beads and all composite numbers by white beads.

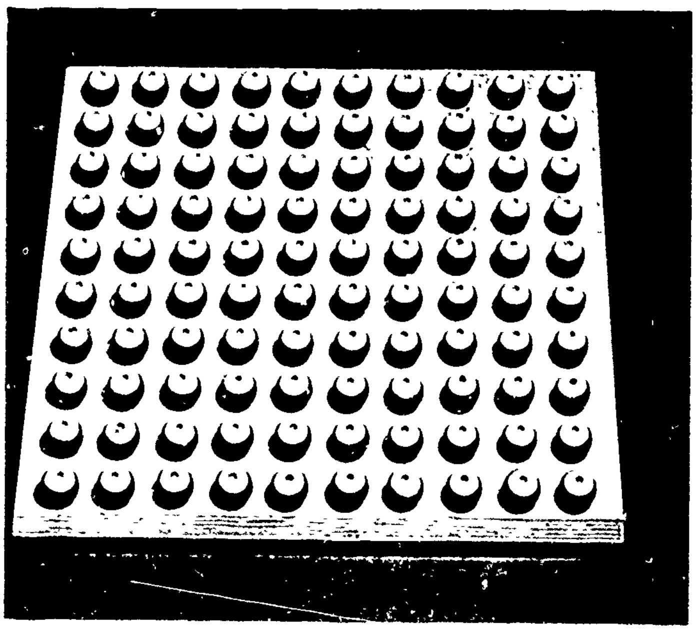
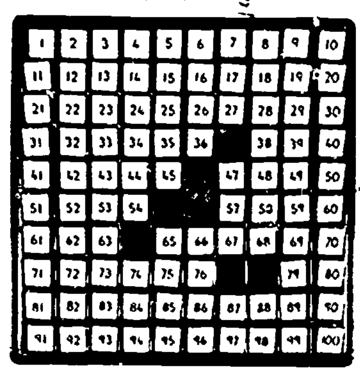


FIGURE 9.29

FIGURE 9.30



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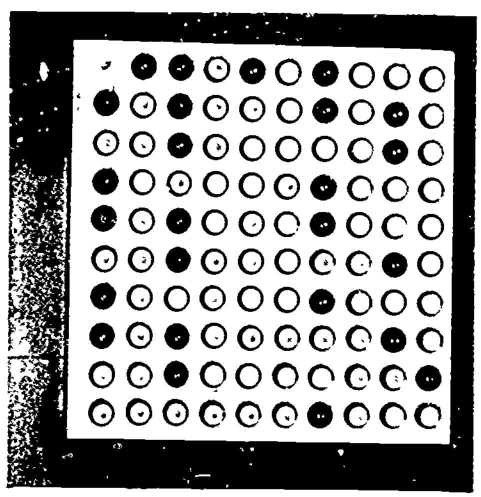


FIGURE 9.31

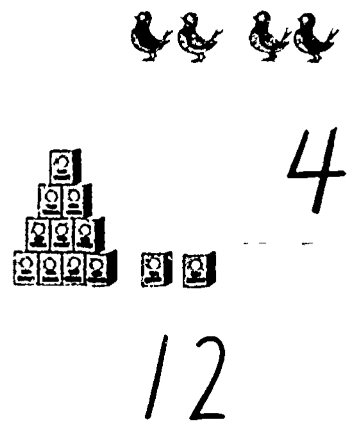


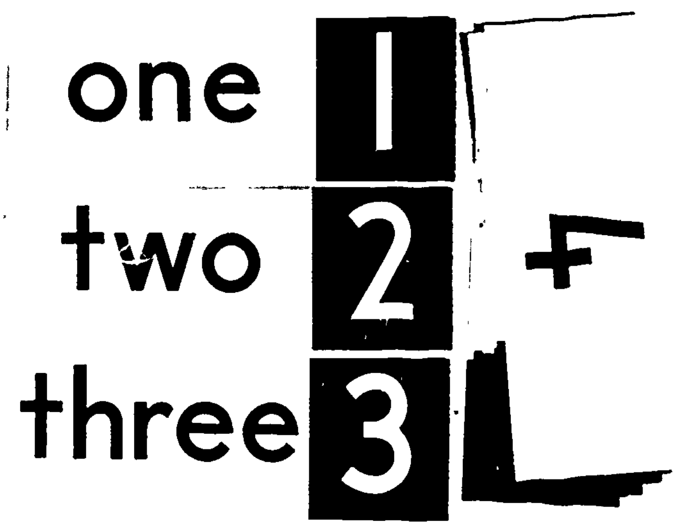

FIGURE 9.32

## CARDS AND CHARTS

Cards and charts for teaching mathematics have retained their popularity through the years. They are appealing to teachers not only because of their permanency but also because of their adaptability to individual and group instruction and to various methods of instruction. Many teachers feel that the use of cards and charts as two-dimensional representations of the physical world serves as a bridge between the concrete and the abstract (Figure 9.32).

This bridge is often referred to as the "semi-concrete" level. It is not known whether learning is enhanced by using cards and charts. The complete lack of research on this problem permits the teacher to select or reject activities involving these materials as he sees fit.

There are many uses of cards and charts that are unrelated to the concrete-semiconcrete-abstract sequence. Demonstration cards can be used to relate numerals and words that name numbers (Figure 9.33). Expanded notation cards can be used in conjunction with a flannel board to show the meaning of place value (Figure 9.34). At the manipulative level, notation cards can be used with a $10 \times 10$ tray. In Figure 9.35 a child has represented 34 concretely by placing 3 ten bars and 1 four bar on the tray. The correct notation cards, "30" with a "4" card placed over "0" in the units column, are shown at the bottom of the illustration.




*Courtesy of Responsive Environments Corporation*

FIGURE 9.33

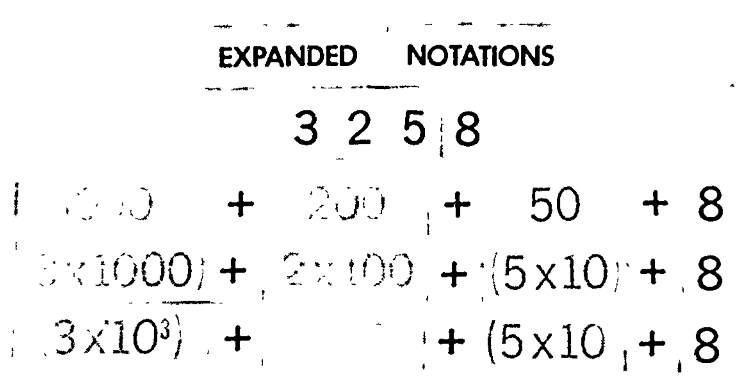
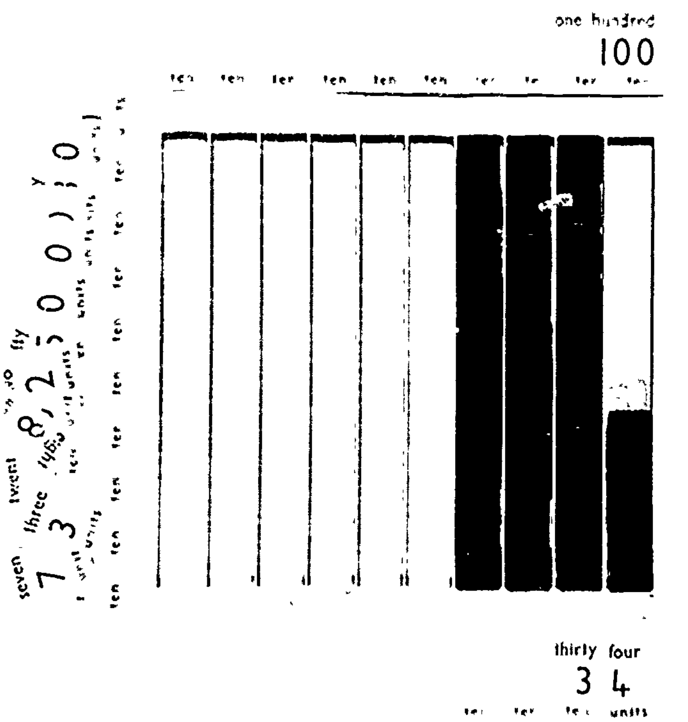



FIGURE 9.34




*Courtesy of Responsive Environments Corporation*

FIGURE 9.35

Drill or flash cards remain popular devices for promoting memorization of basic facts. It is important to remember that drill on basic facts should never take place before the child understands the nature of the operation on which he is to practice.[1] Other uses of cards, for building interest as well as learning concepts, are described in the section on games.

Two other important uses of charts that deserve mention at this time are pattern analysis and enrichment. One type of chart that can be used in pattern analysis activities is a matrix addition chart. A preliminary use of such a chart is to record addition facts as they are developed. For example, when the fact 2 + 3 = 5 was developed in the rabbit story, this fact could have been recorded on the chart as illustrated in Figure 9.36. Thereafter the chart could be used by pupils as a reference when a question relating to this fact occurred.

As the chart is expanded, children can observe developing patterns. One important use of patterns is in introducing the zero facts, in which at least one addend is zero, the identity element in addition. Since these sums are often difficult to objectify, children can predict the appropriate sum by analyzing the pattern (Figure 9.37).

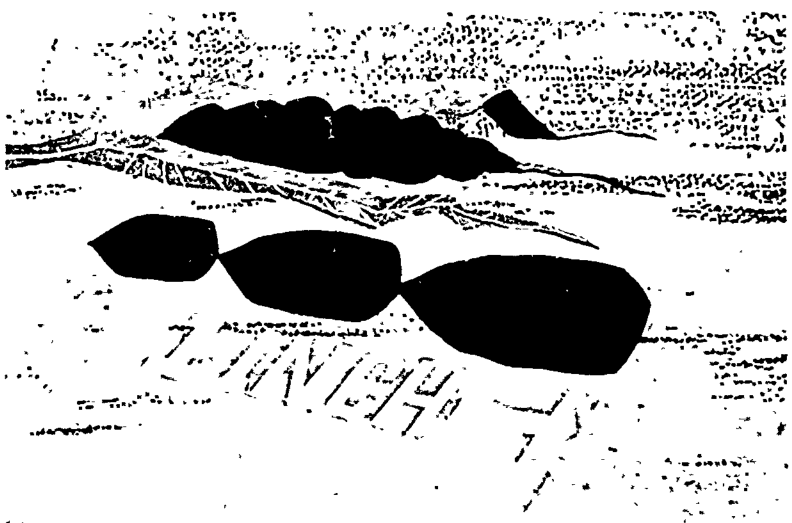
| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| 1 | | | | | | | | | | |
| 2 | | | | 5 | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | | | | | |
| 5 | | | | | | | | | | |
| 6 | | | | | | | | | | |
| 7 | | | | | | | | | | |
| 8 | | | | | | | | | | |
| 9 | | | | | | | | | | |

FIGURE 9.36

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | | | | | | |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | | | | |
| 4 | 4 | 5 | 6 | 7 | 8 | | | | | |
| 5 | | 6 | 7 | 8 | | | | | | |
| 6 | | 7 | 8 | 9 | | | | | | |
| 7 | | 8 | 9 | | | | | | | |
| 8 | | 9 | | | | | | | | |
| 9 | | | | | | | | | | |

FIGURE 9.37

1. See the discussion on drill and repetitive practice in *The Learning of Mathematics: Its Theory and Practice*, Twenty-first Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The Council, 1953).



*Courtesy of Ford Motor Company*

FIGURE 9.38. *In the first quarter of the fourteenth century King Edward II established an inch to be the length of three full, dry barleycorns from the center of the ear, laid end to end.*

*Courtesy of Ford Motor Company*

FIGURE 9.39. *During the sixteenth century the legal rod was determined by lining up the left feet of sixteen men as they came out of church on Sunday morning.*

After the chart has been completed, children can pursue a pattern search for interesting relationships such as properties of the addition operation and so on. Charts for other operations can be purchased or prepared. Charts can also be used to examine characteristics of nondecimal numeration and finite mathematical systems.

The use of charts for enrichment is illustrated by Figures 9.38 and 9.39. These two charts can be used to help children gain an understanding of the historical development of mathematics, and this, in turn, can lead to an appreciation of concepts being learned.

Other specific suggestions for the use of cards and charts are made by Deans, Drasin, Ingraham, Machlin, Michalov, Overholser, and Williams (84–90).

## MEASUREMENT DEVICES

Yardsticks, metersticks, chalkboard compasses and protractors, balances, and individual student rulers, compasses, and protractors should be available as standard classroom equipment for use in developing measurement concepts. For example, pupils might be given a number of foot-square pieces of cardboard and asked to cover the floor with these square shapes. After finding the area of the floor in this fashion, they can be asked to find the dimensions of the floor using a yardstick and then asked to compute the area of the floor by using the formula $A = lw$. A student who has had such experience is not apt to confuse a linear unit with a square unit and will have a good idea of area.

Sets of materials like the Lake and Islands Board shown in Figure 9.40 are available and serve a similar purpose. The set shown in the picture contains unit cubes, unit flat pieces, and a large piece of masonite on which irregular shapes are painted. Concepts involving both area and volume can be developed using such a set.

A trundle wheel (Figure 9.41) can be used to measure distances. By placing the wheel so that the starting mark is down and pushing the wheel along a marked line, distances can be

measured by counting rotations of the wheel. By using a trundle wheel the child has opportunity to gain an intuitive idea of the relation between the diameter and circumference of a circle. Problems that involve finding distances by using a trundle wheel are easily visualized and solved by the pupil.

Sets of devices for measuring capacity are available in many different shapes and sizes. One such set is shown in Figure 9.42. Other types are discussed in the section on models of geometric relationships.

*Photos on this page, courtesy of Responsive Environments Corporation*

FIGURE 9.41

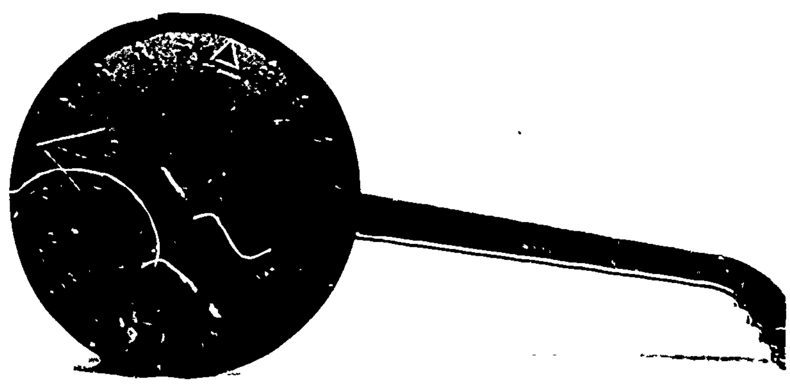


FIGURE 9.40

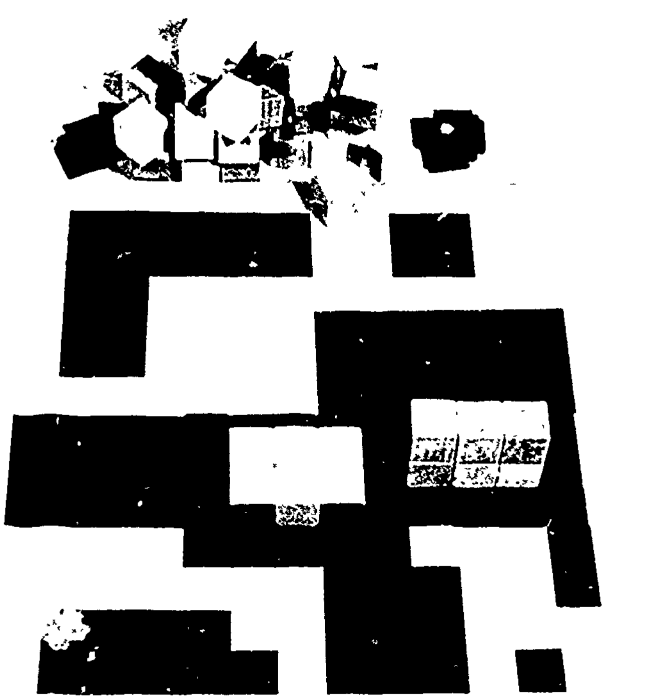


FIGURE 9.42

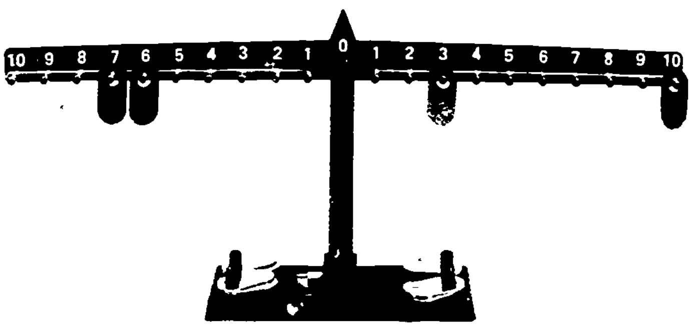



FIGURE 9.43

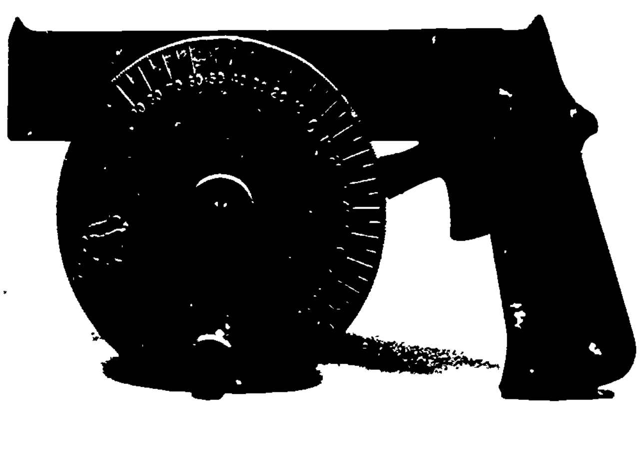


FIGURE 9.44

*Photos on this page, courtesy of Responsive Environments Corporation*

Balances such as the one shown in Figure 9.43 can be used for various purposes. In the primary grades the device can be used to illustrate, for example, that $7 + 6 = 3 + 10$, as shown in the picture. At a later stage, the balance can be used with problems involving levers. In the latter case, the illustration could be interpreted as showing that

$$(1 \times 7) + (1 \times 6) = (1 \times 3) + (1 \times 10).$$

More than one weight can be hung on a peg. Two weights placed on the 6 peg at the left would balance three weights placed on the 4 peg at the right, since

$$2 \times 6 = 3 \times 4.$$

The unusual looking device shown in Figure 9.44 is a clinometer. This device can be used to measure angles of elevation and angles of depression.

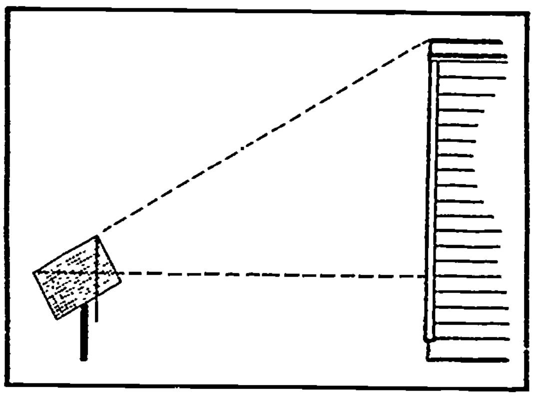
The clinometer is often combined with a hypsometer. The hypsometer is a simple device that can be used to determine the heights of objects such as buildings and trees, and horizontal distances. The drawing in Figure 9.45 shows a homemade hypsometer combined with a clinometer. The hypsometer part consists of a graph chart fastened to a piece of cardboard mounted on a pole. The angle the hypsometer makes with the pole can be fixed by using a bolt with a wing nut. A soda straw glued to the graph chart along the upper edge serves as a sighting tube, and a fishline and sinker suspended from the upper right-hand corner of the graph chart serve as a plumb line.

To find the measure of an angle of elevation with the homemade combination device, simply sight through the sighting tube and read the angle measure the plumb line points out on the protractor.

As you can see from the diagram, the scale at the right of the graph chart indicates horizontal distances from an object and the scale at the bottom indicates heights.

To find the height of a building, set up the device at some convenient distance from its base, say 50 feet. Sight to the top of the building through the soda straw and locate the intersec-

tion of the plumb line and the 50 (foot) line on the right-hand scale of the graph chart. The dotted lines on the graph chart indicate where to read the measure of that part of the height of the building that is *above* the device. Since the number pointed out on the bottom of the graph chart is 23, the height of the building is 23 feet plus the height of the device.



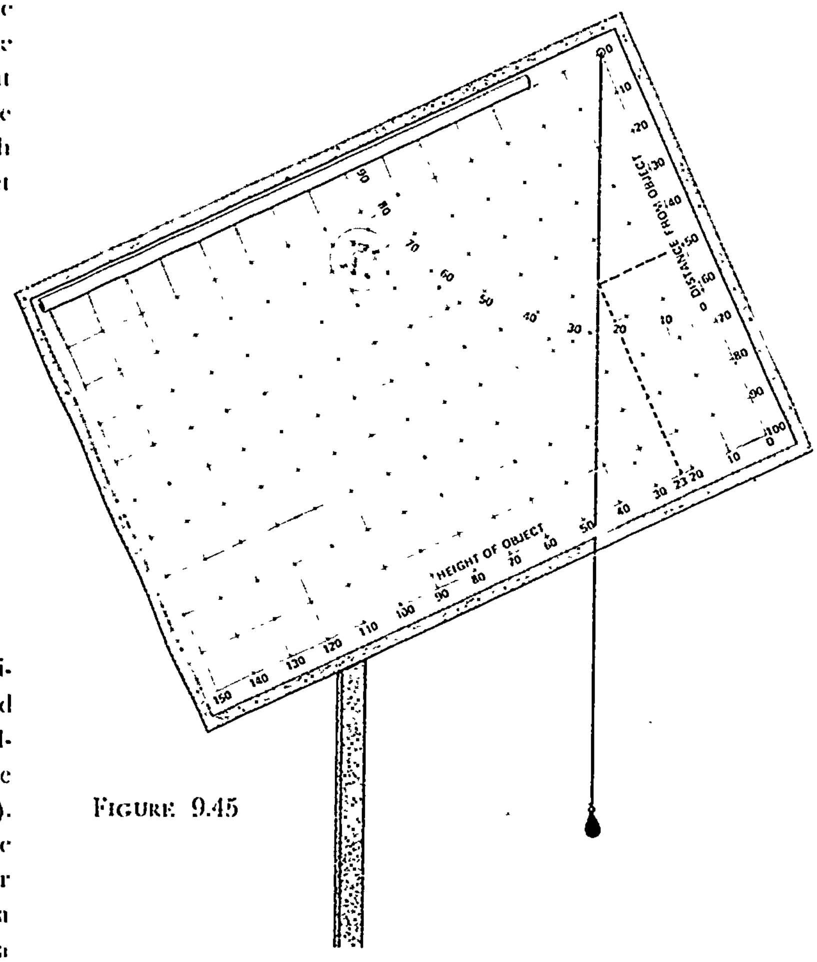



FIGURE 9.45

Other appropriate devices for field work activities include the angle mirror, the alidade and plane table, and the transit. Shuster and Bedford describe methods of constructing these devices and suggest activities for their use (96).

The results of a limited study by McClintic show a preference by kindergarten children for the use of a unit type of measure such as a cubical counting block, rather than a rod or a tape, to obtain overall length dimensions (94). Bourne reports that experiences with young children seem to show that ideas of three-dimensional space or volume appear to develop more easily than ideas of area (91). He suggests that to establish a firm concept of volume, an activity such as packing solid wooden cubes into rectangular containers be used. Similar activities can be used to develop a firm concept of area by considering the amount of space in a plane region (area) rather than the amount of space occupied in three dimensions (volume). A suggested activity is counting the number of objects of identical size needed to cover a plane region.

In a study of beginning first-grade children's concepts, Mascho concluded that measurement should be taught in situations in which pupils can actually use the unit of measure involved, thus beginning at the concrete level (93). He suggests that measurement environments rich with concrete materials be provided by the first-grade teacher in order that pupils can solve everyday in-school measurement problems. Parker proposes the use of a project to motivate pupils in making various measurements around the school (95). Such a project tends to increase pupils' skill in using actual measuring tools.

Swart recommends the development of a measurement laboratory that would include material to provide experience in measurement (97). He lists some materials available for less than ten dollars and describes how such a laboratory could be used.

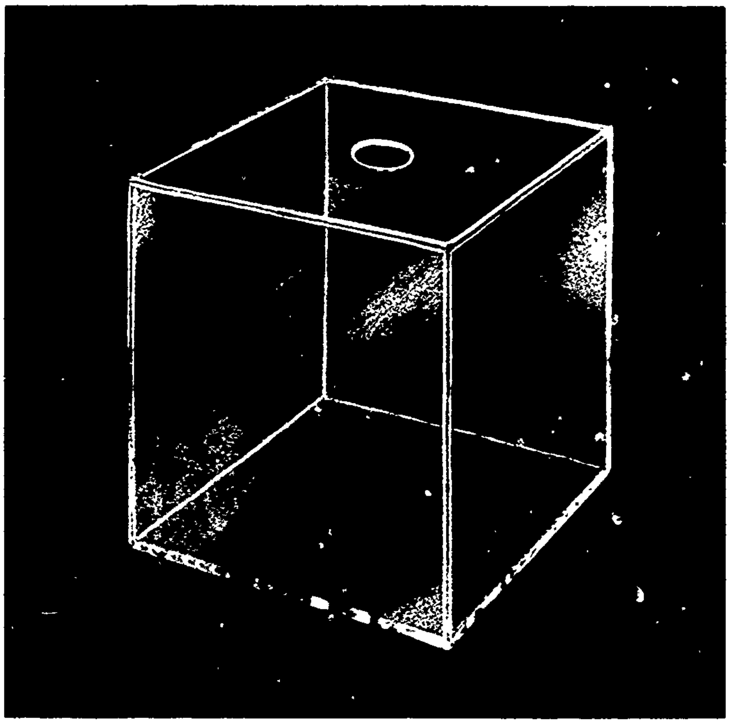


FIGURE 9.46

## MODELS OF GEOMETRIC RELATIONSHIPS

Geometric models include the familiar wooden cubes, cones, cylinders, and spheres and, in addition, many other sophisticated wooden, metal, and plastic models. Some models such as the one shown in Figure 9.46 can be filled with sand or water and used when studying volume.

Flat blocks such as those shown in Figure 9.47 can be used to help pupils visualize work with areas and volumes and to make designs.

Plastic blocks similar to those shown in Figure 9.48 are useful in design work, comparison of areas, and in working with rational numbers.

Geometric puzzles such as those shown in Figure 9.49 provide interesting enrichment material.

Angle boards such as those shown in Figure 9.50 are helpful in providing visualization in working with angle measure.

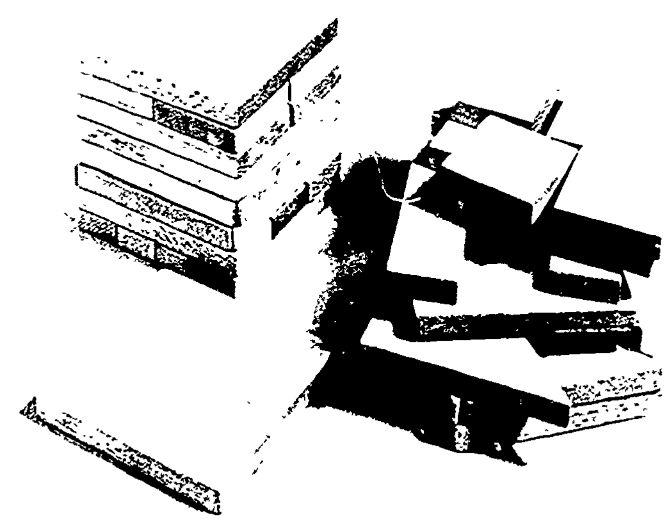
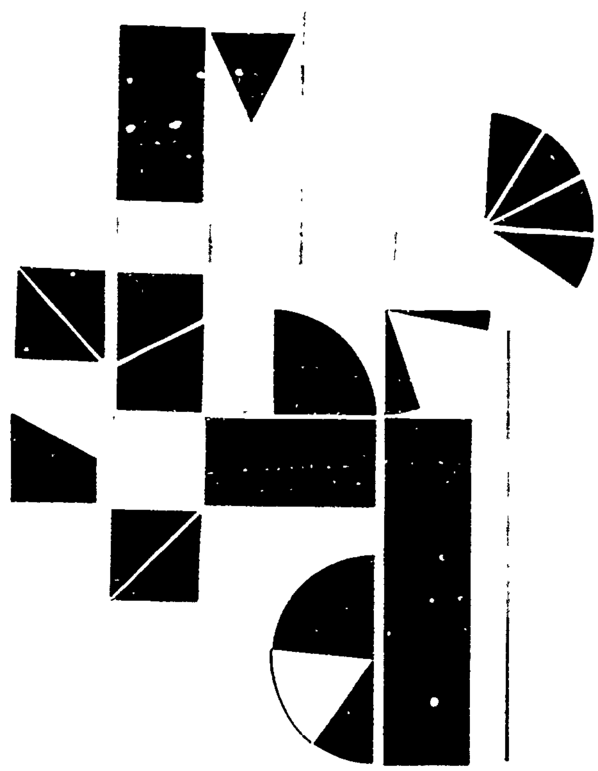
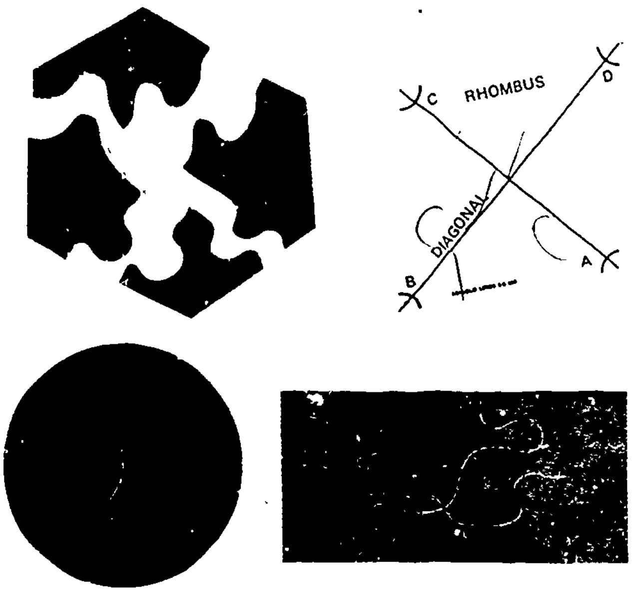


FIGURE 9.47

FIGURE 9.48



*Photos on this and the following page, courtesy of Responsive Environments Corporation*



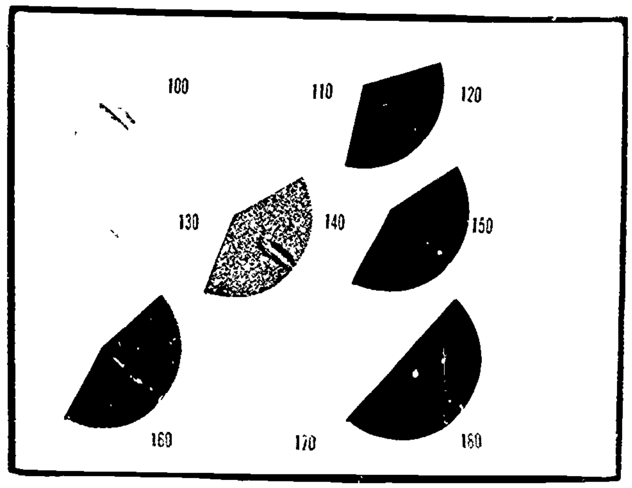




FIGURE 9.50

FIGURE 9.49

Johnson describes a large number of geometric concepts that can be illustrated by paper folding (101). Included are the basic geometrical constructions, the building of three-dimensional paper models, and the formation of regular polygons by tying knots in strips of adding machine paper tape.

In two articles on geometry in the primary grades, Vigilante notes that at this level geometry should be approached informally, using a theory of instruction based on multisensory experiences (108; 109). These experiences should provide opportunities for exploration, manipulation, and development of spatial perceptions. Smith concurs by stating that children require things they can handle, see, feel, work with, and produce (105).

The construction and use of three-dimensional models is a recommended activity. Buck suggests the building of regular tetrahedra out of D-Stix (99). Black describes a program using large wooden solids to help students find various shapes in their environment and discover basic properties of shapes (98). Wahl provides a set of patterns designed for ease of cutting and pasting (110). Woods and Hoff feel that experiences with mathematical models for geometries of more than three dimensions should be provided at the elementary level to reduce the difficulty pupils have with ideas involving more than three dimensions (113).

Combinations of pegboards and string have been suggested by many. Smith suggests using pegs and rubber binders with the pegboard to provide a successful medium for portraying geometric ideas (106). Hewitt used a length of venetian-blind cord to fashion geometric figures and then tied knots at intervals of one foot to obtain a measurement tool to enable pupils to find the lengths of the sides and diagonals of polygons (100). Using colored thread and a pegboard or heavy cardboard, Knowles had pupils construct geometric figures using only straight lines (102). Major outlines a different, axiomatic approach to geometry in the upper elementary grades, an approach based on rings and strings used with a pegboard (103).

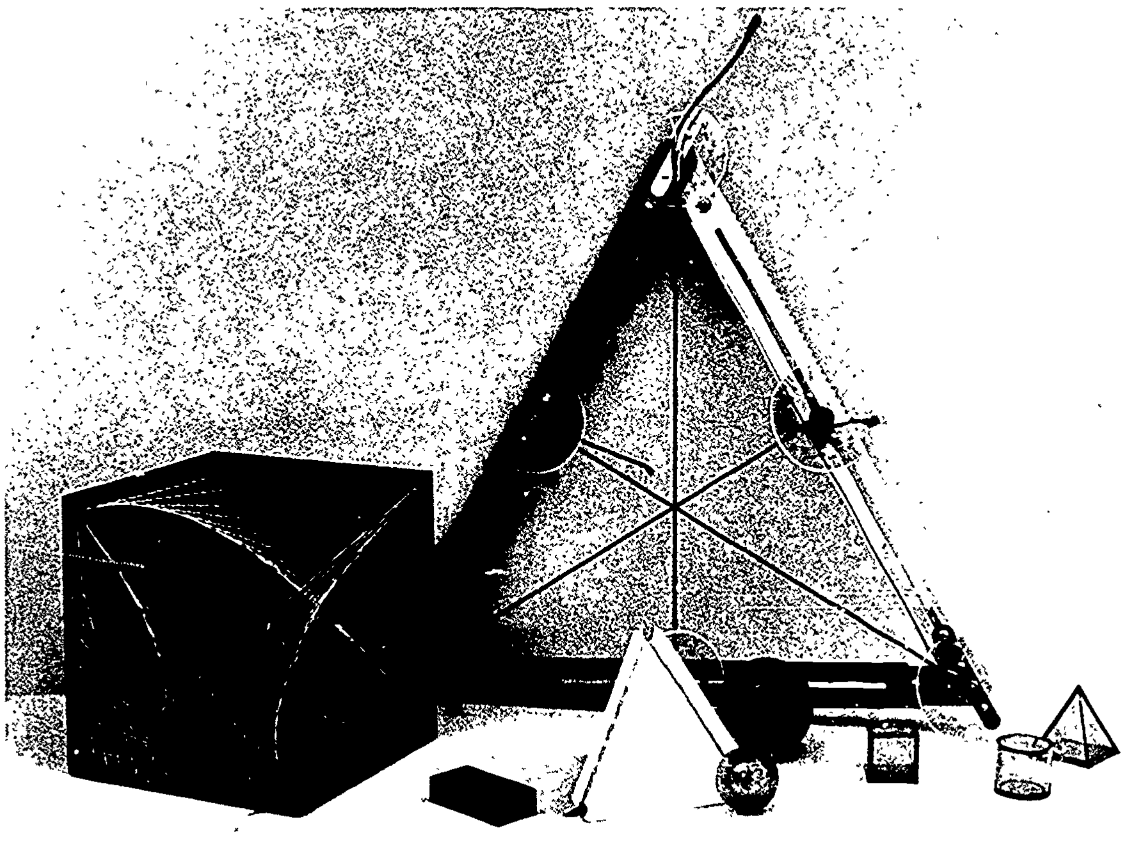


FIGURE 9.51

Curve stitching, as shown at the left in Figure 9.51, is always popular with pupils and provides an excellent opportunity for creative design work. At the present time, many mechanical drawing devices are available for purchase in variety stores and in the toy sections of department stores. The Spirograph and the Magic Designer are two devices that children can use to create their own designs by using the colored-ink pens and the rotating parts included with the devices.

Metal and plastic models that incorporate the use of elastic are useful in illustrating geometric propositions. The one shown in Figure 9.51 can be used to illustrate that the medians of a triangle are concurrent. Since this model can be adjusted to show an equilateral triangle, children can observe that the medians, altitudes, and angle bisectors of an equilateral triangle all coincide. The model can be easily adjusted to show that this is not the case if the triangle is isosceles or scalene. Such manipulation results in a saving of teacher time because the geometric figures do not have to be constructed on the chalkboard.

Other types of devices have also been suggested. Uncapher developed an "Object-a-Screen" on which geometric figures can be represented by different arrangements of light behind a translucent plastic screen (107). Walter created Mirror Cards to provide a means of informally obtaining geometric experiences that combine spatial insight and play (111). Richards reports many possibilities of using Tinkertoys in constructing models of geometric figures (104).

## GAMES AND PUZZLES

The changing structure of the elementary classroom from a teacher-centered, dogmatic orientation to one of a child-centered, learning laboratory where learning is interesting, enjoyable, and often fun, has considerably altered the role of games and puzzles. In days gone by, games and puzzles were reserved for kindergarten and early primary grades with the exception of their use at other grade levels on the day preceding a holiday as a "something to do" activity. They were not viewed as an integral part of the curriculum and were seen by teachers and administrators as a device employed by weak teachers incapable of structuring a legitimate lesson.

Today a strong case can be made for including games and puzzles as a means of attaining desirable outcomes of mathematics instruction. They seem to be especially well-suited for programs of drill and practice, providing for individual differences, building desirable attitudes, and encouraging problem solving. Although commercially prepared games are becoming more and more popular and their numbers are rapidly increasing, the creative nature of elementary teachers and pupils provides sources of ideas for commercial companies. Typical of teacher-made games is Mathmagicland (Figure 9.52).[2]

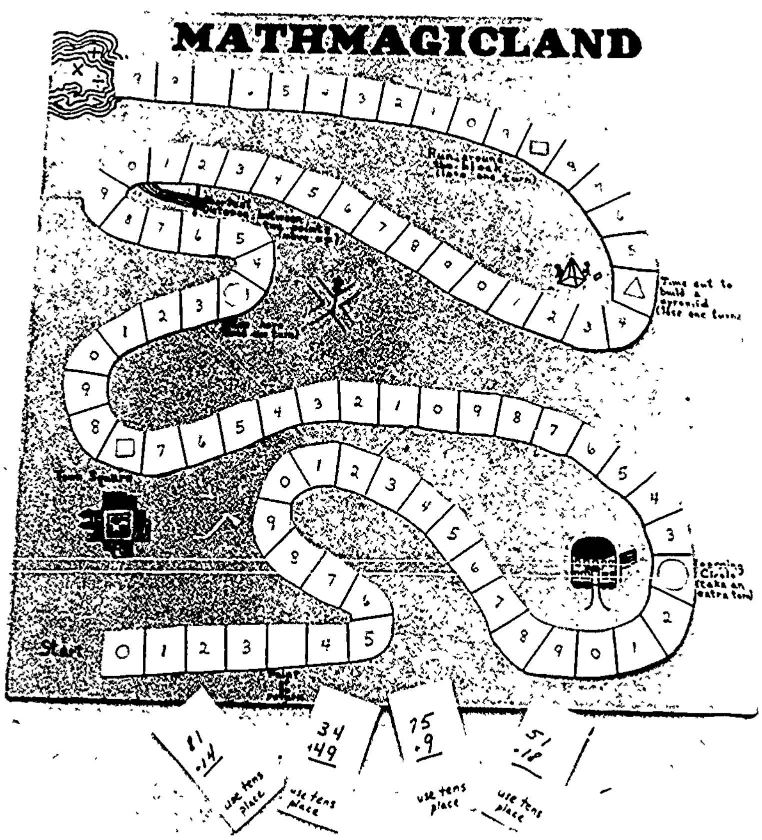



FIGURE 9.52

### MATHMAGICLAND

*Objective:*

To give practice in basic facts of addition and subtraction at the primary level and multiplication at the intermediate level

*Materials:*

Four markers; white cards containing basic facts without answers; geometric shapes; red cards containing basic facts without answers; "tracks" for addition and subtraction, and multiplication; and the board

*Rules:*

1. The game may be played by 2–4 players.
2. Moves are determined by drawing a white card.
3. Each player begins at the place marked "Start." Each player chooses one card, and the player with the card showing the greatest basic fact starts the game. The cards drawn are then buried at the bottom. The play is then from left to right.
4. Play begins when the first player draws a white card. He computes the fact and moves his marker to the nearest numeral that designates the answer. (Example: $2 + 4 = 6$.) The player moves his marker to the first "6."
5. The numbers in the primary track are arranged in sequence from 0 to 9 and then begin repeating the same sequence. Interspersed in the track are frames containing geometric shapes. The numbers in the in-

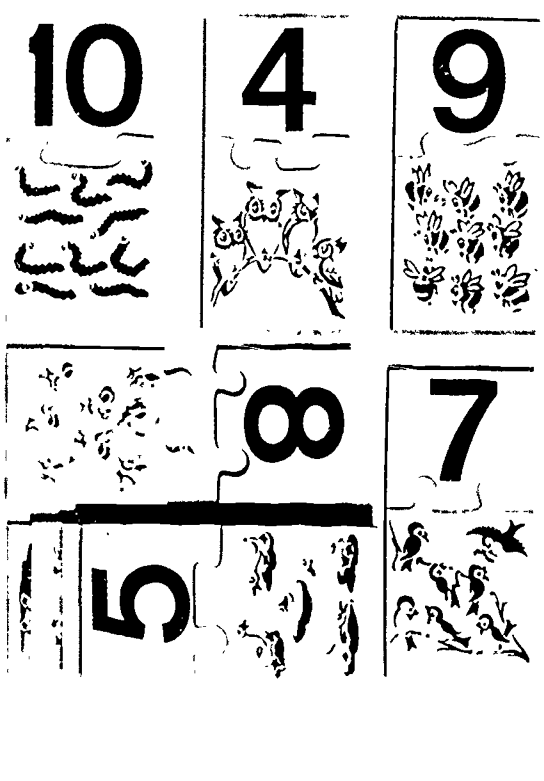
2. Candy Takkunen, Judy Kaplan, Peggy Mahan, and Ruth Alverson, four former students of the authors of this chapter, are responsible for originating this game.

termediate level are multiples of a certain number (e.g., 3: 3, 6, 9, 12, . . .). These numbers are not in sequential order so that children cannot compute them by checking the board.

6. Play continues with each person drawing a white card and moving his marker to the next sequence of numbers until one player reaches Mathmagicland.
7. If a player chooses a card containing a geometric shape instead of a basic fact, he moves his marker to that shape directly and follows the directions there, if any.
8. If a player's marker should land on a frame already occupied, this player must choose a red card, give the basic fact, and then move his marker backward to the nearest numeral that designates the answer.
9. A variation at the primary level is to have facts with answers greater than 9 and have the children move according to the figure in the tens place instead of the ones place.

Figure 9.53 illustrates the use of a puzzle situation to help young children match sets of objects with numbers.

FIGURE 9.53




A popular commercial game that provides pupils with activities involving addition is dominoes. Figures 9.54 and 9.55 illustrate two of many variations of the domino game. Practice is provided in this case with fractions, decimals, and percents.

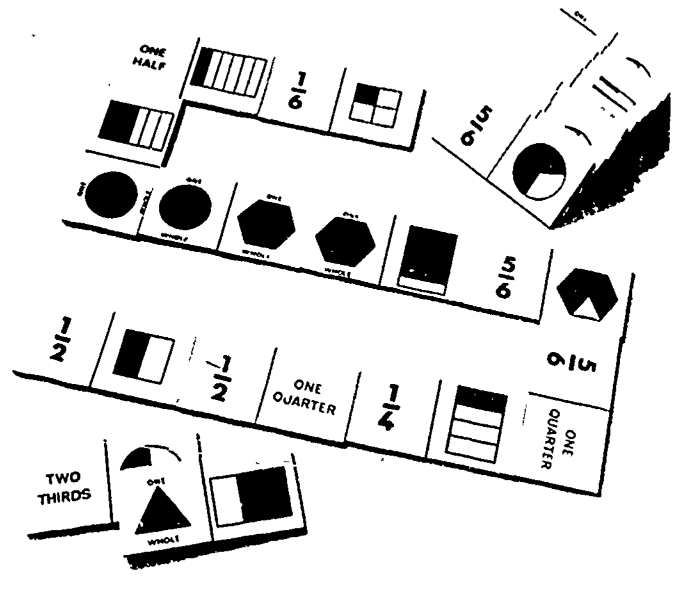



FIGURE 9.54

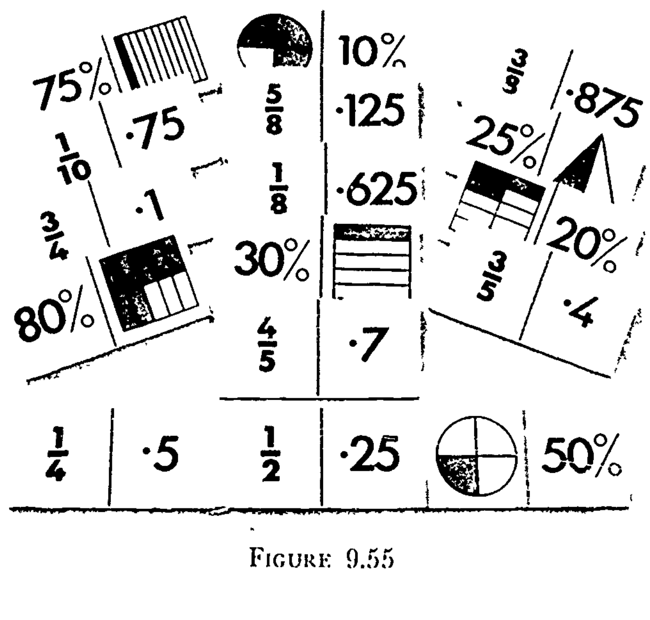



FIGURE 9.55

*Photos on this page, courtesy of Responsive Environments Corporation*

An exciting card game that combines the ability to compute with the ability to solve problems is Krypto (Figure 9.56). Five cards are dealt to each player, and at a given signal a single card, the Krypto card, is turned up from the top of the remaining cards in the deck. By using the four fundamental operations with all five cards in a player's hand and using each card once and once only, the player's answer must match the Krypto card. The player who successfully solves his problem calls out "Krypto" and must then explain his process to the other players. If his explanation and computation are correct, he receives a total score equal to the sum of the numbers represented in his hand and the Krypto card. If his work is incorrect, this amount is subtracted from his score. In the hand shown in Figure 9.56, the Krypto card can be equaled in the following way:

$10 \div 2 = 5.$
$5 + 4 = 9.$
$9 + 15 = 24.$
$24 - 13 = 11$ (the Krypto card).


FIGURE 9.56

*Courtesy of Krypto Corporation*

Manipulative puzzles such as tangrams and Instant Insanity illustrate possible roles of puzzles in problem solving. Shown in Figure 9.57 are the seven pieces of the Chinese tangram puzzle. The object of this puzzle is to put the seven pieces together to form a square. The challenge in the Instant Insanity puzzle is to arrange the four cubes in a row so that no two faces having the same color face up, down, away from the observer, or toward him (Figure 9.58).[3]

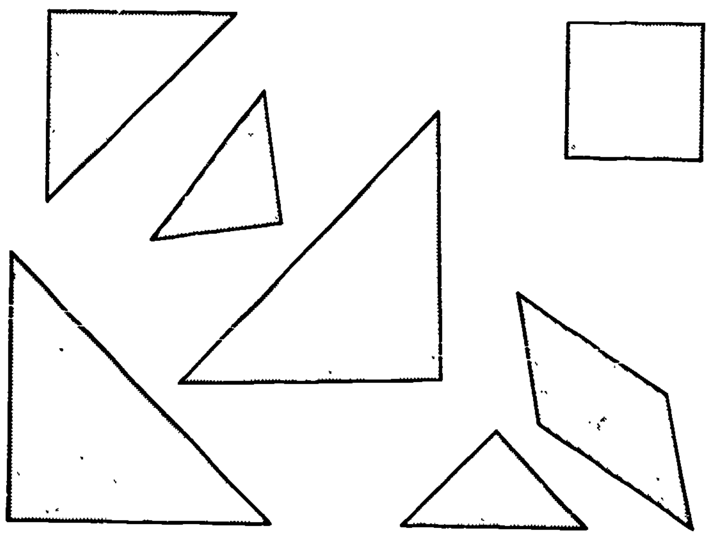


FIGURE 9.57

3. For a solution to the Instant Insanity puzzle see "A Note on Instant Insanity," by T. A. Brown, *Mathematics Magazine*, September 1968, pp. 167–69.

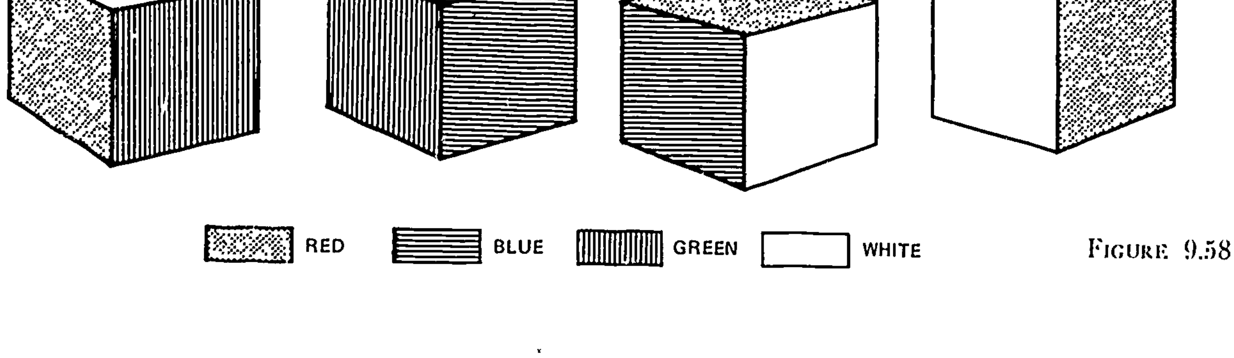



FIGURE 9.58

There are many sources of information on types and uses of both commercial and teacher-made games and puzzles. These include Calvo, Deans, Dohler, Haggerty, Inbody, Johnson, Jordan, Parker, Ruderman, Shurlow, and Timmons (114–25).

## SPECIAL COMPUTATIONAL DEVICES

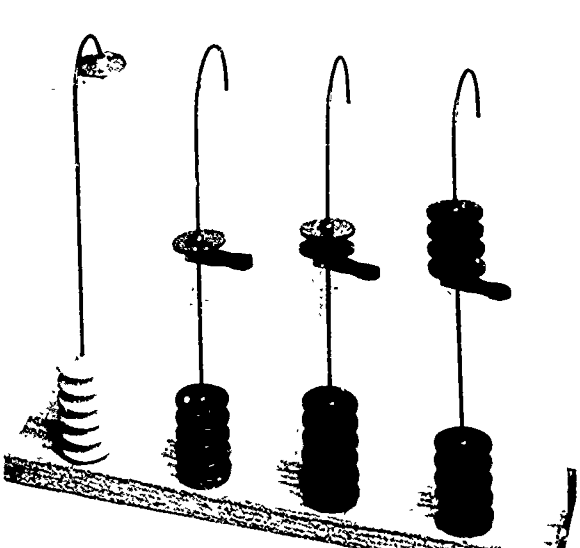
Computational devices have been used by men throughout history to increase the speed, accuracy, and efficiency of mathematical calculations. The ancients found the abacus to be an effective device for doing calculations, and the widespread use of the abacus in the Orient today confirms that it has continued to be an effective calculating device. Shown in Figure 9.59 is a modern version of the abacus, designed for classroom use.

The historical significance of the abacus as a computing device justifies the use of class time for exploration of its use.

In the culture of twentieth-century America simple desk calculators that do not require years of training to operate and are nearly always available when needed have been developed (Figure 9.60). Students should have ample opportunity to use desk calculators.

The extensive use of the number line in modern mathematics programs has drawn new attention to such devices as the addition-subtraction slide rule. Figure 9.61 shows an inexpensive demonstration addition-subtraction slide rule constructed with two yardsticks. To operate this slide rule, locate the first addend on the bottom rule. Slide the top rule to the right so that the index (zero end) of the rule is aligned above the first addend. Find the second addend on the top rule and read the number on the bottom rule that is aligned with the second addend. The number indicated on the bottom rule is the sum. For example, the slide rule pictured in Figure 9.61 is set to add 8 and 5. The 8 is located on the bottom rule and the top rule is moved to the right so that the index is opposite 8. Now 5 is found on the top rule and the sum of 8 and 5, which is 13, is found on the bottom rule. To subtract one number from another, align the two numbers and read the answer opposite the index. The difference, 17 − 9, can be found in

FIGURE 9.59



*Courtesy of Ideal School Supply Company*

FIGURE 9.60


*Courtesy of Clary Business Machines Company*

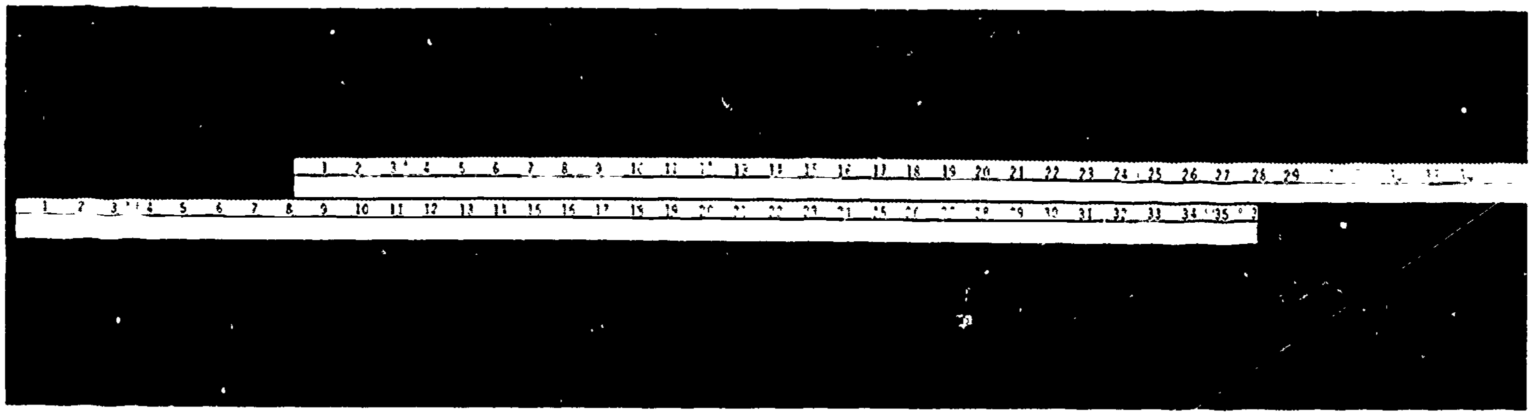
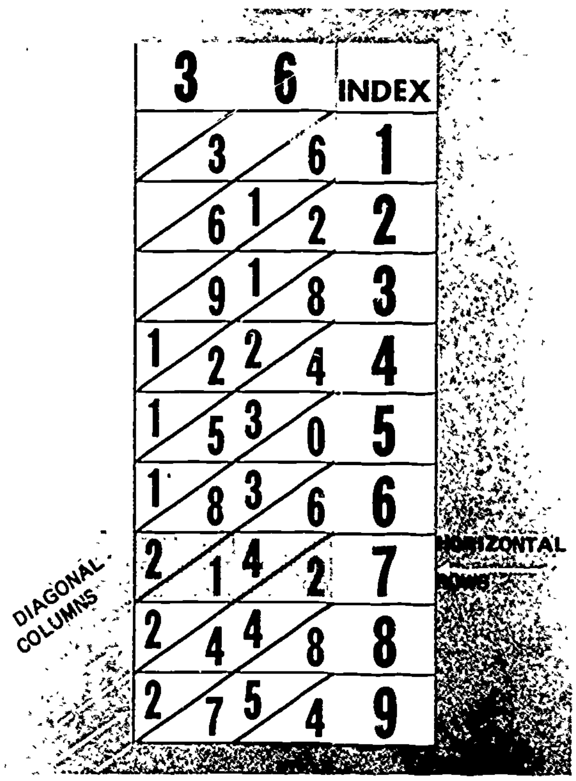


FIGURE 9.61

Figure 9.61. A slide rule made with yardsticks can also be used for addition and subtraction of certain rational numbers. Ordinary foot rulers are appropriate for making addition-subtraction slide rules for use by pupils.

Another example of a computing device is called Napier's rods. This device, which is an adaptation of the lattice method of multiplication, consists of ten rods and an index rod. The rods are usually referred to as the 0-rod, 1-rod, 2-rod, and so on. Pictured in Figure 9.62 are the 3-rod, the 6-rod, and the index rod. A complete set of rods is pictured in Figure 10.3 in the next chapter. The rods may be in the form of rectangular prisms or rectangular cardboard strips as in Figure 9.62. The top cell of each rod is labeled by the numeral that names the rod. This numeral also represents the first of two factors of a product. The remaining cells on each rod are labeled by numerals that represent the products of the first factor and the numbers from 1 to 9. The diagonal slash in each cell separates the tens and units digits of each product. To find the product 3 × 6, place the index rod next to the 6-rod as shown in Figure 9.62. Then locate 3 on the index rod, and in the same horizontal row read 1/8 on the 6-rod. This is the product, 18.

To find the product 7 × 36, read the numbers in the horizontal row to the left of the multiplier 7 on the index rod. Each *diagonal column* in the row represents *one* digit in the product. If there is more than one digit in a diagonal column, the numbers represented by them must be added. In the present example, the first digit on the right of the desired product is 2. The second diagonal column contains two digits, 1 and 4. Since 1 + 4 = 5, the second digit of the product from the right is 5. The diagonal column on the left contains a 2, which is the third digit of the product. Hence, the product is 252.




*Courtesy of Ideal School Supply Company*

FIGURE 9.62

The special calculator devices described thus far are for use with the decimal system of numeration. Except for the desk calculator, each of the devices can be adapted for use with nondecimal numeration systems.

It is not important for the child to memorize the basic facts of addition or multiplication when studying nondecimal numeration systems, but it is necessary for him to have these facts available in exploring and discovering characteristics of such numeration systems. A set of nondecimal Napier's rods or a nondecimal abacus can be most helpful in handling computational situations.

Numerous forms of circuit boards have been designed which allow students interesting practice periods on basic facts. The Tabletamer, a printed circuit that operates with a flashlight battery, is one such device. The basic Tabletamer device is pictured in the upper left-hand part of Figure 9.63. It can be used by students to practice multiplication facts. To check the product of two numbers, the student plugs in the left-hand electrode below the pair of numbers to be multiplied and plugs in the right-hand electrode below a possible product. If the answer is correct, the circuit is completed and a green light flashes on at the top of the board. Plastic overlays can be used to change the basic device so that it can be used to provide practice

*Courtesy of AIM Industries, Inc.*

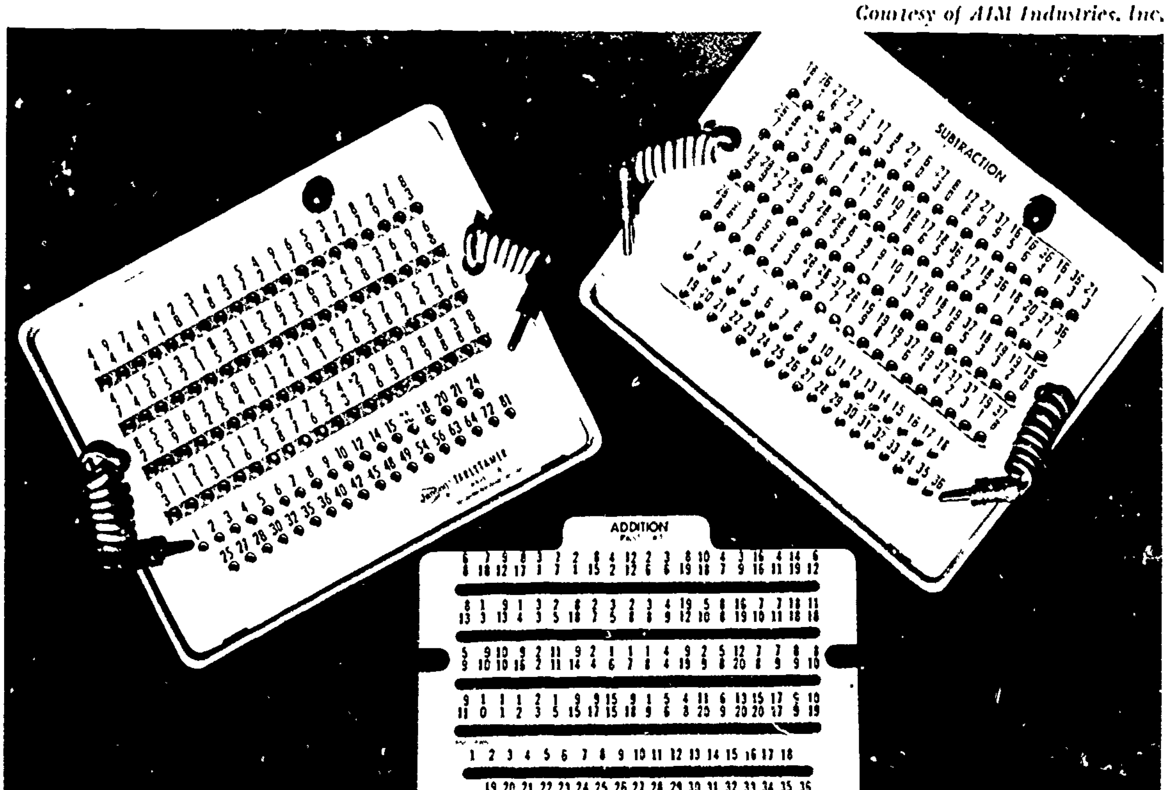



FIGURE 9.63. *Pictured at the upper left is the basic Tabletamer device for practicing multiplication facts. At the upper right is the same device with a subtraction overlay. An addition overlay is shown at the bottom. The same device is used for all three kinds of facts.*

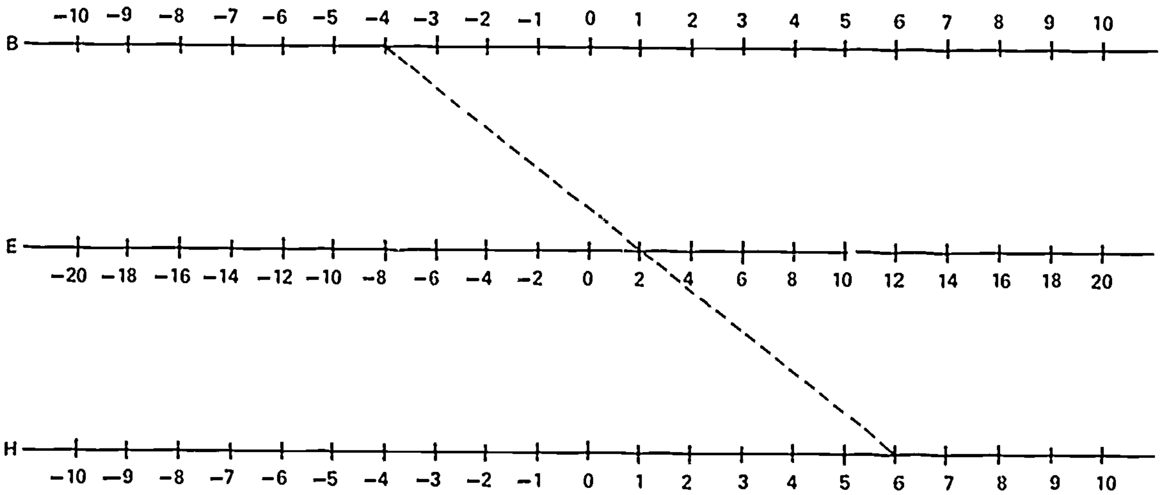



FIGURE 9.64

with addition and subtraction facts.

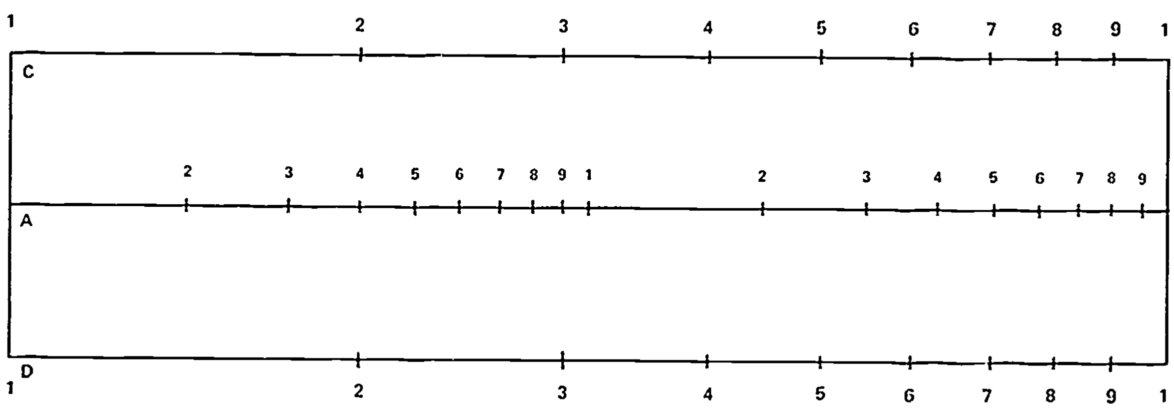
Another special computational device that has wide application in science and industry as well as in the classroom is the nomograph. It allows the user to do rapid calculations by reading numbers from scales drawn on graph paper or notebook paper. The diagram in Figure 9.64 illustrates a nomograph that is designed to add and subtract integers.

It has three parallel scales, B, E, and H. Each scale is perpendicular to the line that includes the zero point of each scale. The B and H scales are the same distance from the E scale. The length of the unit segment on the B scale is the same as on the H scale. The unit segment on the E scale is half as long as on the B and H scales.

The relation among the coordinates of the three intersection points determined by a line that crosses the three scales is $B + H = E$. The diagram in Figure 9.64 illustrates the example $-4 + 6 = 2$.

A multiplication and division nomograph can be produced rather easily. Obtain a standard slide rule and copy the C, A, and D scales from the slide rule on a diagram as shown in Figure 9.65. If you place a straightedge across the three scales in the diagram, the points of intersection are related by the formula $C \times D = A$. To multiply two numbers, locate one number on the C scale and the other on the D scale. The line joining these two points will cross the A scale at the point representing the product.

FIGURE 9.65



The use of the abacus as a computational device has been recommended by many authors. Hamilton explains its use in addition and multiplication (11); DeVault extends its use to multiplication (19); and Smith discusses addition with the abacus (62). Jenkins presents a unique story idea of the abacus as a computing tool (136). Healy describes how the abacus can be used in the upper elementary grades to strengthen understanding of all the operations, including work with decimals and common fractions (53). Other suggestions for using the abacus as a computational device are presented by Cunningham, Peterson, and Spross (18; 140; 31). These views are perhaps best summarized by Bernstein, who points out that the advantage of the abacus as a computational device is that pupils can use visual, tactual, and kinesthetic senses (4).

The number line as a computational device has frequently been discussed in the literature, and most classroom teachers are familiar with it. Ashlock, Clark, Cochran, Coon, Overholt, Schmickrath, Sganga, and Shaw all suggest ways in which the number line can be used (126; 128–30; 139; 143–44; 146). A note of caution regarding the introduction of the number line too early is given by Gibney (133). He suggests that much experience with concrete objects must precede any attempt to deal with number symbols, as the change from objects to symbols on the number line appears to be difficult for many children.

A pupil-made slide rule is suggested by Gramlich as an effective computational device (134). The advantages he lists include pupil interest, carry-over to out-of-school activities, an introduction to measurement in physical science, and quick estimation of reasonable answers to a problem.

Kreitz and Flournoy suggest using Napier's rods to stimulate interest in multiplication and to compare placement of partial products in this and the conventional methods of multiplication (137).

Other forms of computing devices are described by Hyde and Nelson and by Lawlis (135; 138).

## CONCLUSIONS REGARDING RESEARCH

The examination of the literature and research in each of the nine classes of aids serves to reinforce the comments made at the beginning of the chapter. The use of manipulative devices in teaching elementary school mathematics is an accepted procedure and appears to be supported by learning theorists and educators alike. Two dangers should be noted in passing. The first danger is that of a teacher's wholesale acceptance of one all-purpose device. Most proponents of the use of manipulative devices agree that a wide variety of devices should be used.

The second danger arises from the belief that devices teach mathematics. Devices themselves do not teach mathematics. It is their use under the guidance of a wise teacher that determines their effectiveness in facilitating pupils' learning. This means that it is necessary for the teacher to have a thorough background in the pedagogical as well as the mathematical principles involved in the use of manipulative aids.

Thus the question does not appear to be one of using manipulative devices or not using them. Rather, the crucial question is which devices should be used with particular pupils in teaching certain concepts. On this question more research is needed, since existing research gives little direction.

## EVALUATION OF MANIPULATIVE DEVICES

Every lesson in mathematics requires some sort of evaluation plan to determine the extent to which the objectives of the lesson have been attained. Lessons that involve the use of manipulative devices are no exception. Teachers select manipulative devices because they feel that the devices can make a major contribution to the learning process. Whether they do or not cannot be determined unless evaluation is planned. As a general guideline for evaluation, the teacher should state his objectives clearly in terms of pupil behavior.

Manipulative devices, however, present a special problem. Since many devices require a sizable investment of money, school systems cannot afford to purchase, test, and perhaps discard devices. Large investments in manipulative devices are not unlike investments in textbooks. With textbook selections, schools generally adopt an evaluative procedure, develop evaluative criteria, and then examine the books on the market to find the series that best fits the criteria. A similar procedure should be followed to justify a heavy investment in manipulative devices. The following suggested evaluation form can serve as such a criterion for school systems; it can also be used to help the individual teacher make decisions about individual items for his classroom.

### *Evaluation Form for Manipulative Devices*

DATA

1. Usual name of the device
2. Photograph, drawing, or picture of the device
3. Verbal description
4. Commercial sources
5. Approximate cost
6. Uses that can be made of the device

AVAILABILITY AND USE

7. Can the device be made by a pupil?
8. Can the device be made by the teacher?
9. Is the device designed to be used primarily by the teacher for demonstration purposes?
10. Is the device designed to be used primarily by the pupil?
11. Is the device designed to be used by both teacher and pupil?
12. Is the device designed to be used by someone other than teachers and pupils?
13. If the device is to be used principally by the teacher, will he be able to operate it easily and understand its use?
14. If the device is to be used mainly by the pupil, will it withstand wear?
15. Can representative learner behaviors be anticipated when this device is used?
16. Can its effectiveness be judged or measured?
17. Can this device be used if the discovery method is employed to teach the concept involved?

OUTCOMES AND PURPOSES

18. Can this device be used to motivate and awaken or broaden interest?
19. Can it be used to reinforce learning?
20. Can it be used to improve communication?
21. Can it be used to clarify concepts?
22. Can it be used to help pupils see relationships?
23. Can it be used to help pupils develop skill in using methods of reasoning?
24. Can it be used to help pupils develop a way of thinking?
25. Will the use of the device enlarge pupils' vocabularies?
26. Will the use of the device develop pupils' appreciation of mathematics?
27. Can the device to be used to develop pupil interest?
28. Can the device be used to enhance pupil initiative?
29. Can the device be used to awaken creativity or to arouse imagination?
30. Can the device be used in making summaries or in reviewing work?
31. Can the device be used as a guide to show steps in a process or procedure?
32. Can the device be used for purposes of recreation and enjoyment?

## OUTLOOK FOR THE FUTURE

The increased development and use of mathematics laboratories in the elementary school call for a classroom setting that contains a multitude of manipulative devices for teacher demonstration and for pupil experimentation. Without question, projects like the Nuffield experiment in England[4] and numerous related experiments in this country will have a heavy impact on the schools of tomorrow. They call for a complete

4. See the Nuffield Foundation Mathematics Project series (New York: John Wiley & Sons).

restructuring of the methodology of content presentation, accompanied by changes in school organization. Further impetus for the laboratory setting has come from materials developed by institutions and companies like the Learning Center and Creative Playthings, both of Princeton, New Jersey, which have carefully assembled kits of materials for pupils to learn mathematical concepts through experimentation.

There is a growing willingness on the part of local school systems to provide financial support for the purchase of manipulative devices. The impetus provided by the federal government with the enactment of the National Defense Education Act in the late fifties, followed by the Elementary and Secondary Education Act of 1965, has since carried over to local school districts which are now budgeting monies on a regular basis for equipment and improvement of facilities.

Learning needs of pupils have resulted in the development of a variety of specialized programs. Among these are Project Headstart, programs for the culturally disadvantaged, and programs for slow and gifted learners. All of these call for increased use of manipulative devices. The National Council of Teachers of Mathematics, in its project for development of special units of instruction for slow learners, typifies these efforts. The approach used with pupils having particular learning difficulties is largely experimentation with manipulative devices.

The involvement of large business enterprises in the education field offers three main advantages. First, the development of new technology will have a direct path to the educational establishment. The lag between these technological advances and their educational uses will be reduced to a minimum. Second, the educational institutions, recognizing their needs for new technology, have direct access to companies with highly skilled personnel who can create the technological advances needed. Third, such participation by private enterprise can awaken interest in local and state governments to provide greater financial support to local schools.

More psychologists have directed their attention to problems of human learning. As this interest continues to grow, new information on the learning process will increase the efficiency of the pupil who learns and the teacher who guides the learning. As pointed out in the beginning of the chapter, knowledge of the learning process has led to recommendations for greater use of manipulative devices.

Finally, teachers have demonstrated an ever increasing willingness to experiment with new techniques, strategies, methodologies, and content. The climate of the years prior to 1960 was not favorable to experimentation. As elementary teachers became better acquainted with newer programs and methods and began to understand the changing goals and expected outcomes of a school mathematics program, their interest in children and their desires to do the best for them freed the teachers from the chains of stagnated programs and teaching techniques. Elementary teachers have not only become capable of carrying out limited research and experimentation with manipulative devices but have demonstrated their interest and desire to do so.

The role of manipulative devices in the elementary school mathematics classroom of the future looks bright indeed.

5. See *Experiences in Mathematical Ideas*, 2 vols., each with *Teaching Package*, ed. Arnold M. Chandler (Washington, D.C.: The Council, 1970).

## SOURCES OF MANIPULATIVE MATERIALS FOR ELEMENTARY SCHOOL MATHEMATICS

The companies whose names appear in the list below produce or distribute manipulative devices that are appropriate for elementary school mathematics instruction. The full names and addresses of the companies are given in the Appendix. The classes of materials available from each company are indicated by x's according to the sorting scheme used in this chapter:

1. Demonstration boards and devices
2. Place value devices
3. Colored beads, blocks, rods, and discs
4. Number boards
5. Cards and charts
6. Measurement devices
7. Models of geometric relationships
8. Games and puzzles
9. Special computational devices.

| PRODUCER OR DISTRIBUTOR | DEMONSTRATION DEVICES 1 | PLACE VALUE DEVICES 2 | COLORED BEADS, BLOCKS, ETC 3 | NUMBER BOARDS 4 | CARDS AND CHARTS 5 | MEASUREMENT DEVICES 6 | GEOMETRIC MODELS 7 | GAMES AND PUZZLES 8 | COMPUTATIONAL DEVICES 9 |
|---|---|---|---|---|---|---|---|---|---|
| ABC School Supply, Inc. | x | x | x | x | x | x | x | x | x |
| Academic Industries, Inc. | | | | | | | x | | |
| The Advancement Placement Institute | | | | | x | | | x | |
| Aero Educational Products | | | | | | | | | x |
| Aestheometry, Inc. | | | | | | | x | | |
| AIM Industries, Inc. | | | | | | | | | x |
| Arithmetic Clinic | | | | | | | | x | |
| Arithmetical Principles Association | | | | | | | | | x |
| Associated School Distributors, Inc. | x | x | x | x | x | x | x | x | x |
| Avalon Hill Company | | | | | | | | x | |
| M. C. Ballard Company | | | | x | | | | | |
| Beckley-Cardy Company | x | x | x | x | x | x | x | x | x |
| Ben-G-Products, Inc. | x | | | | | | | x | |
| Channing L. Bete Company, Inc. | x | | | | x | | | | |
| Book-Lab, Inc. | | | | | | | x | | |
| Stanley Bowmar Company, Inc. | | | | | x | | | x | |
| Milton Bradley | x | x | x | x | x | x | x | x | x |
| Bremner Multiplication Records, Inc. | | | | | | | | | x |
| Cadaco, Inc. | | | | | | | | x | |
| Caddy-Imler Creations, Inc. | | | | | | | | x | x |
| Cambosco Scientific Company, Inc. | | | | | | x | | | |
| Carr Plastics, Inc. | | | | | | x | x | | |
| Cenco Educational Aids | | | | | | | x | | x |
| Champion Publishing Company | | | | | | | | x | |
| Childcraft Equipment Company, Inc. | | | | | | | x | x | x |
| Robert R. Clamage, P. A. | | | | | | | | x | |

| PRODUCER OR DISTRIBUTOR | DEMONSTRATION DEVICES 1 | PLACE VALUE DEVICES 2 | COLORED BEADS, BLOCKS, ETC. 3 | NUMBER BOARDS 4 | CARDS AND CHARTS 5 | MEASUREMENT DEVICES 6 | GEOMETRIC MODELS 7 | GAMES AND PUZZLES 8 | COMPUTATIONAL DEVICES 9 |
|---|---|---|---|---|---|---|---|---|---|
| Claridge Products and Equipment, Inc. | x | | | | x | | | | |
| John Colburn Associates, Inc. | | | | | x | x | x | | |
| Cooper Brothers Company | | | | | | | x | | |
| Cooperative Recreation Service, Inc. | | | | | | | | x | |
| Corbett Blackboard Stencils | x | | | | | | | | |
| Creative Playthings / Learning Center | x | x | | x | | x | | | |
| The C-Thru Ruler Company | | | | | | x | | | |
| Cuisenaire Company of America, Inc. | | | x | | | | | | |
| The Curta Company | | | | | | | | | x |
| Custom Fabricators, Inc. | | | | | x | | | | |
| Daintee Toys, Inc. | x | x | | | x | | | | |
| Dana and Company, Inc. | | | | | x | | | | |
| The Denny Press | x | | | x | | | | | x |
| Ralph D. Doner, Mathematical Puzzles | | | | | | | | x | |
| Dyna-Slide (*See* Science Related Materials) | | | | | | | | | |
| Eckel and Ballard | | | | | x | | | | |
| Edmund Scientific Company | | x | | | | x | x | x | x |
| Educational Aid Publishers | x | | | | | | | | x |
| Educational Development Laboratories | | | | | | | | | x |
| Educational Fun Games | | | | | | | | x | |
| Educational Playthings | x | x | | x | | | | | |
| Educational Supply and Specialty Company | x | x | x | x | x | x | x | x | x |
| EduKaid of Ridgewood | | x | x | | | | | | x |
| Encyclopaedia Britannica, Instructional Materials Division | | | | | | | x | | |
| Encyclopaedia Britannica Educational Corporation | | | x | | | | x | | |
| E.S.R., Inc. | | | | | | | | | x |
| E T A School Materials Division | | | x | | | | | | |
| Exclusive Playing Card Company | | | | | | | | x | |
| Exton Aids | x | | | | | | | | |
| Fearon Publishers | | | | | x | | | x | |
| Fortune Games | | | | | | | | x | |
| Franklin Teaching Aids | x | x | | | x | | | | |
| Frederick Post Company | | | | | | x | | | |
| Hans K. Freyer, Inc. | | | | | | | x | | |
| Gamco Products, Inc. | | | | x | | | x | | |
| The Gangler-Gentry Company | | | | | | | | x | |
| Garrard Press | | | x | | | | | x | |
| Gel-Sten Supply Company, Inc. | x | x | x | x | x | x | x | x | x |
| Genius Supply Company | | | | | x | | | | |
| Geodestix | | | | | | | x | | |
| Geyer Instructional Aids Company | x | | | | | x | x | | |

| PRODUCER OR DISTRIBUTOR | DEMONSTRATION DEVICES 1 | PLACE VALUE DEVICES 2 | COLORED BEADS, BLOCKS, ETC. 3 | NUMBER BOARDS 4 | CARDS AND CHARTS 5 | MEASUREMENT DEVICES 6 | GEOMETRIC MODELS 7 | GAMES AND PUZZLES 8 | COMPUTATIONAL DEVICES 9 |
|---|---|---|---|---|---|---|---|---|---|
| Ginn and Company | x | x | | | x | | | | |
| G W School Supply Specialists | x | x | | | x | | | x | x |
| Hall and McCreary Company | | | | | | | | x | |
| J. L. Hammett Company | x | x | x | x | x | x | x | x | x |
| Harcourt Brace Jovanovich, Inc. | x | x | | | | x | x | | x |
| Miles C. Hartley | | | | | | | x | | |
| Hayes School Publishing Company, Inc. | | | | | x | | | | x |
| Helberg Enterprises, Inc. | x | | | | | | | | |
| Herder and Herder | | | x | x | | | x | | |
| Holt, Rinehart and Winston, Inc. | x | x | x | | x | | | | |
| Houghton Mifflin Company | | x | x | | | | x | | |
| Hubbard Scientific Company | | | | | | | x | | |
| Hudson Products | | | x | | | | | | |
| Ideal School Supply Company | x | x | x | x | x | x | x | x | x |
| Imout | | | | | | | | x | |
| Instructional Aids, Inc. | | x | | | x | | | | |
| Instructional Materials Company | | | | | | | x | | |
| Instructo Products Company | x | x | | | x | | | | |
| Jacronda Manufacturing Company | x | | | | | | | | |
| Jell's Arithmetic Games | | | | | | | | x | |
| The Judy Company | x | x | x | x | | x | | | x |
| Kalah, Inc. | | | | | | | | x | |
| The Kendry Company | | | | | | | x | | |
| Kenworthy Educational Service | | | | | x | | | x | |
| Kindrey Manufacturing Company | | | | | | | x | | |
| Kohner Brothers, Inc., Tryne Game Division | | | | | | | | x | |
| Kraeg Games | | | | | | | | x | |
| Krypto Corporation | | | | | | | | x | |
| Lanco | x | | | | | | | | |
| Lano Company | | | | | | x | x | | x |
| LaPine Scientific Company | | | | | | | x | | |
| The Learning Center | x | x | x | x | | x | x | | x |
| Little Red School House | x | x | | | x | | | | |
| E. S. Lowe Company, Inc. | | | | | | | | x | |
| 3D Magna-Graph Corporation | | | | | | | x | | |
| Mainco School Supply Company | x | x | x | x | x | x | x | x | x |
| Math Master Labs, Inc. | x | x | x | x | x | x | x | x | x |
| Math Media Division, H and M Associates | | | | | | x | x | x | |
| Mathaids Company | | | | | | | x | | |
| Mathematical Pie, Ltd. | x | x | x | x | x | x | x | x | x |
| Midwest Publications, Inc. | x | x | | | x | | | | x |
| Miles Kimball Company | | | | | | | | x | |
| Minnesota Mining and Manufacturing Company | | | | | | | | x | |

| PRODUCER OR DISTRIBUTOR | DEMONSTRATION DEVICES 1 | PLACE VALUE DEVICES 2 | COLORED BEADS, BLOCKS, ETC. 3 | NUMBER BOARDS 4 | CARDS AND CHARTS 5 | MEASUREMENT DEVICES 6 | GEOMETRIC MODELS 7 | GAMES AND PUZZLES 8 | COMPUTATIONAL DEVICES 9 |
|---|---|---|---|---|---|---|---|---|---|
| Models of Industry, Inc. | | | | | | | | x | |
| Moyer Division, Vilas Industries Limited | | | x | x | | | | | |
| McGraw-Hill Book Company, Educational Games and Aids | | | x | | x | | | | |
| Nasco Mathematics | | x | | x | | | x | x | |
| Nifty Division, St. Regis Paper Company | x | x | | x | x | x | x | x | |
| F. A. Owen Publishing Company | | | | | x | x | | | |
| Pacific Coast Publishers | | | | | x | | | | x |
| Palfreys School Supply Company | x | x | x | x | x | x | x | x | x |
| Physics Research Laboratories, Inc. | | | | | | | x | | |
| The Plaway Games | | | | | | | | x | |
| Playskool Manufacturing Company | | | x | | x | | x | x | |
| The Plymouth Press | | | | | x | | | | |
| The Charles T. Powner Company | | | | | | | | x | |
| Psychological Service | | | | | | | | x | |
| Quantum Corporation | | | | | | | | x | |
| Responsive Environments Corporation, Learning Materials Division | x | x | x | x | | x | x | | |
| St. Paul Book and Stationery Company | x | x | | x | x | | x | | x |
| Sargent-Welch Scientific Company | | x | | | | x | x | | |
| School Material Company | x | x | x | x | x | x | x | x | x |
| School Products Company, Inc. | | | | | | x | | | x |
| School Service Company | x | x | x | x | x | x | x | x | x |
| Science Productions | | x | | | | | | | |
| Science Related Materials, Inc. | | | | | | x | x | | x |
| Science Research Associates, Inc. | x | x | | | | | | | x |
| Science Seminars, Inc. | | | | | | | | x | |
| Sciences Materials Center | | | | | | x | | | |
| Scientific Educational Products Corporation | | x | | | | | | | x |
| Scott, Foresman and Company | | | x | | x | | | x | |
| Scritchfield Manufacturing Company | | | | | | | x | | |
| Selective Educational Equipment (SEE), Inc. | | x | x | | x | x | x | x | x |
| Self-Teaching Flashers | | | | | | | | x | |
| Sigma Enterprises, Inc. | | | | | | | x | | |
| Skool-Aids Corporation | | | | | x | | | | |
| The Speed-Up Geometry Ruler Company, Inc. | | | | | | | x | | |
| Standard Education Society, Inc. | | x | | | | | | | x |
| STAS Instructional Materials, Inc. | x | | x | | | x | | | |
| Steck-Vaughn Company | | | x | | x | | | | |
| Summit Industries | | | | | | | | | x |
| Systems for Education, Inc. | | | | | | | | | x |
| Tal-Cap, Inc. | | | x | | | | | | |
| Teacher's Aids | x | x | | | | | | | |

| PRODUCER OR DISTRIBUTOR | DEMONSTRATION DEVICES 1 | PLACE VALUE DEVICES 2 | COLORED BEADS, BLOCKS, ETC. 3 | NUMBER BOARDS 4 | CARDS AND CHARTS 5 | MEASUREMENT DEVICES 6 | GEOMETRIC MODELS 7 | GAMES AND PUZZLES 8 | COMPUTATIONAL DEVICES 9 |
|---|---|---|---|---|---|---|---|---|---|
| Teachers Publishing Corporation | | | | | x | | | x | |
| Teaching Aids | x | x | x | x | | | | | |
| Touch, Inc. | | | | | x | | | | |
| Fern Tripp | | | | | | x | | | |
| TUF (*See* Avalon Hill Company) | | | | | | | | | |
| Charles E. Tuttle Company, Inc. | | x | | | | | | | x |
| United Chemical and School Supply Company | x | x | x | x | x | x | x | x | x |
| Viking Company | | | | | | | x | | |
| Vision Incorporated | x | | | x | | | | | |
| Vis-X Company | | | | | | x | x | | |
| Wabash Instrument Corporation | | | | | | x | | | |
| Walker Products | | | | | | | x | | |
| Walker Teaching Programs and Teaching Aids | | | | | x | | x | | |
| Wang Laboratories, Inc. | | | | | | | | | x |
| Weber Costello | | x | x | x | x | x | | | |
| Webster Division, McGraw-Hill Book Company | | | x | | x | | x | x | |
| Webster Paper and Supply Company | x | | | | | | | | |
| Welch (*See* Sargent-Welch Scientific Company) | | | | | | | | | |
| Western Publishing Company, Inc. | | x | x | | | | | x | |
| WFF 'N PROOF | | | | | | | | x | |
| W. H. M. Company | | | | | | x | | | |
| World Wide Games | | | | | | | | x | |
| L. M. Wright Company | | | | | | | | | x |
| Xerox Corporation, Curriculum Programs | | | x | | | | x | | |
| Yoder Instruments | | x | | | | | | x | |
| Zinni Educational Materials | | | | | | | | | x |

## SELECTED BIBLIOGRAPHY

For the reader's greater convenience, entries in this bibliography have been separated into several groupings, as shown below:

A. General References, entries 1–38
B. Demonstration Boards and Devices, entries 39–42
C. Place Value Devices, entries 43–66
D. Colored Beads, Blocks, Rods, and Discs, entries 67–79
E. Number Boards, entries 80–83
F. Cards and Charts, entries 84–90
G. Measurement Devices, entries 91–97
H. Models of Geometric Relationships, entries 98–113
I. Games and Puzzles, entries 114–25
J. Special Computational Devices, entries 126–46.

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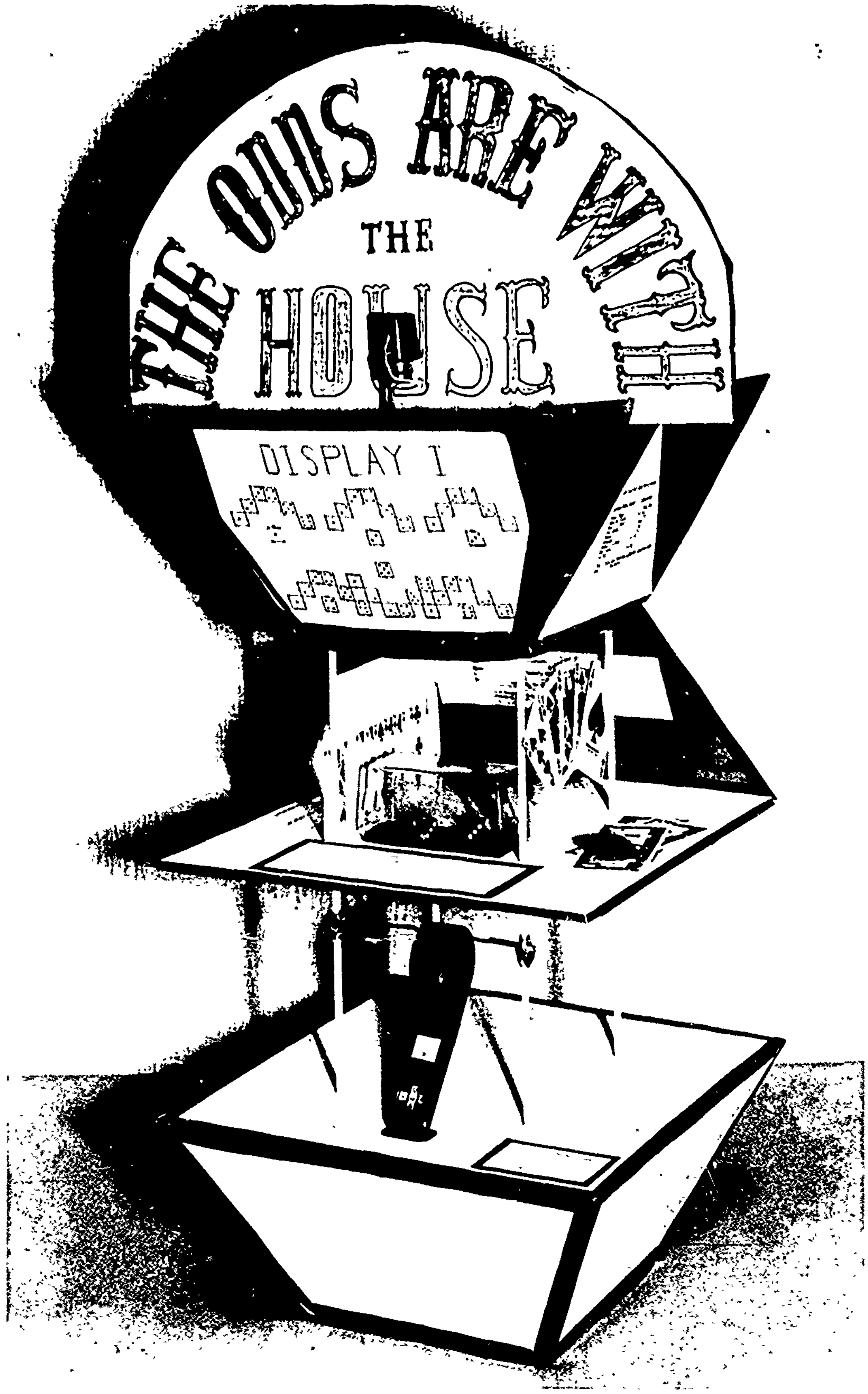
# 10. PROJECTS, EXHIBITS AND FAIRS, GAMES, PUZZLES, AND CONTESTS


THE ODDS ARE WITH
THE
HOUSE
DISPLAY I

CHAPTER 10

# MATHEMATICS PROJECTS, EXHIBITS AND FAIRS, GAMES, PUZZLES, AND CONTESTS

*by*

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This chapter describes how projects, exhibits, fairs, games, puzzles, and contests can be used in a school mathematics program. Suggestions are given for the preparation and evaluation of projects and exhibits. Illustrated examples are presented of projects made by students of varying abilities and interests. Following the chapter there is a list of sixty-seven project topics with references for each and a list of sources of mathematical games and puzzles.

◄A STUDENT PROJECT as it appeared on display at the Annual Meeting of the National Council of Teachers of Mathematics held in Minneapolis

# 10. MATHEMATICS PROJECTS, EXHIBITS AND FAIRS, GAMES, PUZZLES, AND CONTESTS

Every student who studies mathematics is capable of creative activities. Such activities include beginning manipulative experiences by the primary student, independent investigations by the college-bound student, and other exploratory activities undertaken by students at various levels of a school mathematics program.

The kinds of creative activities in which professional mathematicians engage and the excitement they experience in working at the frontiers of mathematical knowledge can be a part of every student's mathematical education. Mathematics projects, exhibits, fairs, games, puzzles, and contests are ideally suited to foster the excitement of creativity.

## THE NATURE OF A MATHEMATICS PROJECT

A mathematics project is an individual or small group activity that involves the preparation of a paper, the production of a model or display, or a combination of these two activities. A project may begin with a student's desire to explore a topic or idea in greater depth than regular classwork permits, or it may be assigned by the teacher. Again, a mathematics project may be related to a student's work in science or some other school subject. Occasionally a project may grow out of a student's out-of-class hobby.

In some respects a mathematics project resembles a scientific investigation. It begins with a statement or recognition of a problem. It continues with collection of pertinent information and, if a model or display is involved, procurement of needed construction materials. A project is completed when the student finishes writing a report, demonstrates the use of a model, or explains a display. (As usually conceived, a display is a thematic arrangement of models, illustrations, and printed messages that are combined to project an idea to an observer.)

Summarized below are the major steps that characterize the development of a mathematics project:

1. The student defines the problem or topic he is going to investigate, or describes the model or display he intends to build.
2. The student extends himself beyond his regular assignments to collect pertinent information for a paper or to procure construction materials for a model or display. This step may involve the use of the library, and in the case of a model or display, the student may need to use materials or equipment from the industrial arts, crafts, or commercial department.
3. The completed project is given recognition and evaluated. Exhibits and fairs provide opportunities for displaying projects and judging their merits. If a project is put on display in a department store, hotel, library, museum, bank, airport, or other public place, the community also plays a part in the evaluation.

## PREPARATION OF A PROJECT

Undoubtedly the most important factor in getting outstanding projects from students is the attitude of the teacher toward projects. A teacher who is energetic in stimulating students to start projects, resourceful in providing needed assistance, and supportive of students' efforts will usually have students who develop significant projects.

There are a number of techniques a teacher can use to motivate students to start projects.

One technique is to put completed projects on display and invite students to examine them. If no suitable collection of projects is available, an effective substitute is a display of photographs and drawings, or the teacher might show slides that depict projects produced by students in former years. This chapter contains nearly fifty

pictures of different projects. These pictures are part of a collection of 300 pictures that are on file in the Mathematics Resource Center of the Saint Paul Public Schools.

Giving students lists of topics with accompanying references is another useful technique. References for sixty-seven different project topics are given at the end of this chapter.

Still another technique is to take students on field trips to possible sources of ideas for projects. Such sources might include a bank, a computer company, an industrial museum, factories, farms, fisheries, and so forth. Trips of this kind give students contact with applied mathematics.

Collecting and maintaining a file of ideas for projects through which students can browse is one of the more compelling motivational techniques a teacher can use to get students started. Ideas for projects should be filed according to subject or topic (e.g., algebra, geometry, number systems, probability, etc.). Listed below are some typical items that might be included in a project idea folder:

1. Questions to be investigated
2. A brief description or outline of a possible project
3. A crude sketch or drawing of a model
4. A progress report of an unfinished project or a report of an attempt to produce a project that didn't materialize
5. Names of people who may have ideas
6. Newspaper clippings that contain the germ of an idea of a project.

Perhaps the most effective way of motivating students to start projects is to give them an opportunity to read résumés of projects written by students who completed the projects. Résumés of this kind should ordinarily be brief. To be useful, a résumé should contain a description of the project, a clarifying photograph or drawing, a list of the references and materials used, a statement about the cost of materials and time spent on the project, the names of advisers consulted, and suggestions for improvement. Below are two examples of résumés prepared by students. The first one refers to the project pictured opposite the title page of this chapter.

My project, entitled "The Odds Are with the House," is a 24-inch-high four-sided display. The top third houses the main body of the display. Three of its four sides open to reveal three different aspects of the laws of chance as applied to gambling: random selection, dice, and poker hands. The center third contains a device that deals "poker hands," using dice for cards. This can be operated by the observer. The bottom third serves as a base. I attempted to show statistically that the odds are indeed with the house.

The only materials I had to buy for the project were two rolls of silver pipe-wrapping tape, which cost 85c apiece, and four 24-inch dowel rods, ¼ inch in diameter, which cost a nickel apiece. This came to $1.90. The other materials (playing cards, dice, poker chips, cardboard, etc.) are from the household.

*Malcolm Ritter*
*Saint Paul, Minnesota*

My project is a display called "Dressing Up Mathematics." The display contains a Möbius dress along with a pamphlet, "Dressing Up Mathematics," which explains the dress's construction. Also included in the display are a Klein bottle, a Möbius strip, the book *Exploring Mathematics on Your Own* by William H. Glenn and Donovan A. Johnson, and an article entitled "Paul Bunyan vs. the Conveyor Belt" which explains how Paul used the odd properties of the Möbius strip in a conveyor belt.

I became interested in this project when reading the part on topology in *Exploring Mathematics on Your Own*. Since I also enjoy sewing, the dress idea was perfect.

Although my project is quite involved, it cost me only about $10, which was for the material in the dress. All the other articles were either borrowed or taken from home.

By doing this project I got a new dress, a new pattern for other clothes, and a wonderfully exciting experience. I also got a better grade and an understanding of one-sided objects. Objects like this, either in theory or material substance, aren't always what they seem. Such things may be thought out in the mind but are surprisingly different in solid form.

*Wendy Johnson*
*Litchfield, Minnesota*


FIGURE 10.1. "Dressing Up Mathematics" display

Establishing tentative deadlines for each of the following stages of development of a project will usually promote steady progress toward its completion once a student has started:

1. Finishing the preliminary reading
2. Preparing an outline of the proposed project, or a diagram if the project involves a model
3. Having a conference with the teacher
4. Concluding the project.

Time should be provided for the student to work on his project by making available uncommitted modules of instructional time or by setting aside certain days of the week. Permitting a student to work on his project after he has completed his daily assignments or instead of working on his assignments, holding special sessions after school, and, of course, permitting all work to be done at home are ways in which additional time can be made available for projects.

If models or displays are involved, a workbench or sheets of Masonite to cover desk tops should be available. It may be possible to arrange with the industrial arts department for needed supervision so that students can use hand tools and power tools. Similar arrangements can be made with the science, crafts, and commercial departments.

To ensure success for each student, the teacher should give frequent encouragement and make constructive suggestions when the need arises. A student's enthusiasm is often in direct proportion to the enthusiasm of those who help him. If it happens that a student's project is beyond his capabilities, either he should be tactfully redirected to another project or subsequent phases of his project rescheduled to permit advisers and consultants to become involved.

## THE ROLE OF PROJECTS IN THE LEARNING OF MATHEMATICS

Teachers of mathematics have long recognized the value of projects that capitalize on literary, graphic, dramatic, and mechanical abilities of students. Doing a project provides the student with an opportunity to explore a mathematical concept while seeking ways of expressing and illustrating it.

In addition, projects serve to stimulate interest in mathematics, to widen and deepen a student's understanding of mathematics, and to give the student a feeling of relevance through active participation in the learning process. Projects also have other, more specific, values.

Doing a project helps a student learn to appreciate the importance of defining a problem precisely. It is not uncommon for a student to discover that a problem may need to be redefined after working on it for a while.

Moreover, doing a project helps a student learn how to use the library effectively, how to obtain information from various sources, and how to approach knowledgeable people.

If a project involves the production of a model, doing the project helps the student visualize the concept that forms the basis of the model. The process of constructing a model assists the student in bridging the gap between the real world and the world of abstractions.

Doing projects gives able students opportunities to work beyond the level of regular classwork and offers less able students opportunities to perform at levels commensurate with their abilities.

Doing projects that involve applications helps students recognize mathematical properties in the world about them. The student who undertakes a project soon learns that there are more places than just books in which to find information about mathematics.

Projects help teachers make better evaluations of the mathematical potentialities of their students. Teachers who regularly have students make projects are occasionally amazed at the high level of achievement of certain students who are not successful in other kinds of learning activities.

Projects stimulate other projects. Preparation of any project usually generates ideas for related projects through refinement, association, extension, revision, or correlation.

Students, teachers, and the community have a tendency to react favorably toward projects and

to support each other in producing them. In some communities engineers and other professionals willingly assume responsibility for working with students on highly technical projects that require equipment and facilities to which only these people have access.

A student who completes a project may be called upon to defend his scholarship and craftsmanship in front of his peers or before a group of judges. This gives the student an incentive to prepare a report and to learn to cope with the nervousness associated with making an oral presentation.

A spirit of creativeness can be nurtured by encouraging students to do projects. There are instances where mathematics projects started by students while in school enabled them to make important contributions to mathematics when they were still young.

Finally, students who prepare projects experience a pride of accomplishment, a spirit of companionship, and the exhilaration that comes with creating something original. For many students preparing a mathematics project is later viewed by them as one of the most satisfying experiences of their education.

## EXHIBITS AND FAIRS

Every student who makes a sincere effort to produce a project should receive some kind of recognition. The extent to which a student feels his creative efforts are recognized will often influence his future achievement in mathematics.

There are many ways of giving recognition to students who produce projects. Certainly judicious praise should be given for good work. For a project that represents a sincere effort but may not have turned out very well, commendation for the effort put forth should accompany tactful suggestions for improving the project.

Exhibiting a project in a school display case or in a store window is an acceptable way of giving recognition. Drugstores, banks, hotels, and other commercial establishments usually welcome an opportunity to provide window space for local school exhibits. When a student's project is exhibited, his name should be prominently displayed. This gives the student recognition and provides an incentive for other students.

Mathematics fairs also serve to give students recognition by providing publicity for students who produce outstanding projects. This stimulates competition and encourages students to become involved in challenging projects.

A major problem in conducting a fair is that projects must be given relative rankings and awards made accordingly. The difficulty is that criteria for judging different kinds of projects are not universal. For example, it makes little sense to compare a scholarly paper with a working model that illustrates a mathematical concept. The two projects represent different kinds of student involvement and different final products, with merits that are impossible to compare.

One solution to this problem is to separate projects entered in a fair into different categories. A typical breakdown of projects can be made by first separating all projects into grade-level or subject-area groups and then dividing the projects in each of these groups into the following categories:

1. Paper only
2. Model or display only
3. Paper accompanied by a model or display.

With this set of categories it is possible to establish evaluative criteria that are valid and fair for the different kinds of projects that students prepare.

Below is a sample set of criteria that can be used for judging projects at a fair. This set is by no means definitive and may be modified to meet local needs.

1. Criteria for a paper only
   - *a*) The mathematical topic should be significant.
   - *b*) The background investigation should be thorough.
   - *c*) Illustrations and diagrams should be sharp and large enough to show detail.
   - *d*) The exposition should be clear.

e) The topic should be developed in a logical manner.
f) The paper should be neat and well organized.
g) References should be listed.
h) The student should be able to answer orally questions pertaining to his project.

2. Criteria for a model or display only
   a) The mathematical principles depicted should be significant.
   b) The mathematical principles depicted should be readily discernible.
   c) The quality of workmanship (woodworking, electrical wiring, art work, etc.) should be adequate for the purpose of the project.
   d) Research of the literature pertaining to the model or display and the mathematical principles involved should be reasonably thorough.
   e) The student should be able to demonstrate the model and answer questions pertaining to its operation.
   f) The display should be attractive and well organized, and the student should be able to explain the theme of the display.
3. Criteria for a paper accompanied by a model or display
   a) The paper should satisfy the criteria listed for "a paper only."
   b) The model or display should satisfy the criteria listed for "a model or display only."
   c) The model or display should reinforce and/or illustrate the topic dealt with in the paper.

This set of criteria is not exhaustive, nor does it specify the weight, or point value, to be assigned to each criterion. The relative weights to be assigned to the different criteria will depend on the subject or grade level of the project being judged. For example, a paper on consumer mathematics might stress applications, whereas a paper on analysis would tend to be more theoretical. Criteria used in judging these two papers should reflect the different emphases. Students should know before they begin work what criteria will be used to judge their projects and what the relative point values are. Criteria used in judging projects are the behavioral objectives that apply to projects.

Before any mathematics fair is held, a team of judges should be selected and a meeting held to discuss criteria for the various categories of projects that are likely to be entered in the fair. To promote uniformity in judging, practice sessions in judging should be arranged. For example, judges can practice by judging projects exhibited in former years. But no matter what set of criteria is adopted, or how dedicated the judges, there will always be some honest differences of opinion. This is natural when a creative effort is being judged.

## EXAMPLES OF STUDENT PROJECTS

There is some topic at every level that can be the source of a project for some student.

Elementary school students can prepare projects to illustrate mathematical concepts from different strands of the elementary school mathematics program. Such projects will usually be manipulative and concrete. Indeed, no written report may be involved. At the junior high school level projects usually tend to be somewhat more sophisticated; however, it is still appropriate for low achievers whose reading and writing abilities are poor to produce projects that are primarily manipulative and concrete. At the senior high school level projects will usually be more abstract and less manipulative than those undertaken by students at lower levels.

There are few restrictions on the kinds of topics that will motivate students to start projects. Almost any activity in which students engage may be expanded into a project.

The topic a student chooses for a project should be appropriate for him; however, appropriateness is very difficult to determine, especially for

highly capable students. It is possible for a junior high school student to get deeply involved with probability and for an able elementary school student to become completely absorbed with computers. In short, if a student has an interest and some ability, it is difficult to determine what is and what is not an appropriate topic for him.

In the commentary that follows, examples of student projects are presented under various topics. A broad variety of skills is shown in conjunction with varying degrees of depth and understanding of mathematics. All of these projects have been presented at affairs sponsored by local schools, at mathematics conferences, at conventions, or at science fairs.

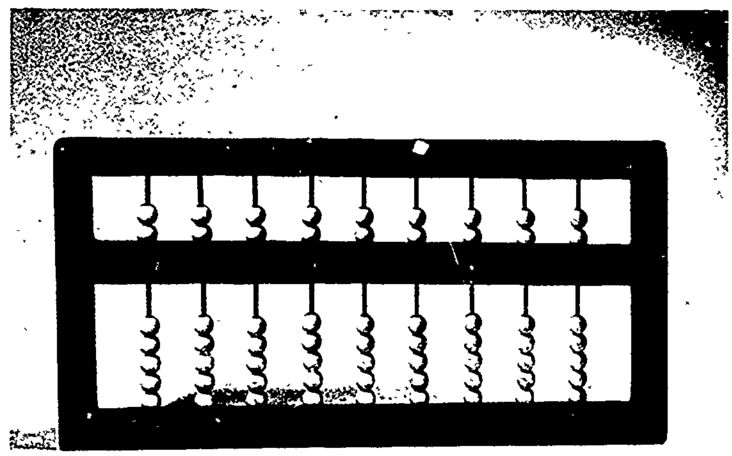


FIGURE 10.2. *Chinese abacus*

### *Computational Devices and Computers*

Over the centuries computational devices have changed from pebbles to computers. Variations include manipulative, mechanical, and electronic devices. (The word *device* is sometimes used to refer to models that have movable parts.) All of these can be projects for students. Abacuses and Napier's bones provide interesting research topics for students in the elementary school. Workable devices are easily constructed and require only a few tools, modest craftsmanship, and a minimum of materials (Figures 10.2 and 10.3).

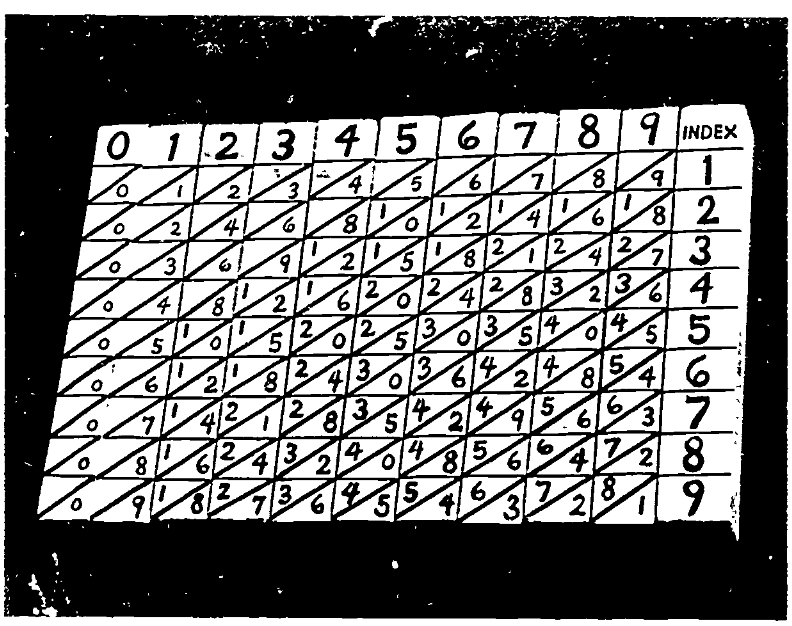



FIGURE 10.3 *Napier's bones*

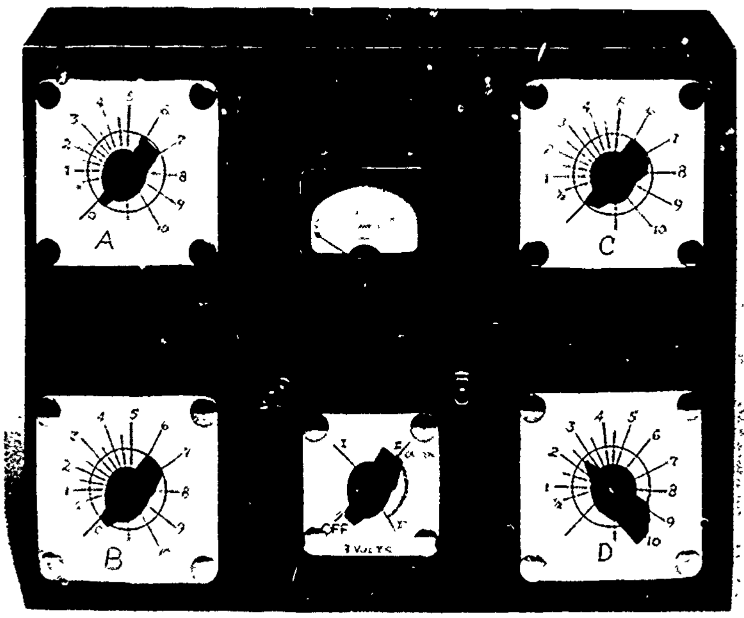


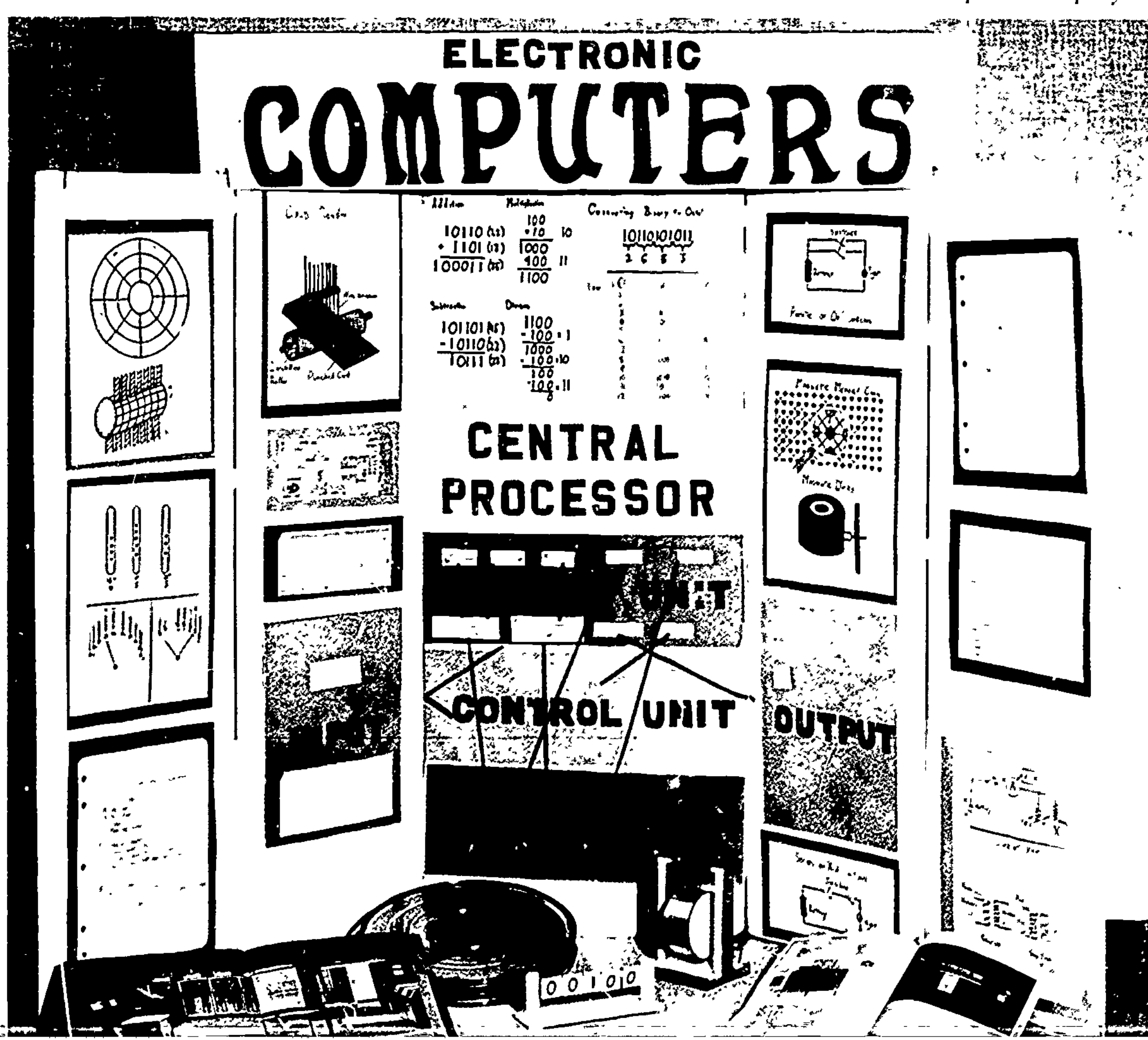

FIGURE 10.4. *Analog computer*

Figure 10.4 pictures a more sophisticated project. The device shown in the picture is a battery-operated analog computer that can be used to solve proportions of the form $\frac{A}{B} = \frac{C}{D}$.

Figure 10.5 shows still another computer project. This one is a display produced by an eighth-grade student "to help," as he put it, "the average person understand and realize the basic operations of a computer."

Figure 10.6 shows a computer project that a student started in ninth grade and finished when he was a high school senior. The machine shown in the picture can do everything from simple

FIGURE 10.5. *Computer display* ▼



addition to playing tick-tack-toe. The system includes a central processing unit, card reader, and remote terminal. Programing is achieved by using a machine language designed by the student who produced the machine.

### *Estimating the Value of π*

Computers have made it possible to obtain the value of $\pi$ correct to thousands of decimal places. Even so, estimating the value of $\pi$ by calculating the quotient of the circumference divided by the diameter for each of a number of different circular objects continues to be an informative project among students in the upper elementary grades (Figure 10.7).

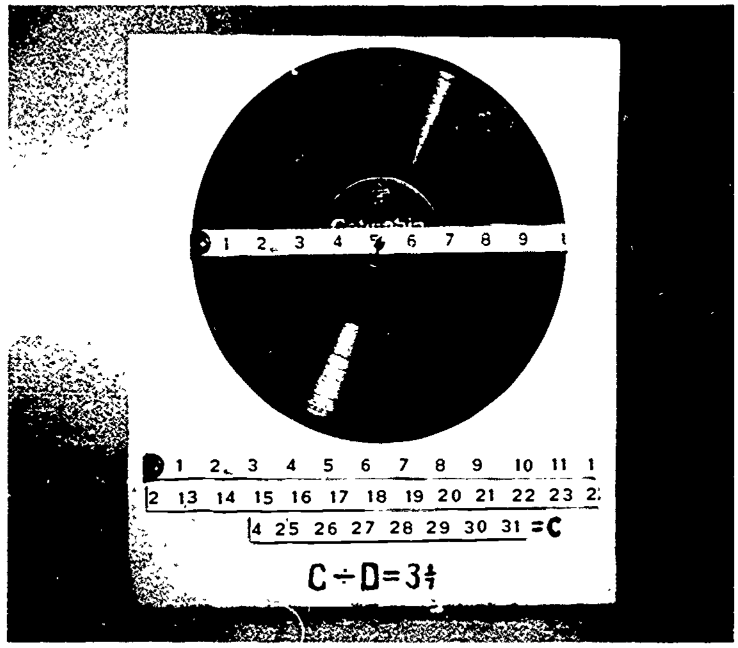
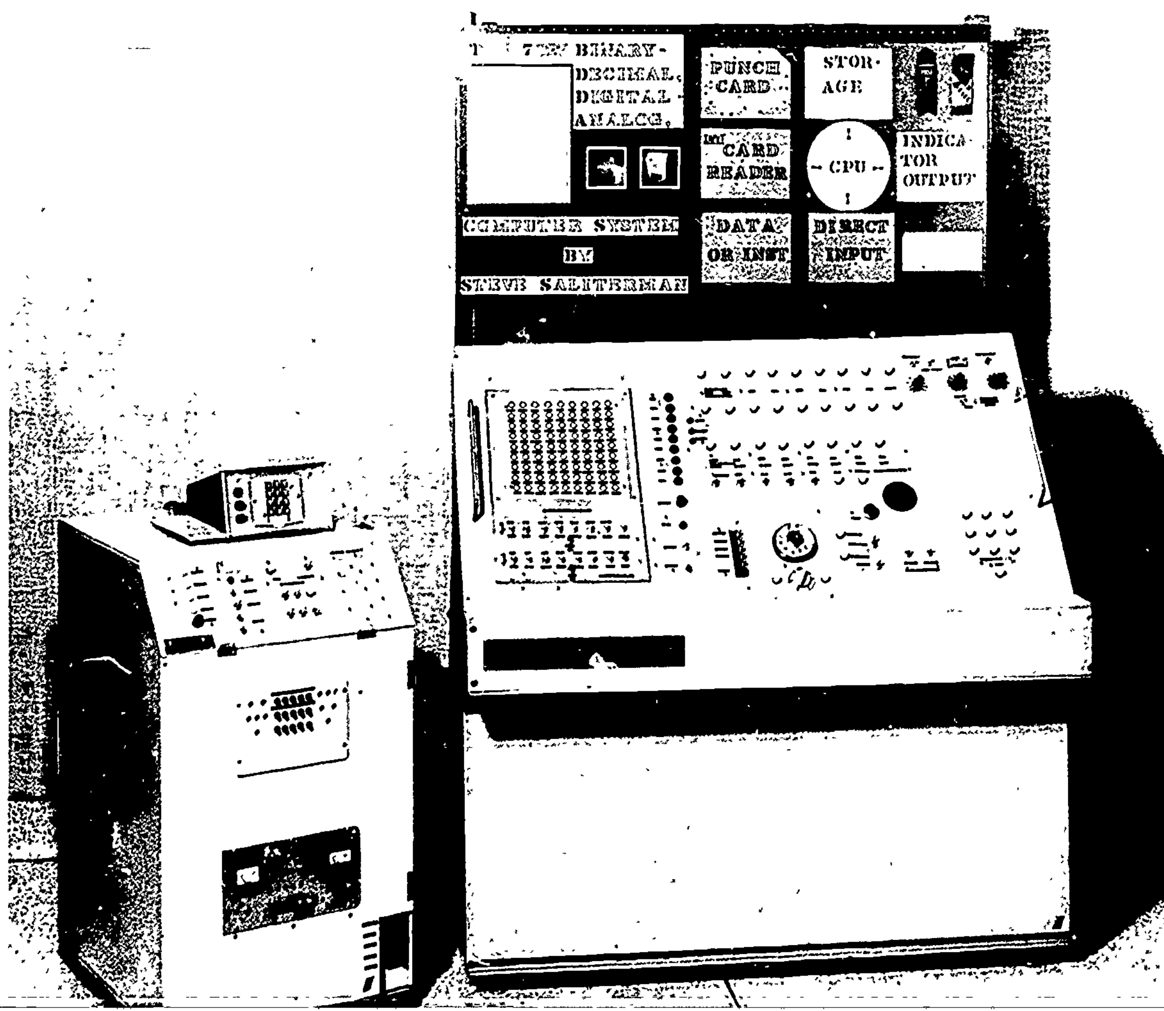



FIGURE 10.7. *Finding an approximation to the value of π*

FIGURE 10.6
*Sophisticated computer built by a high school student* ▼



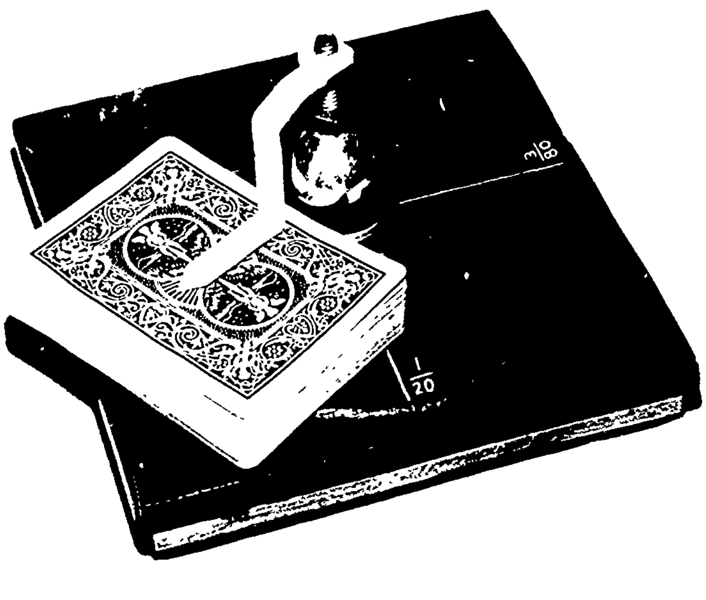



FIGURE 10.8. *A homemade microm ter that can be used to measure the thickness of objects correct to 1/80 of an inch*

*Making Measuring Instruments*

Projects involving the production and use of measuring instruments range from the simple to the sophisticated and can be counted on to kindle interest among students from the lower grades through the high school.

Making a micrometer that can be used to measure the thicknesses of objects correct to $\frac{1}{80}$ of an inch may sound like a complicated project, but it is really quite easy. In fact, students in the upper elementary grades can pursue this project successfully. Figure 10.8 illustrates the idea. The homemade micrometer shown in the picture consists of a bolt and nut with a pointer welded to the nut. The bolt has 20 threads to the inch. This means that one complete turn of the pointer moves it vertically $\frac{1}{20}$ of an inch, one fourth of a turn moves it vertically $\frac{1}{80}$ of an inch, and so on.

Making a spherometer that can be used to measure the radius of a sphere is a challenging project for students of tenth-grade geometry. Figure 10.9 shows two models. One is simply a refinement of the other. Each model consists of a coffee can, open at one end, and a pencil. The pencil is perpendicular to the plane of the closed end of the can and fits through a hole at its center. A scale is marked on the pencil to indicate how far its lower extremity is above or below the plane of the open end of the can. The operational principle of these devices is similar to that of commercial spherometers.

Making a scale that indicates the capacity of a circular cylindrical tank in horizontal position is a fairly sophisticated project even for students of high school trigonometry; however, the project can also be adapted for work with volumes in grades 5 and 6. Figure 10.10 shows a circular cylindrical tank (tomato juice can) in horizontal position with a scale that indicates the volume of the tank for various heights, correct to tenths of a cubic inch. There is a hole at the top into which liquid can be poured. The piece of glass

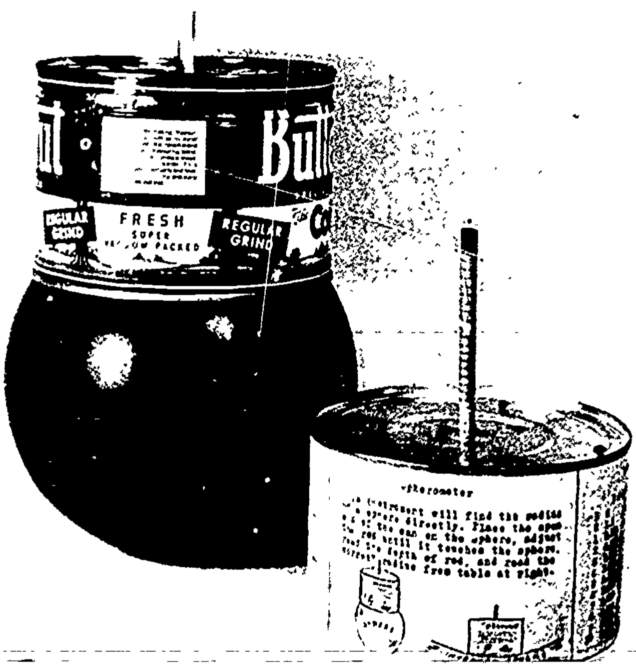



FIGURE 10.9. *Spherometers*

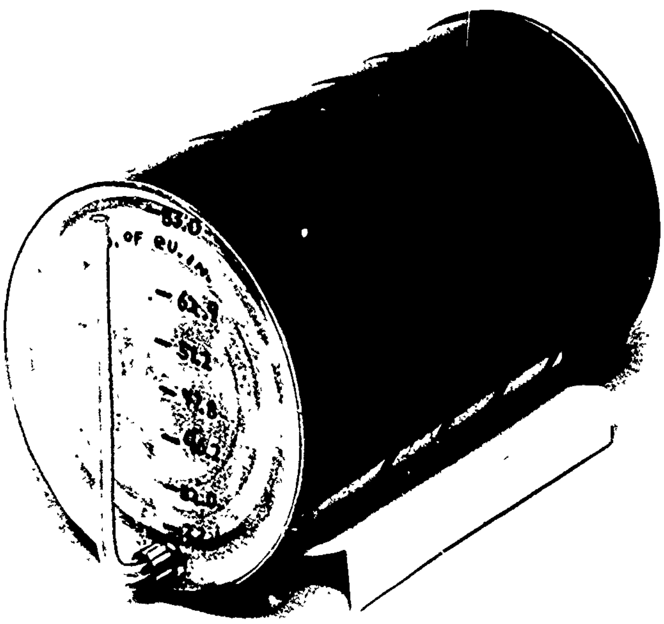


FIGURE 10.10. *Circular cylindrical tank in horizontal position with a gauge on the outside for indicating volumes*

tubing on the outside of the tank is bent and sealed in at the bottom. The exterior part of the glass tubing is aligned with a diameter. Students in grades 5 and 6 can make a scale like the one shown in the picture by pouring measured amounts of liquid into the hole at the top and marking the height of the liquid for different volumes. Another way to make the scale is by computing the volumes of right cylinders that have segments of circles as bases.

## *Applications of Mathematics in Science*

As already stated, frequently a mathematics project is related to a dent's work in science. Figure 10.11 pictures .. asy-to-make periscope. The mirrors are set in parallel planes, thus making the entering rays of light at the top parallel to the sight line below. Producing a device like this and writing a paper explaining its use in contrived situations makes an exciting project for students in the upper elementary grades.

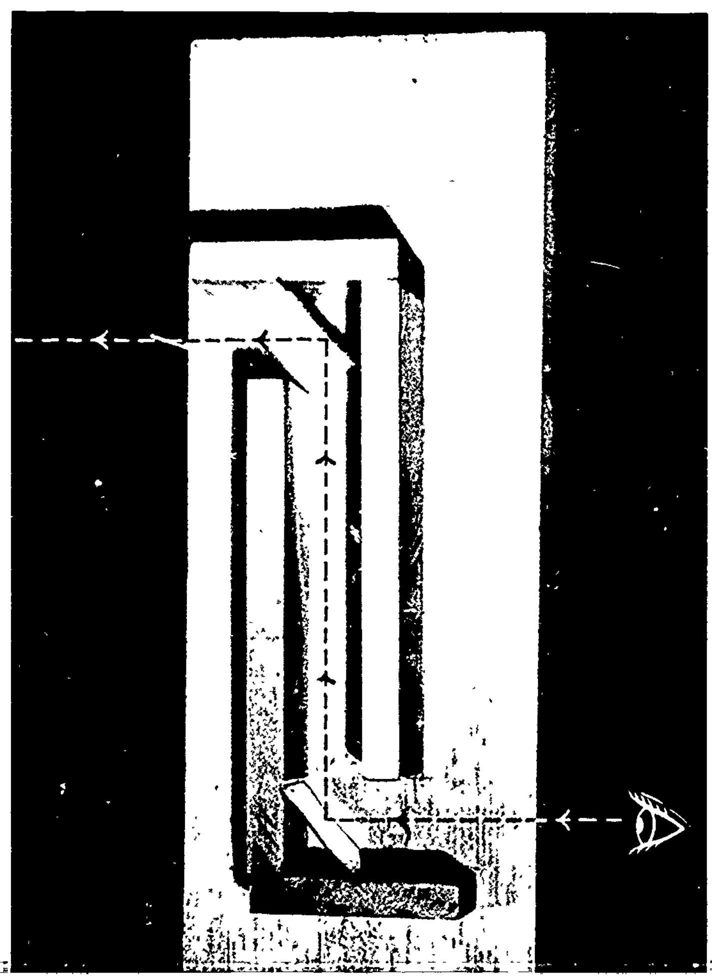


FIGURE 10.11. *Periscope*

Figure 10.12 pictures an apparatus that a student produced to investigate the behavior of springs stretched by weights. The three springs have the same spring constant and identical lengths (when not stretched). The weights shown in the picture are 1 ounce, $\frac{1}{2}$ ounce and $\frac{1}{8}$ ounce, respectively. Finding a formula that relates the average force applied in stretching a spring, the length of stretch, and the spring constant involves a great deal of precise measuring, computing, formulating, and checking.

Figure 10.13 pictures an inclined-plane apparatus designed by a student of trigonometry. The device can be used to relate the variables in the equation $F = W \times \sin \theta$, where $\theta$ is the measure of the angle of inclination of the inclined plane, $W$ is the weight of the load, and $F$ is the downward component of the load measured along the inclined plane. The magnitude of $F$ can be read

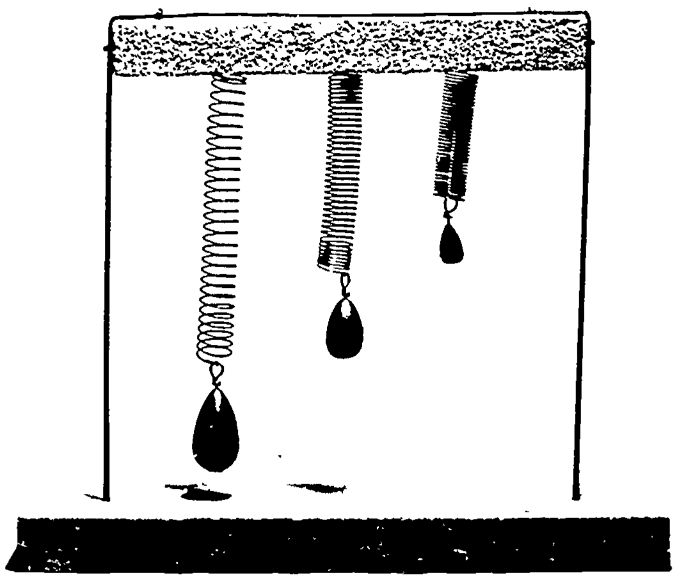


FIGURE 10.12. *Apparatus for studying properties of springs*

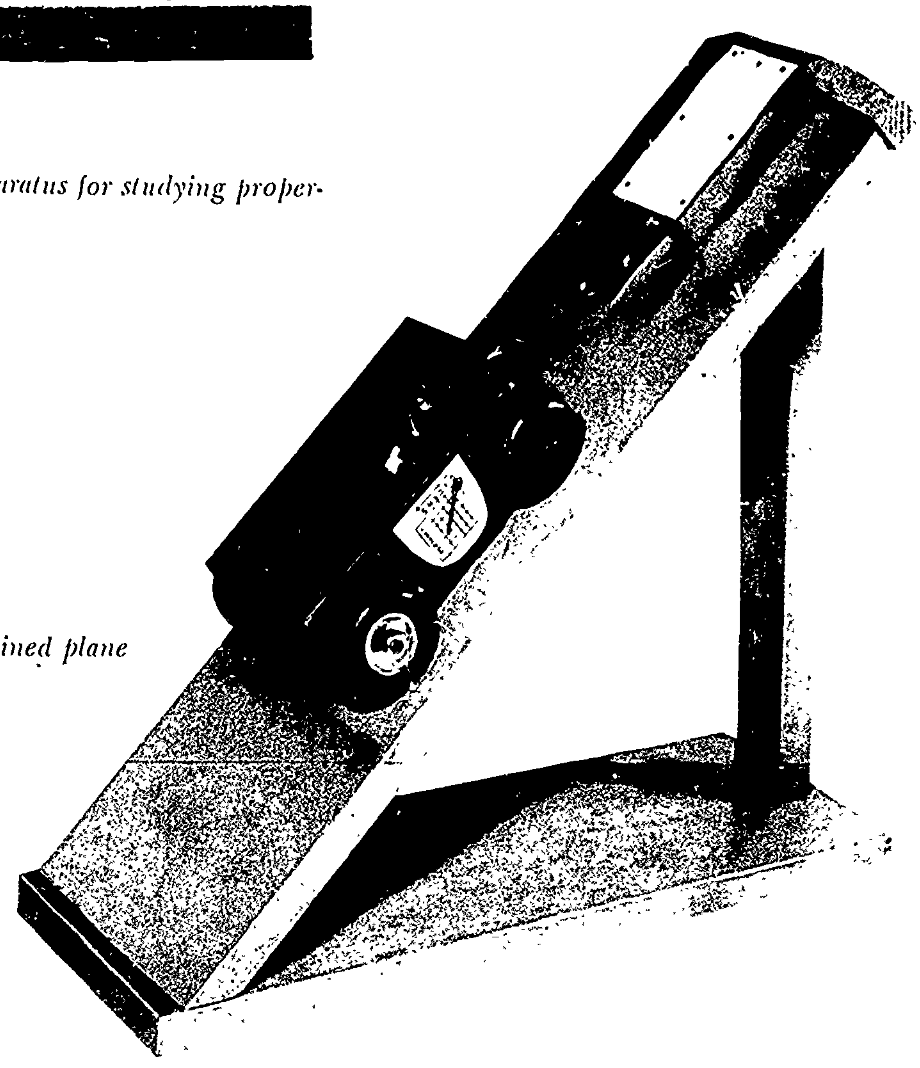


FIGURE 10.13. *Inclined plane*

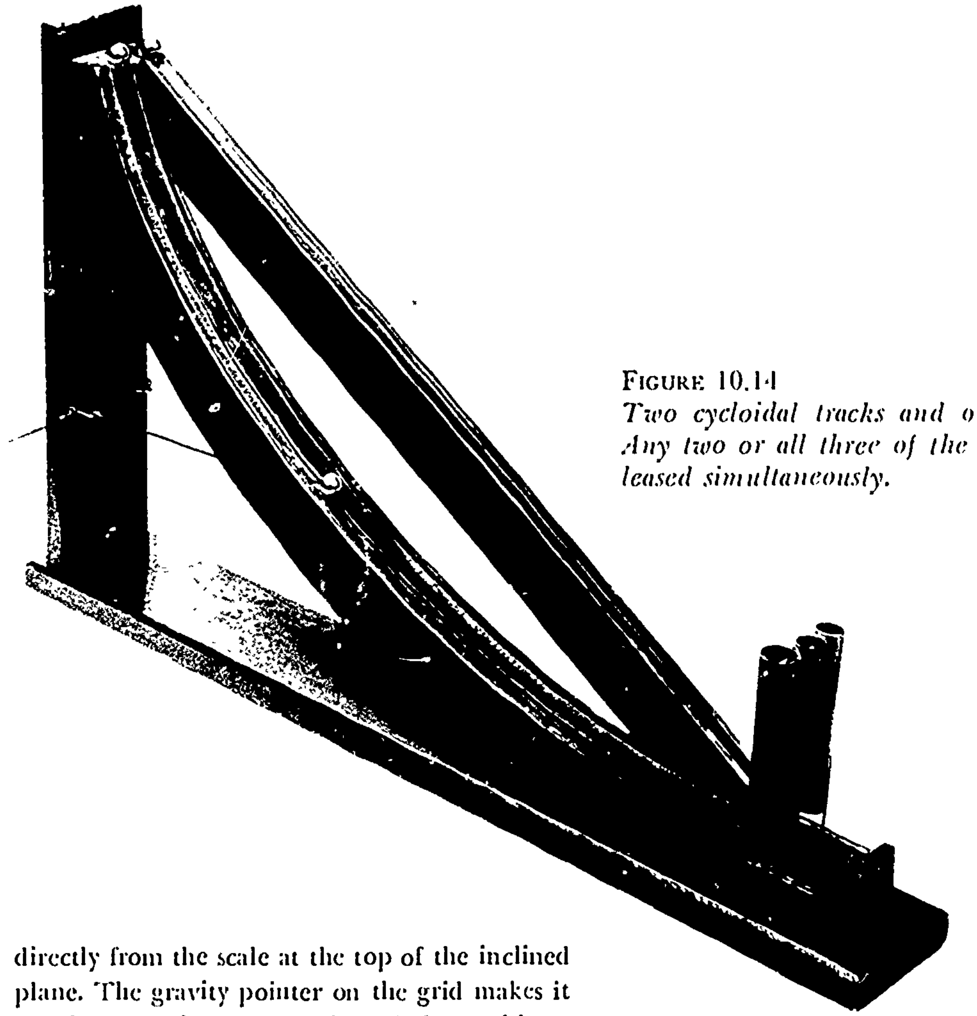


FIGURE 10.14
*Two cycloidal tracks and one linear track. Any two or all three of the balls can be released simultaneously.*

directly from the scale at the top of the inclined plane. The gravity pointer on the grid makes it possible to estimate the value of sin $\theta$ without using a table of sines. A device like this is useful for solving and checking a variety of problems involving the inclined plane.

The cycloid curve has several interesting applications in science that may be investigated by means of projects in mathematics. Such projects sometimes lead to construction of novel devices. Pictured in Figure 10.14 is a device with three tracks on which steel balls can be released. Two of the tracks have the shape of a half arch of an inverted cycloid (from the cusp to the low point). The third track is an inclined plane. If the two steel balls at the top of the device are released at the same instant, the ball traveling down the cycloidal track will reach the bottom before the one traveling down the straight track. It can be proved using differential equations that a cycloidal path under the action of gravity is the brachistochrone, or curve of fastest descent. The cycloid is also the tautochrone, or curve of equal time. If the two steel balls resting at different heights on the two cycloidal tracks are released simultaneously, they will reach the bottom at the same time.

Figure 10.15 pictures a frame composed of two adjacent arches of an inverted cycloid and a pendulum bob held by a string suspended from the point where the two arches meet. When the pendulum swings, the string wraps itself around the cycloidal arches and the bob traces a cycloid. Since the cycloid is the curve of equal time, the period of the pendulum is independent of its amplitude. Various other demonstrations are possible. Projects involving investigations of properties of the cycloid make a strong appeal to capable students.

### *Making Models of Three-Dimensional Figures*

One way of getting students to *do* mathematics is to encourage them to do projects that involve the production of models, and perhaps no subject offers a greater range of possibilities than three-dimensional geometry.

Checking the formula for the volume of a right circular cone is one such project. An appropriate model can be made by pouring a known volume of sand through a funnel, as shown in Figure 10.16.

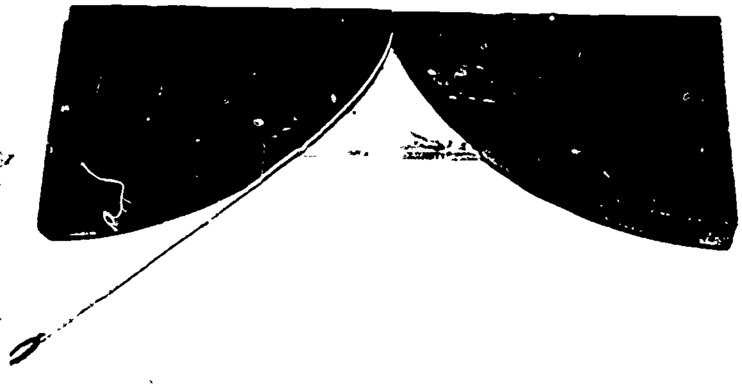


FIGURE 10.15. *Cycloidal pendulum*

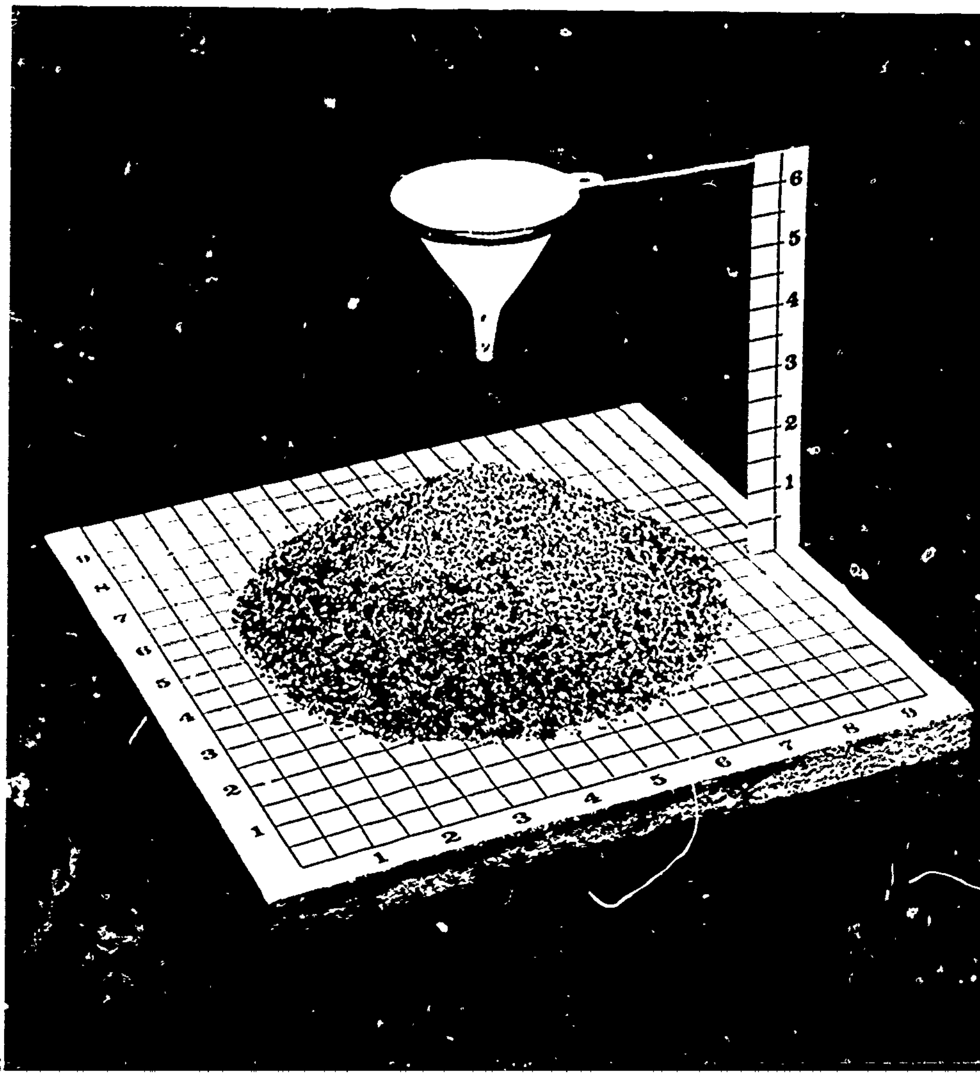
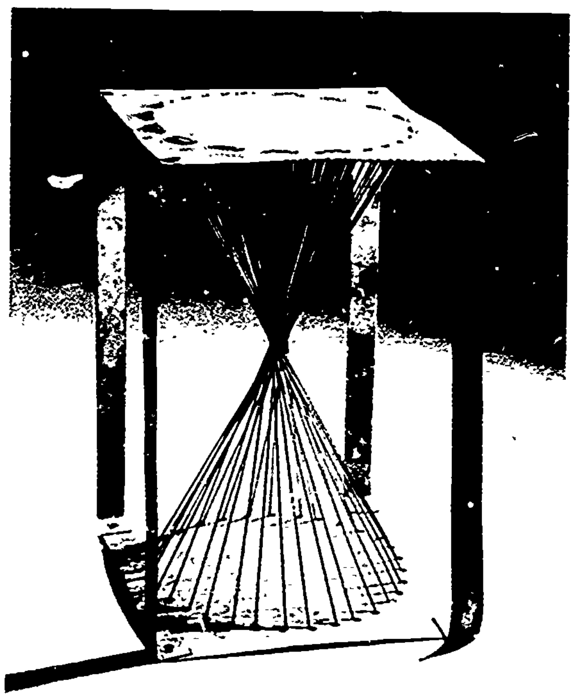



FIGURE 10.16
*A device for making a model of a right circular cone and checking the formula for its volume*

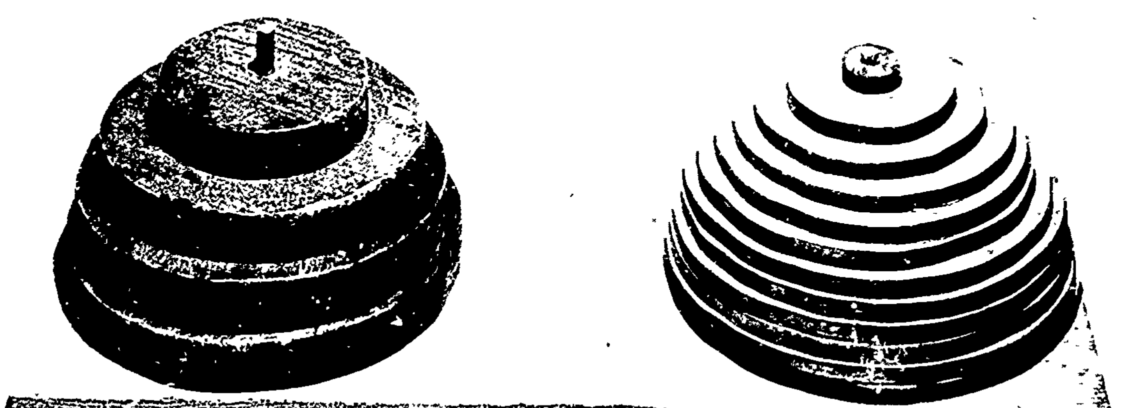


A student need not have a high degree of inventiveness in order to carry out a project that results in a pleasing, meaningful model. For impact in the classroom, compare a commercially constructed model of a cone of two nappes with the model shown in Figure 10.17.

Shown in Figure 10.18 is a three-dimensional model that has implications for calculus. The model illustrates the use of right circular cylinders in making successive approximations of the volume of a hemisphere. A student who attempts to characterize successive approximations of this kind will find himself face to face with the theory of limits.

FIGURE 10.17. *A cone of two nappes*

FIGURE 10.18. *Successive approximations of the volume of a hemisphere*



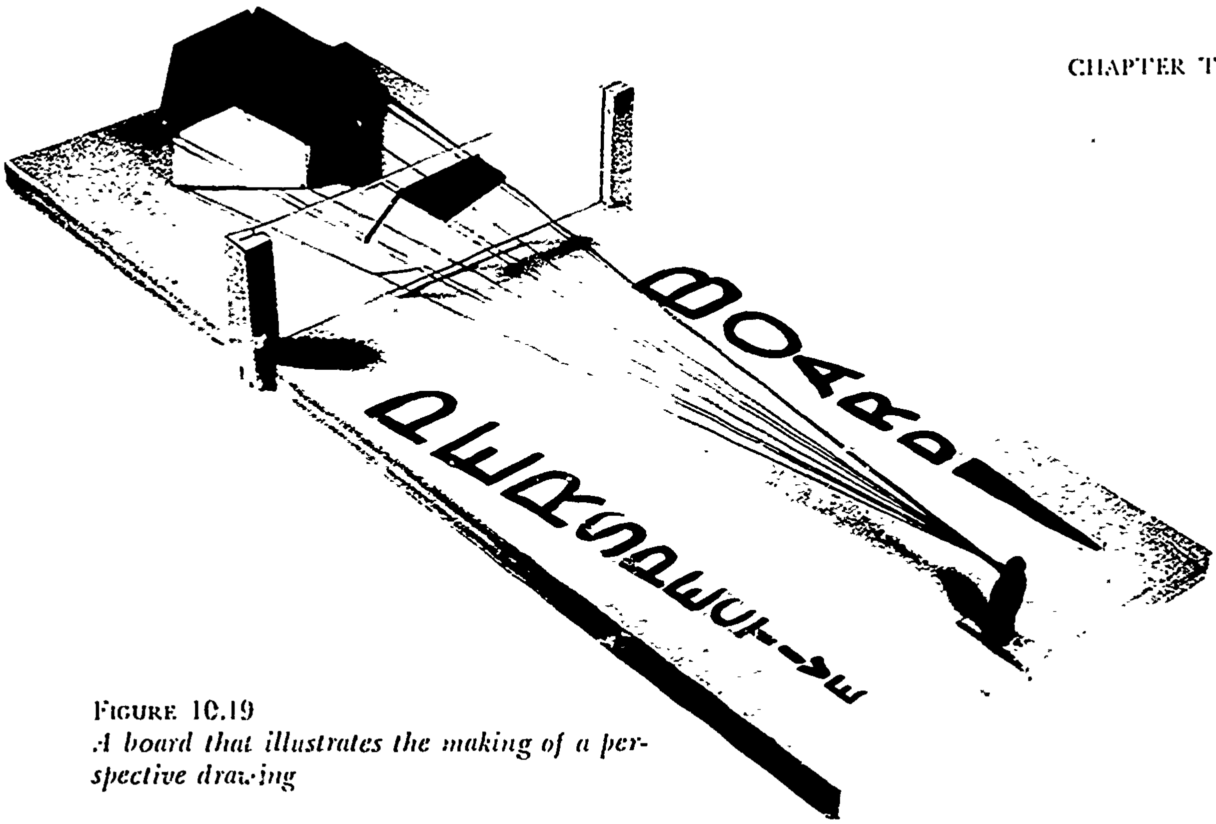



FIGURE 10.19
*A board that illustrates the making of a perspective drawing*

## *Geometry and Art*

Typical of many projects the perspective board shown in Figure 10.19 bridges two disciplines. The creator of this project was a capable student of art who combined his knowledge of three-dimensional geometry with his talents in art.

## *Giving Geometrical Interpretations to Algebraic Expressions*

Asking students to give geometrical interpretations to algebraic expressions can lead to challenging projects. The student who produced the dissectible cubical block in Figure 10.20 used it to give a geometrical interpretation to the proof of the following special case of the binomial theorem:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Figure 10.21 shows a dissectible model that was produced in response to a more imaginative assignment. A student was asked to show by means of a geometrical interpretation that

$$(x + y)(x - y)(2x - y) = 2x^3 - 2xy^2 - x^2y + y^3.$$

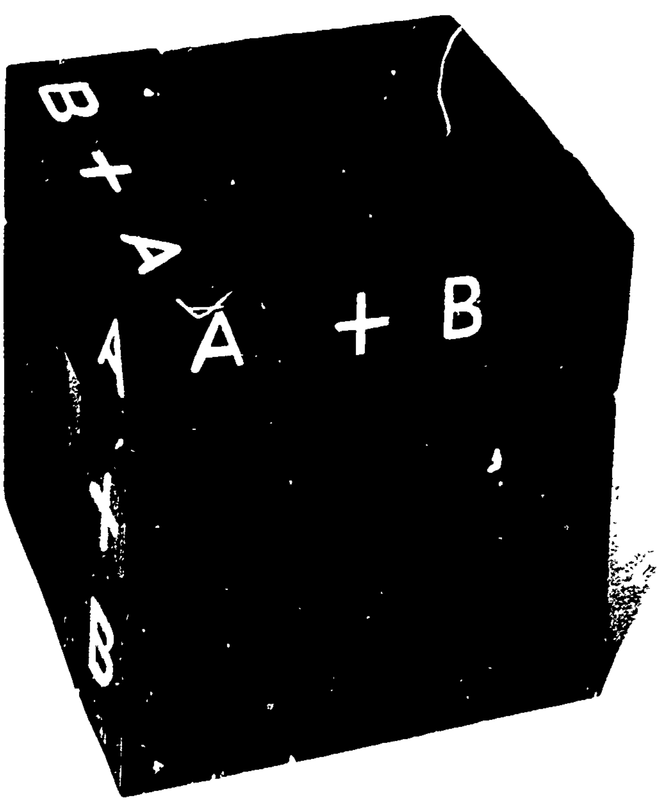



FIGURE 10.20

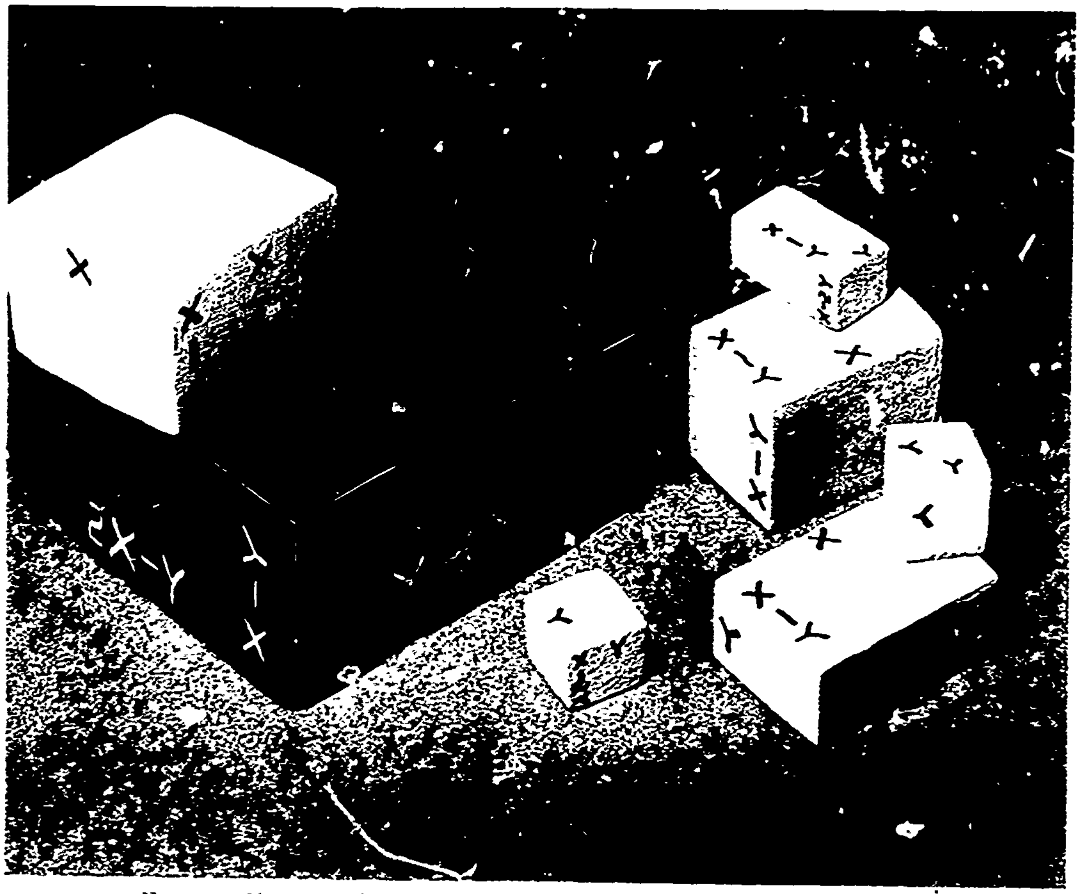
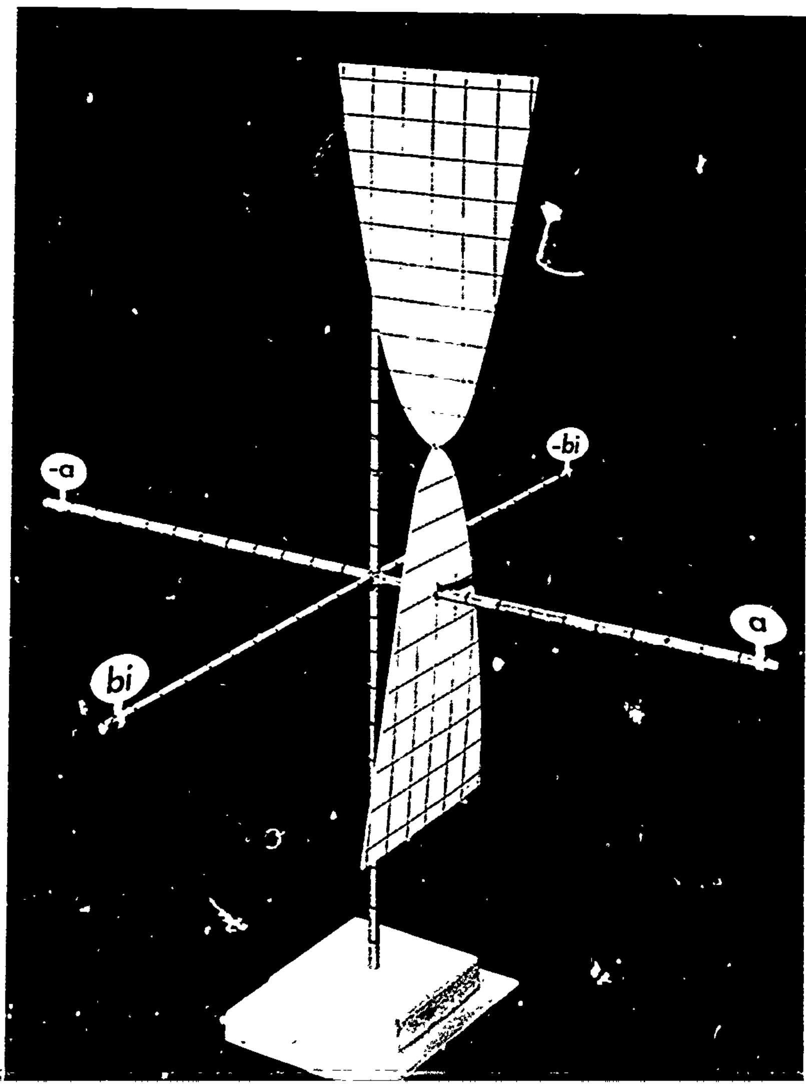


FIGURE 10.21. *The volume of the black box is equal to the product $(x+y)(x-y)(2x-y)$ and also equals the sum of the volumes of the six white blocks.*

FIGURE 10.22




## *Graphical Representation of Complex Roots of an Equation*

The Eighteenth Yearbook of the National Council of Teachers of Mathematics contains a chapter by Howard F. Fehr entitled "Graphical Representation of Complex Roots."[1] Figure 10.22 shows a three-dimensional model of the graph of $y = x^2 - 4x + 8$ that is pictured in the chapter by Fehr. The graph has two branches—one real, one complex—both of which are parabolas. In the model, the real branch bounds the upper plane region; the complex branch bounds the lower plane region. The graph is obtained by solving the given equation for $x$ to obtain

$$x = 2 \pm \sqrt{y-4}$$

and substituting real values for $y$. For $y \geq 4$ the resulting values of $x$ are real, but for $y < 4$ the values of $x$ are complex. This leads to the representation of $y$ values on a real number axis

1. *Multi-Sensory Aids in the Teaching of Mathematics,* Eighteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1945), pp. 130–38.

(the vertical axis in the model) but requires a complex number plane for the representation of x values (the horizontal plane with real axis *a* and imaginary axis *bi*). For $y \geq 4$ the real branch of the graph lies in the plane determined by the y-axis and *a*-axis (this means that real values of *x* lie on the *a*-axis in the complex plane). For $y < 4$, the complex branch of the graph lies in the plane that is perpendicular to the *a*-axis at the point $2 + 0i$, since for all real values of y less than 4 the real part of *x* is 2. It is a simple matter to determine that the complex branch intersects the complex plane at $(2 \pm 2i, 0)$ and that $2 \pm 2i$ are solutions of $x^2 - 4x + 8 = 0$.

Projects involving hybrid real-complex coordinate systems such as the one described are a challenge to good students.

## *Geometrical Constructions and Curve Drawing*

Projects involving geometrical constructions and curve drawing provide stimulating experiences for students of nearly all ages and ability levels. Drawing simple designs with a straightedge and compasses satisfies elementary school students' natural sense of regularity. The creative instincts of students in later grades find satisfaction in designing mechanical devices to draw curves that cannot be drawn with straightedge and compasses.

Figure 10.23 shows a device for drawing a sine curve. The device shown in the picture was invented by a student of second-year algebra. Built into the device is a scheme for determining line values of the sine function mechanically. Pushing or pulling the long driver rod horizontally rotates the two wheels and causes the pencil to trace out the curve.

The student who produced the device shown in Figure 10.24 also had in mind a scheme for tracing a sine curve. As the pointer is rotated a sequence of lights is turned on.

Figure 10.25 pictures the construction of a device that can be used to draw a parabola. The white strip with black dots and the white thumb tack represent, respectively, the directrix and focus of a parabola. If the piece of chalk is held taut against the string that is looped over the top of the triangle and the base of the triangle is moved horizontally along the directrix, the piece of chalk traces a parabola.

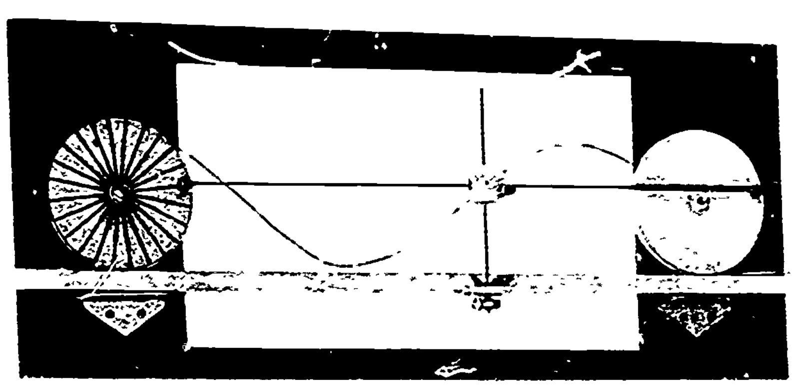


FIGURE 10.23. *Device for drawing a sine curve*

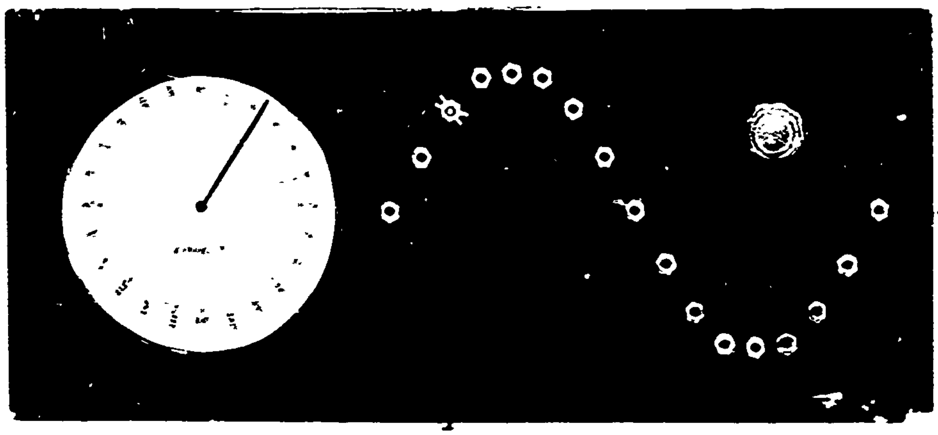
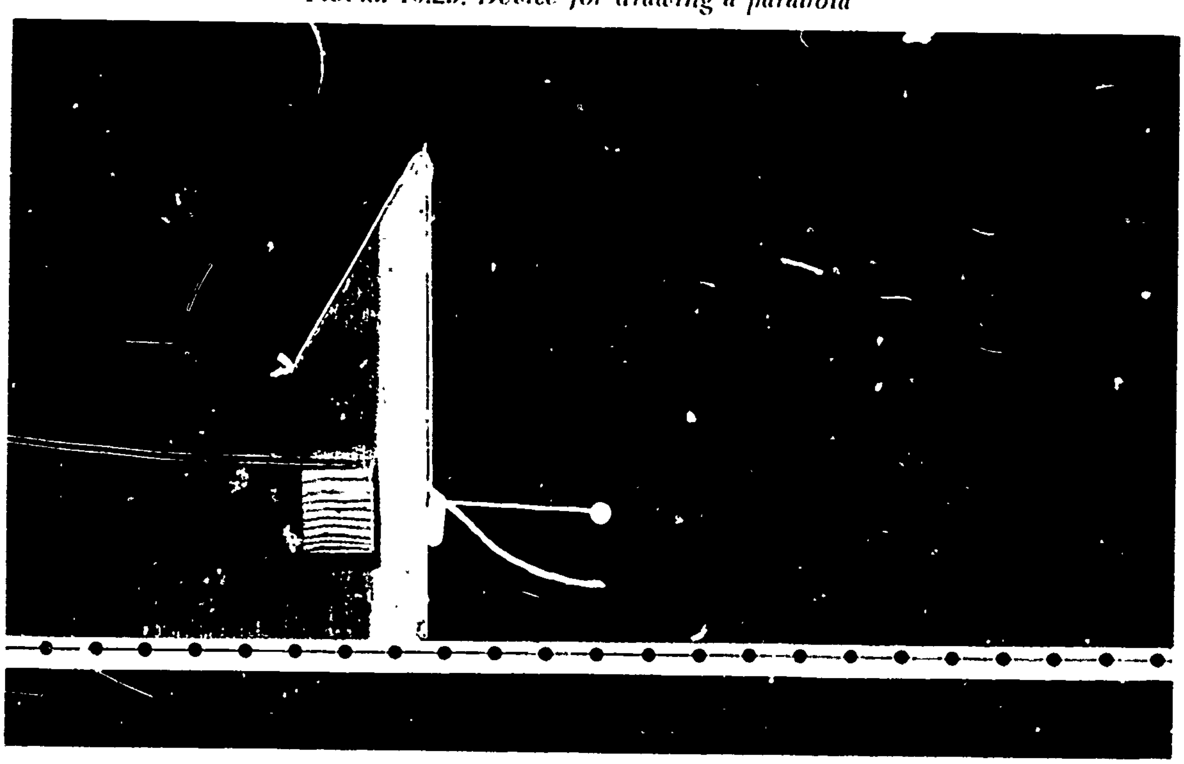


FIGURE 10.24. *Light travels along the sine curve as the pointer is rotated.*

FIGURE 10.25. *Device for drawing a parabola*



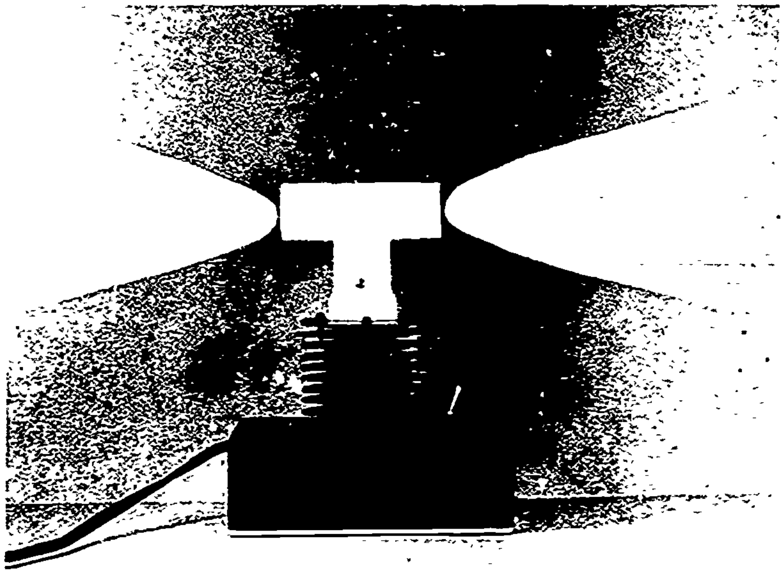


FIGURE 10.26. *A cone (of light) of two nappes is cut by a plane, producing a hyperbola.*

## *An Optical Method of Showing Conic Sections*

If a lighted floor lamp with an open-ended circular cylindrical shade is placed next to a wall, one may readily observe a brightly illuminated region with a boundary that approximates the shape of a hyperbola. The student who made the device pictured in Figure 10.26 tried to make this principle precise. He placed a small bright light in the center of an open-ended circular cylindrical tube, thus causing a conical beam of light to be projected from each end. If the cylindrical tube is held so that its axis is parallel to a wall, the resulting section is a hyperbola. The other conic sections can be formed by varying the angle between the axis of the cylindrical tube and the wall. A range of problems involving properties of the different conic sections can be investigated with the aid of this device.

## *Reflection Properties of the Parabola and the Ellipse*

Projects involving reflection properties of the parabola and the ellipse are good starters for students who have difficulty choosing a topic for a project. Only a few suggestions are needed to help students get started making displays of parabolic reflectors. Demonstrating the reflection property of a parabola is a little harder. One boy constructed the "pool table" shown in Figure 10.27 to demonstrate this property.

The cushion has the shape of a parabola. All channels are parallel to the axis of this parabola and the pocket is at its focus. A ball rolled as indicated in Figure 10.27 will bounce off the cushion into the pocket.

The elliptical "billiard table," on which a ball started from one focus will always rebound through the other focus, can be used to demonstrate the reflection property of an ellipse. The device shown in Figure 10.28 can be used to explain why the ellipse has this property. The string looped around two nails is threaded

FIGURE 10.27. *Device for demonstrating the reflection property of a parabola*

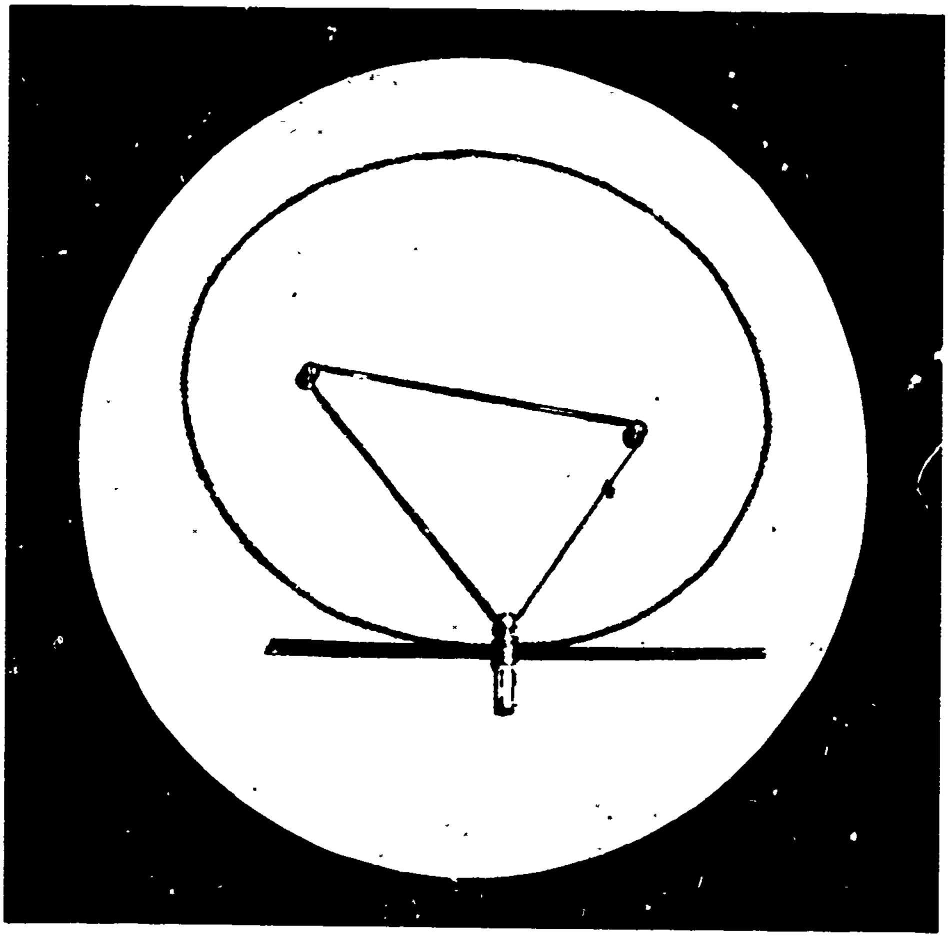


FIGURE 10.28. *A tangent of an ellipse at any point makes equal angles with the focal radii.*

through a vertical plumb that holds a straight wire in horizontal position. As the plywood disc is rolled the plumb traces out an ellipse, and it can be seen that the tangent to the curve at any point makes equal angles with the focal radii.

### *Exponential Curves*

Curves of this kind and their applications are a rich source of projects. The catenary is one such curve. It has the shape of a freely hanging chain or flexible cable (Figure 10.29) and looks deceptively like a parabola, which is the other curve shown in the picture.

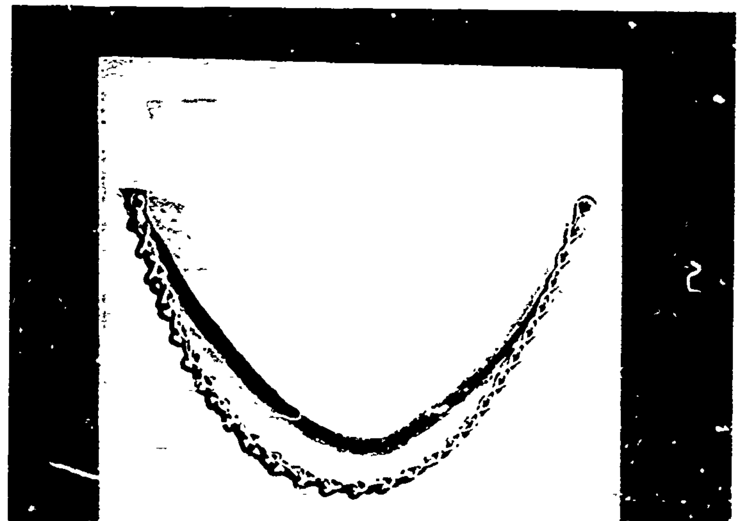


FIGURE 10.29. *Comparison of a catenary (chain) and a parabola (black curve)*

### *Fundamental Principle of Counting*

The projects pictured in Figures 10.30 and 10.31 started with an assignment to a second-year algebra class to collect illustrations of applications involving permutations and combinations. One student came across an illustration of a combination lock that is supposed to have existed around 1560.[2] His discovery led to construction of the wooden combination lock shown in Figure 10.30. The lock that is illustrated has one fixed cylinder and four movable cylinders. Painted on the outside of each movable cylinder are eight stripes of different colors. Of the 4,096 possible alignments of stripes, exactly 1 opens the lock.

2. Franklin W. Kokomoor, *Mathematics in Human Affairs* (New York: Prentice-Hall, 1942), p. 488.

Figure 10.31 shows an electrical device that grew out of the same student assignment as the wooden combination lock. Of the 1,331 possible settings of the three switches, exactly 1 will turn on the light.

### *Probability*

Production of a probability machine is always a popular project. The movement of the shot rolling down a board and the resulting approximation of a normal curve never cease to attract spectators and arouse excitement (Figure 10.32). In the device shown in the picture the configuration of channels just below the reservoir at the top was created by gluing hexagonal ceramic tiles to a piece of plywood.

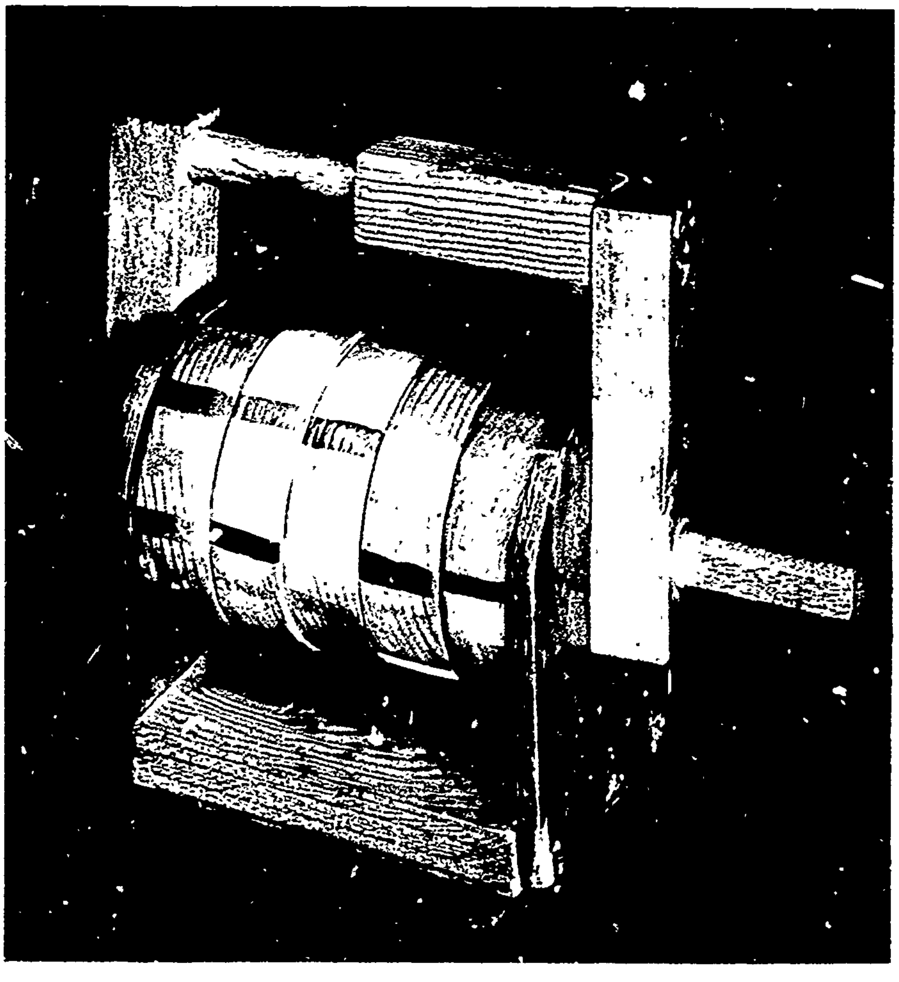


FIGURE 10.30. *Wooden combination lock*

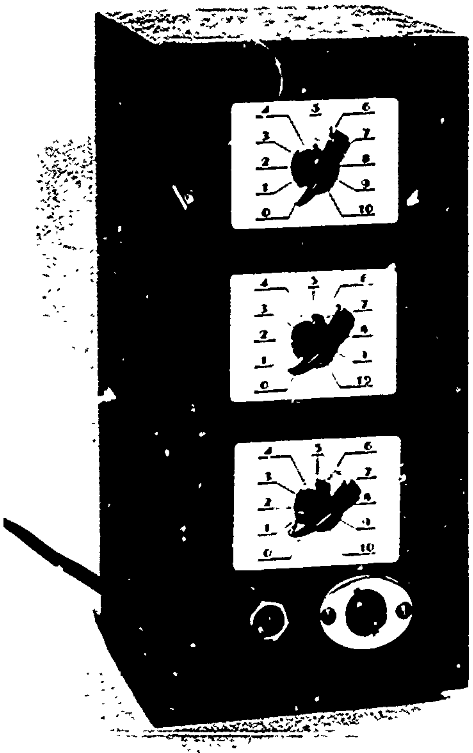


FIGURE 10.31
*Electrical combination device*

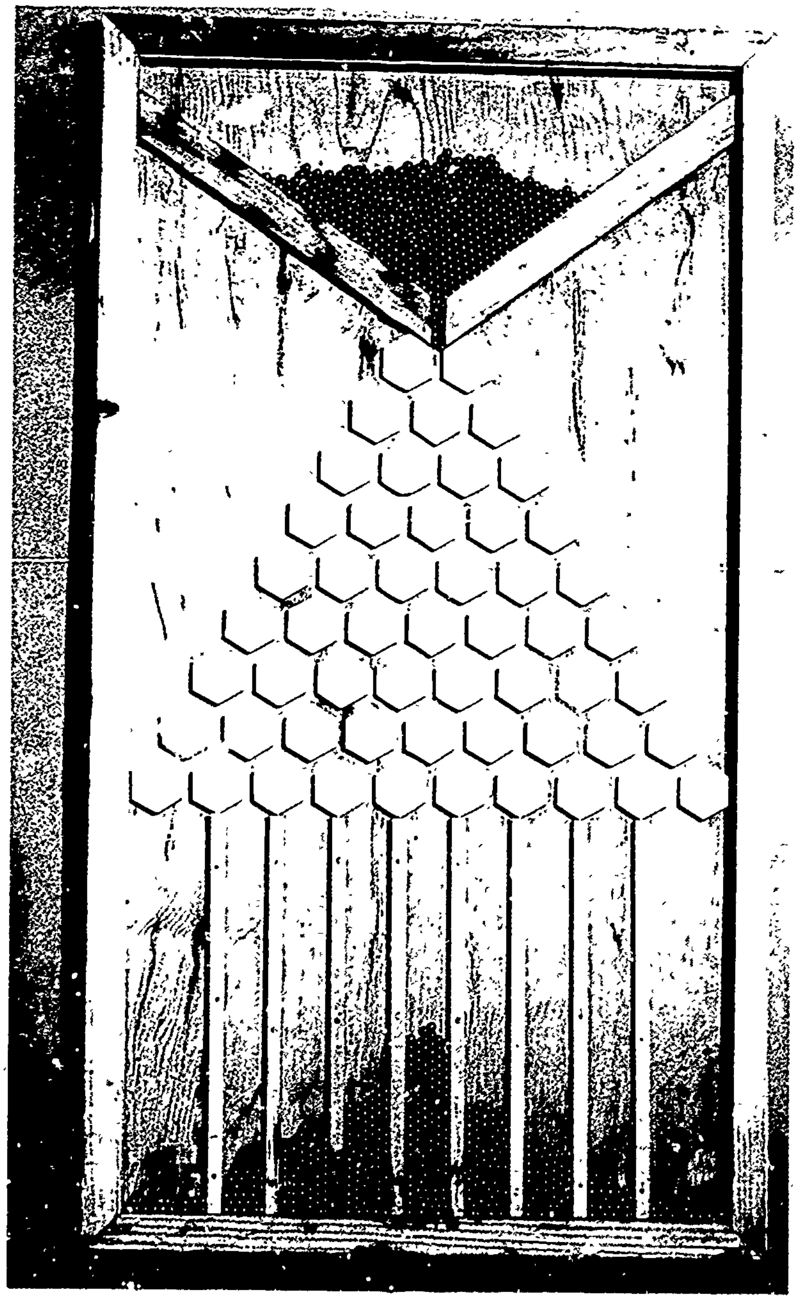


FIGURE 10.32
*Probability machine*

The penny board in Figure 10.33 shows a connection between the expanded form of

$$(h + t)^n$$

for $n = 1, 2, 3, 4$ and the possible outcomes of tossing $n$ pennies. Consider the last equation on the penny board. By the binomial theorem

$$(h + t)^4 = h^4 + 4h^3t + 6h^2t^2 + 4ht^3 + t^4.$$

Shown below is an expansion of $(h + t)^4$ written without using exponents and without combining "like" terms.

$$\begin{aligned}(h + t)^4 = hhhh &+ hhht + hhtt + httt + tttt \\ &+ thhh + thth + ttth \\ &+ hthh + htht + thtt \\ &+ hhth + tthh + ttht \\ &+ thht \\ &+ htth\end{aligned}$$

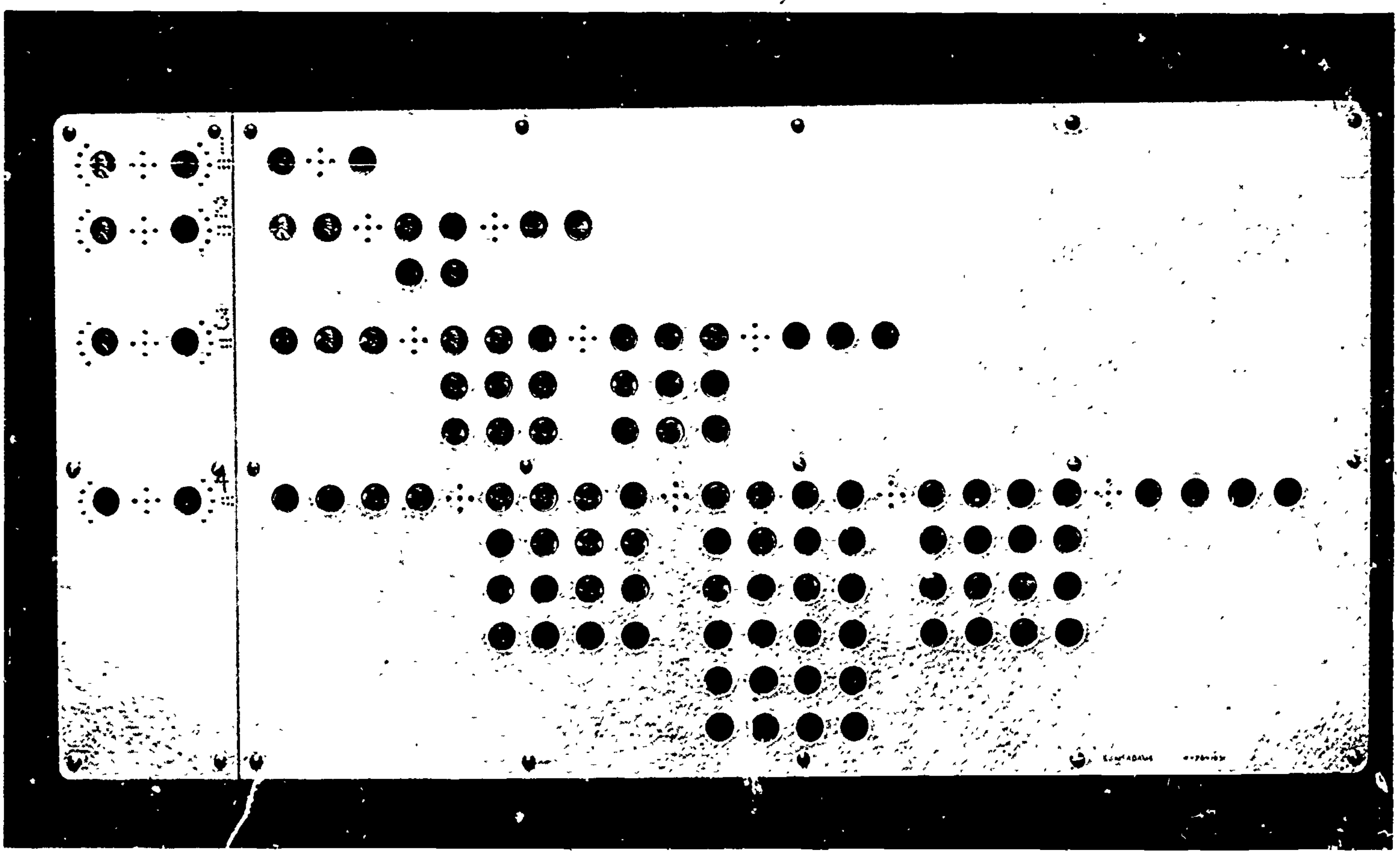
The five rectangular arrays of pennies in the right-hand member of the last equation on the penny board correspond to the five sets of "like" terms in the above expansion with a penny "head" for $h$ and a "tail" for $t$. The sixteen rows in these arrays are exactly the possible outcomes of a single toss of four pennies. The five arrays on the penny board correspond to the following events:

$E_1$: Getting 4 heads
$E_2$: Getting 3 heads and 1 tail
$E_3$: Getting 2 heads and 2 tails
$E_4$: Getting 1 head and 3 tails
$E_5$: Getting 4 tails

From this and a definition of probability it follows that $Pr(E_1) = \frac{1}{16}$, $Pr(E_2) = \frac{4}{16}$, $Pr(E_3) = \frac{6}{16}$, $Pr(E_4) = \frac{4}{16}$ and $Pr(E_5) = \frac{1}{16}$.

If you turn the book to look at the picture bottom side up, you can see a histogram that approximates a normal curve.

FIGURE 10.33. *Penny board*



### *Trisecting an Angle*

The problem of trisecting an angle comes up for discussion perennially, and any hint that such a construction is not possible with an unmarked straightedge and compasses only serves to challenge some student to attempt the impossible. The skillful teacher will know how to redirect efforts of this kind into fruitful projects involving a history of the trisection problem, proof that a classical construction is impossible, and production of trisection devices. Figure 10.34 illustrates the use of a trisection device that was produced by a tenth-grade student of geometry.

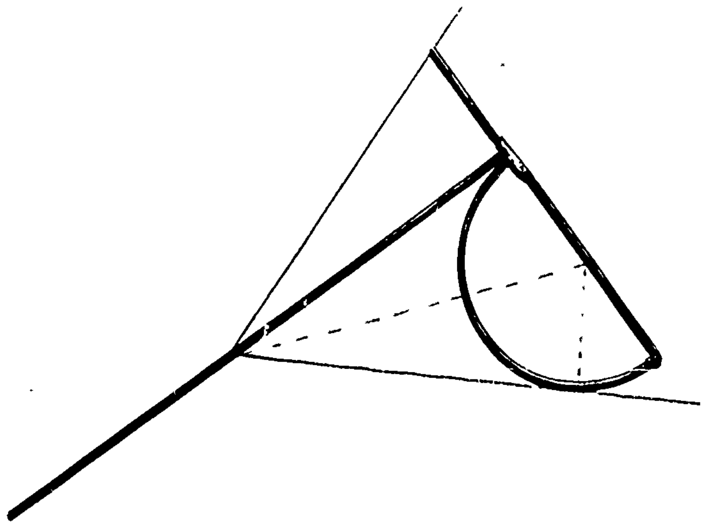


FIGURE 10.34
*The "tomahawk," a simple angle trisector*

### *Non-Euclidean Geometry*

Non-Euclidean geometry is another topic that is rich with possibilities for projects. Figure 10.35 shows a crude model of a pseudosphere, a surface that has constant negative curvature and that can be generated by revolving a tractrix about its asymptote. The asymptote of this surface is a straight line (the axis of revolution) to which all meridians (curves of shortest length) are tangent at infinity. The model shown in the picture was produced by a student of second-year algebra. He used the wire frames of two lamp shades which to him suggested the desired representation of a pseudosphere when fastened together as illustrated in the picture.

FIGURE 10.35
*Model of a pseudosphere*

## *Map Projections*

The topic of map projections provides a range of possibilities for projects by students of varying ability levels. Among the easier possibilities are projects involving models that illustrate gnomonic projections (Figure 10.36) and cylindrical projections (Figure 10.37). The major problems involved in making a map projection are these two: plotting points of a globe on a developable surface and characterizing image points in terms of preimage points.

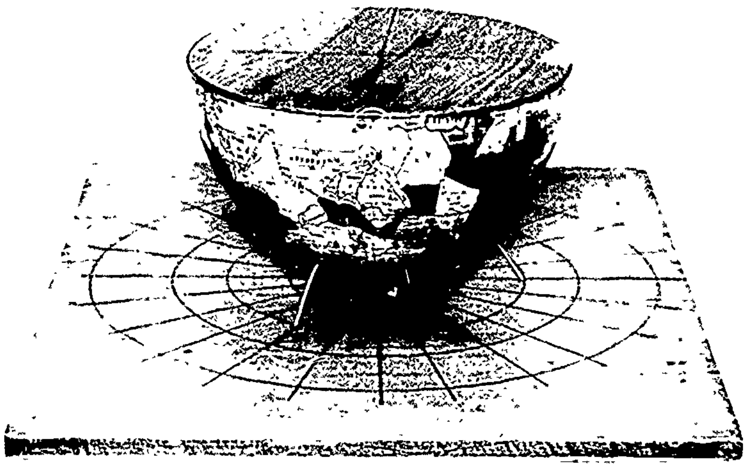


FIGURE 10.36
*Gnomonic projection: a projection of the surface of a globe from its center onto a plane tangent at the "north pole"*

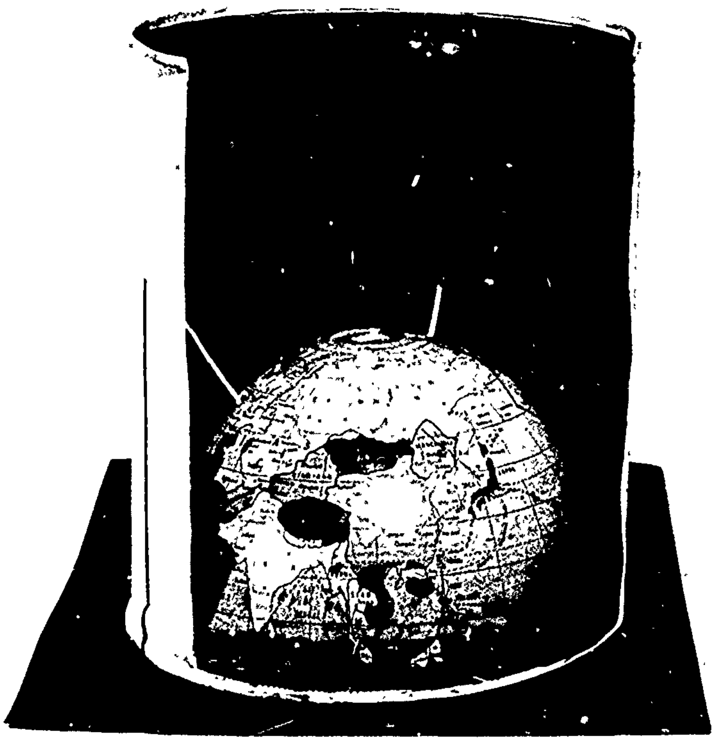


FIGURE 10.37
*Cylindrical projection: a projection of the surface of a globe from its center onto a cylinder tangent to the globe at its "equator"*

## *Sundials*

A study of sundials can be an intriguing project, and an intensive study involves some sophisticated mathematics. A typical sundial consists of two parts, a gnomon for casting a shadow and a dial on which the shadow falls. There are several kinds of sundials—equatorial, horizontal, vertical, declining, and reclining. The name of each is identified with the position in which the dial is set. The construction of a sundial depends on the latitude of the observer as well as the intended position of the dial.

The sundial pictured in Figure 10.38 was produced by a student of trigonometry who lived in Saint Paul, Minnesota (latitude 44° 57′ 19″). It

was constructed to be used with the dial mounted in a horizontal position and is therefore referred to as a horizontal sundial. Although it is not evident from the picture, the gnomon is perpendicular to the dial and intersects it in the 12 o'clock line. The angle formed by the upper edge of the gnomon and the 12 o'clock line is 45°. Since the measure of the angle between the upper edge of the gnomon and the plane of the dial should equal the latitude of the point of observation, the student's error was only 2′ 41″. In the Northern Hemisphere the 12 o'clock line must point north.

The position of any hour line $h$ on a horizontal dial can be determined by using the formula

$$\tan A_h = \sin L \tan B_h.$$

Referring to the formula, $A_h$ is the degree measure of the angle *between* the hour line $h$ (being determined) and the 12 o'clock line, $L$ is the latitude of the observer, and $B_h$ is the degree measure of the hour angle (15 times the number of hours between noon and the hour corresponding to hour line $h$).[3]

A sundial indicates local sun time. Mean solar time can be computed from local sun time by using a correction known as the equation of time.

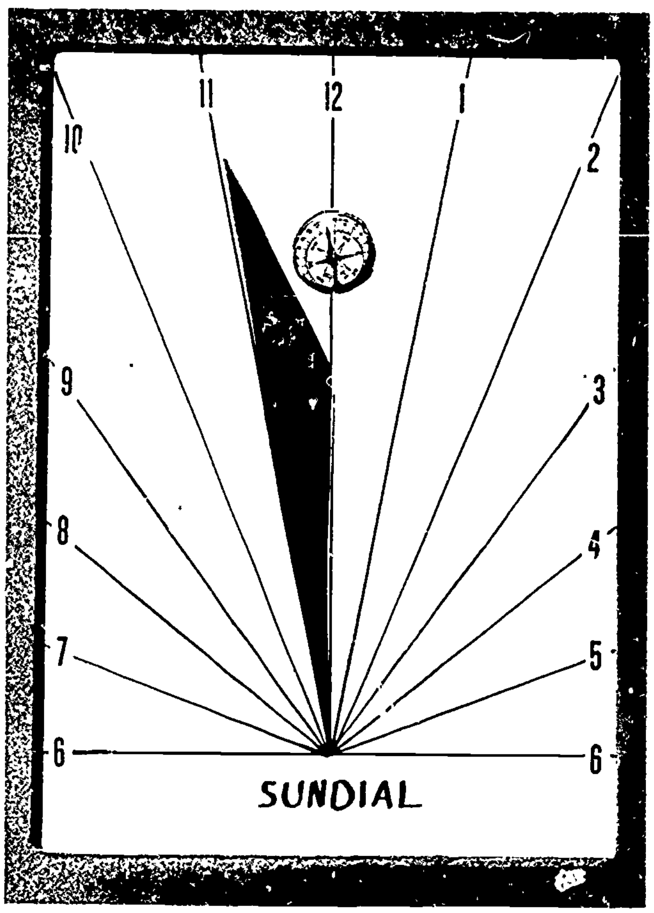



FIGURE 10.38. *Horizontal sundial*

### *Making Designs*

The beauty of mathematics is often reflected in designs based on mathematical curves or figures. Students frequently attempt to express the beauty of mathematics by means of projects involving curve stitching or drawings. Even for students who may be poorly motivated in mathematics, the fascination of a design is often sufficient to generate enthusiasm for artistic expression (Figures 10.39, 10.40, 10.41, 10.42, 10.43, and 10.44).

3. Carl N. Shuster and Fred L. Bedford, *Field Work in Mathematics* (New York: American Book Co., 1935; distributed by Yoder Instruments, East Palestine, Ohio), pp. 88–89.

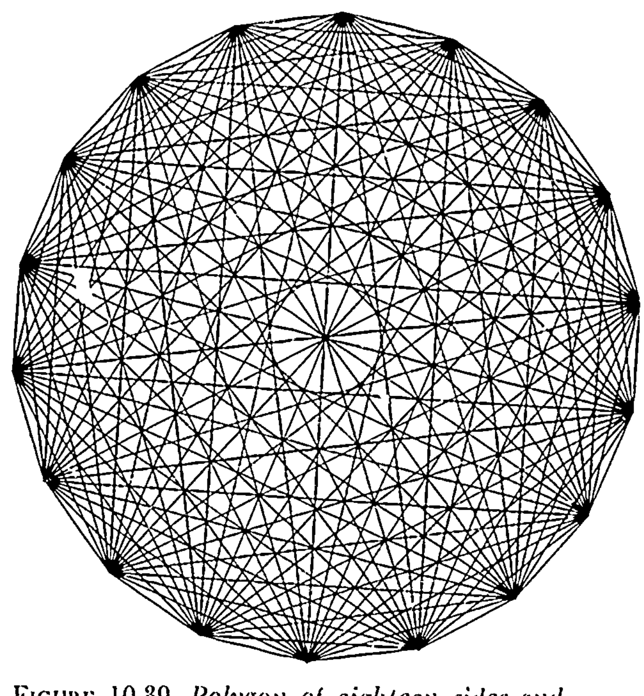


FIGURE 10.39. *Polygon of eighteen sides and all possible diagonals*

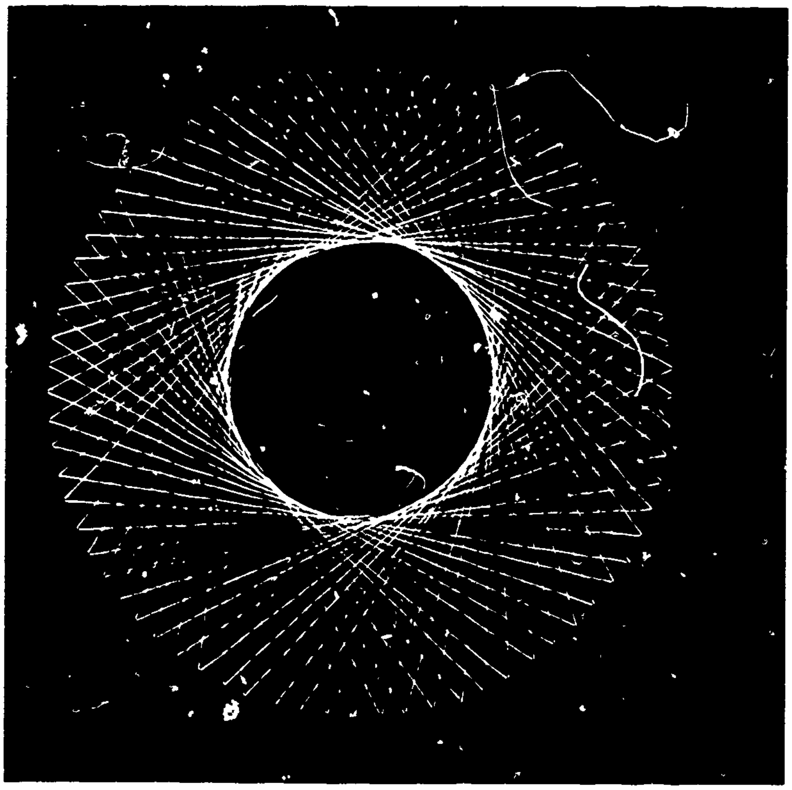
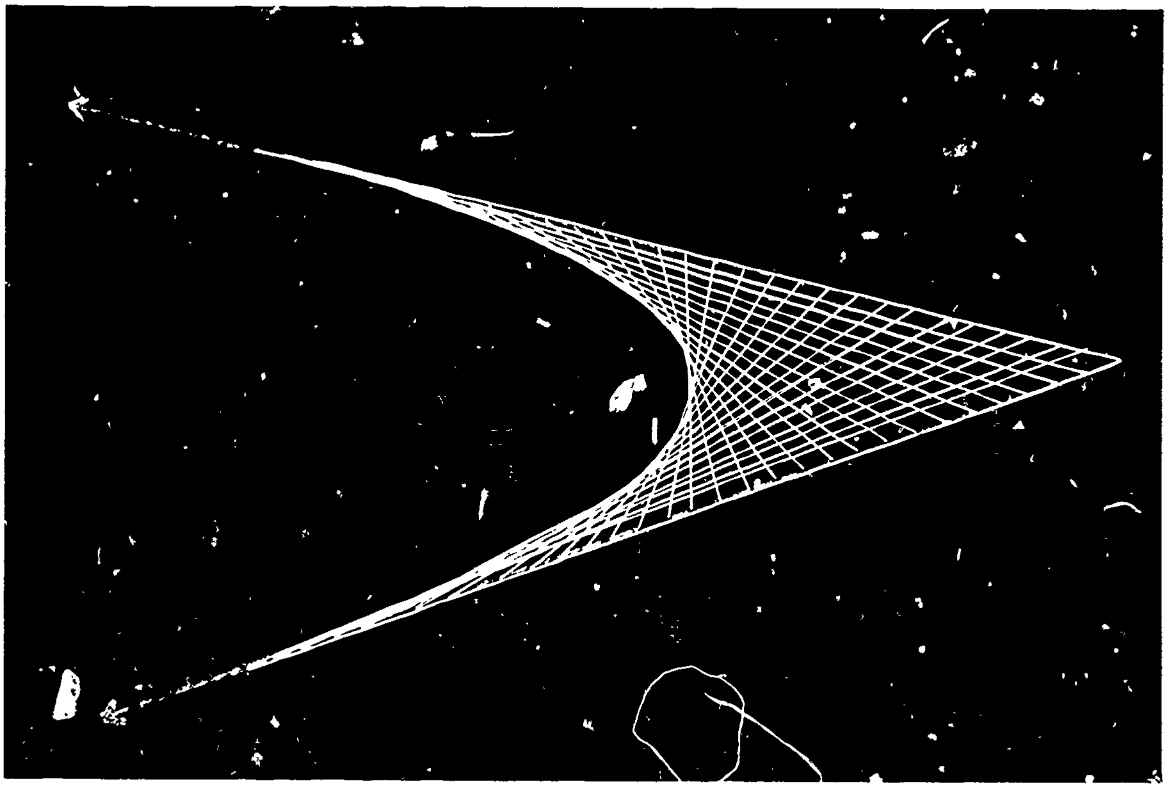


FIGURE 10.40
*Curve stitching: chords of equal length*

FIGURE 10.41
*Curve stitching: envelope of a pa· ·bola*



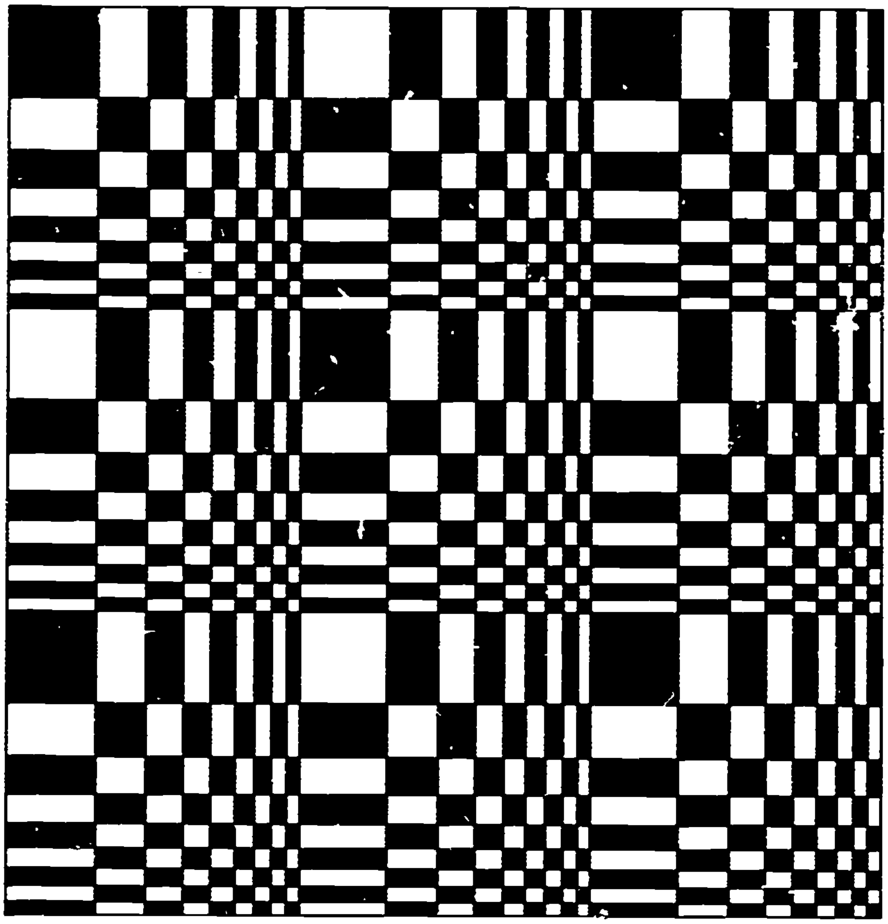
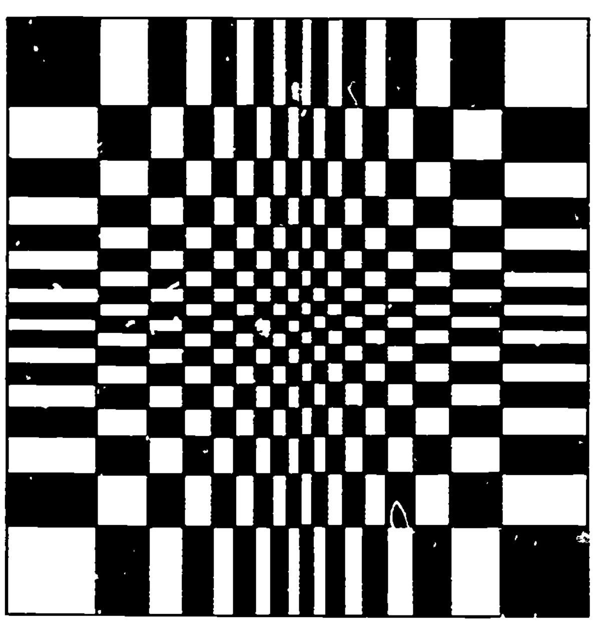
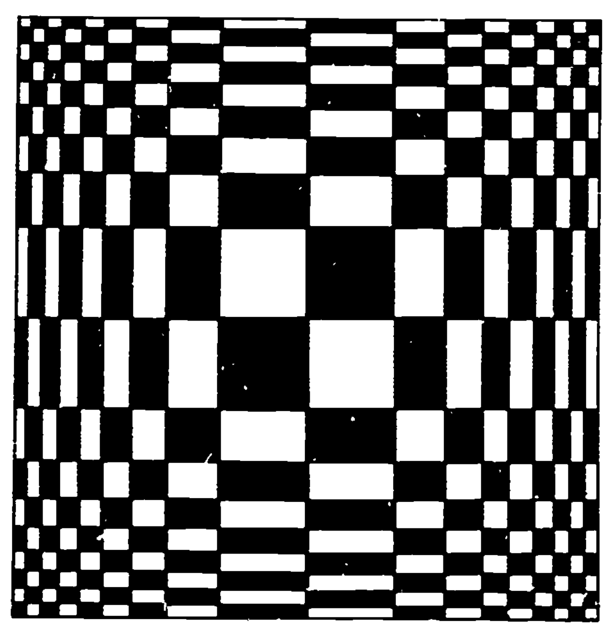
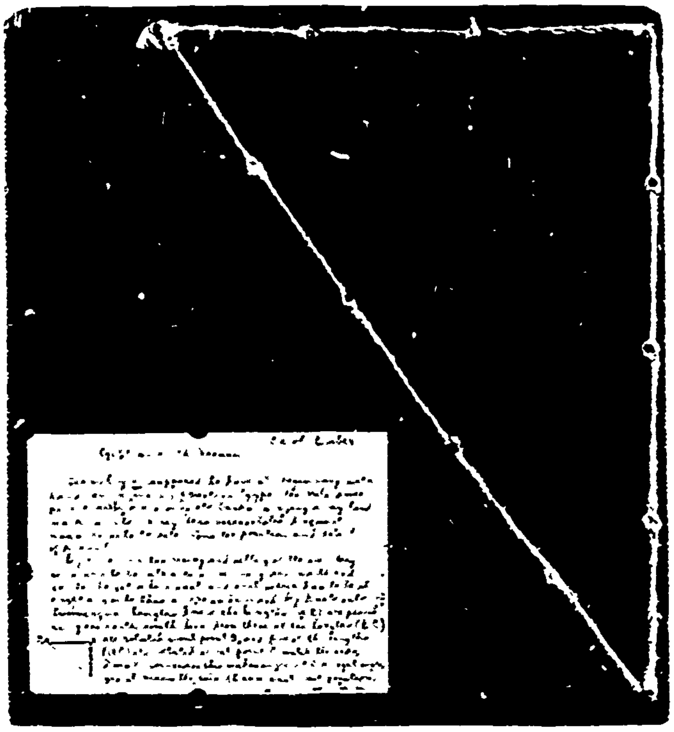


FIGURE 10.42. *Design produced by shading alternate rectangular regions of a 3 by 3 cycle logarithmic grid*

FIGURES 10.43 and 10.44. *Each of the two designs below was produced by rearranging four square regions of the logarithmic grid pattern in* FIGURE 10.42.



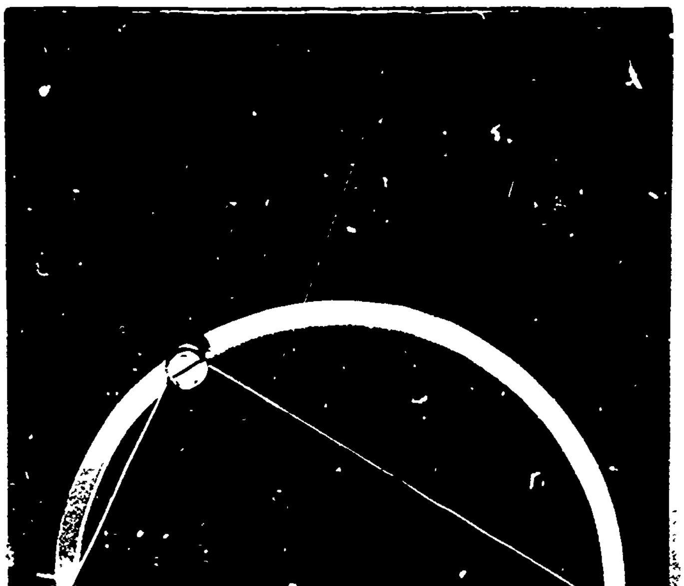




### *A Historical Project*

Constructing a right angle with a piece of rope that is separated by knots into 12 parts of equal length fascinates students because of the simplicity of the construction and because of its historical significance. The girl who produced the display (Figure 10.45) showing that segments with lengths of 3 units, 4 units, and 5 units can be sides of a right triangle probably learned something of the history and lore of mathematics as she researched her project.

FIGURE 10.45
*Laying out a right angle—the story of the "rope stretchers"*



### *Theorem Demonstration Models*

Many easy projects involving theorem demonstration models are possible. The symmetry and regularity of the circle make it a favorite topic for such projects. The model in Figure 10.46 "magically" demonstrates an unusual theorem about angles inscribed in a semicircle. At least one student thought so. To "show" that every angle inscribed in a semicircle is a right angle, he cut a semicircular slot in a piece of plywood, slipped a loose-fitting bolt through the slot, and stretched a rubber binder around the bolt and the two nails at the ends of the diameter.

The Pythagorean theorem is another favorite topic for projects involving theorem demonstration models. Two projects that illustrate intuitive approaches to this theorem are shown in Figures 10.47 and 10.48.

FIGURE 10.46
*All angles inscribed in a semicircle are right angles.*

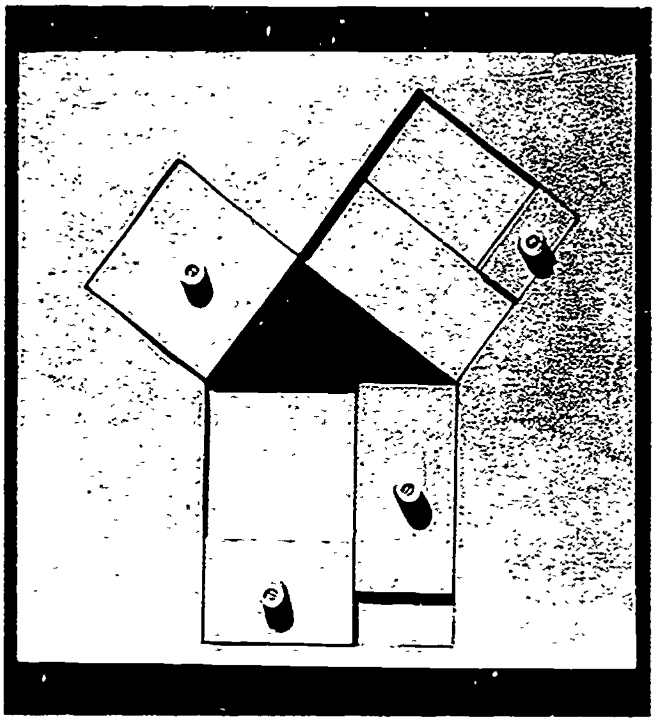


FIGURE 10.47

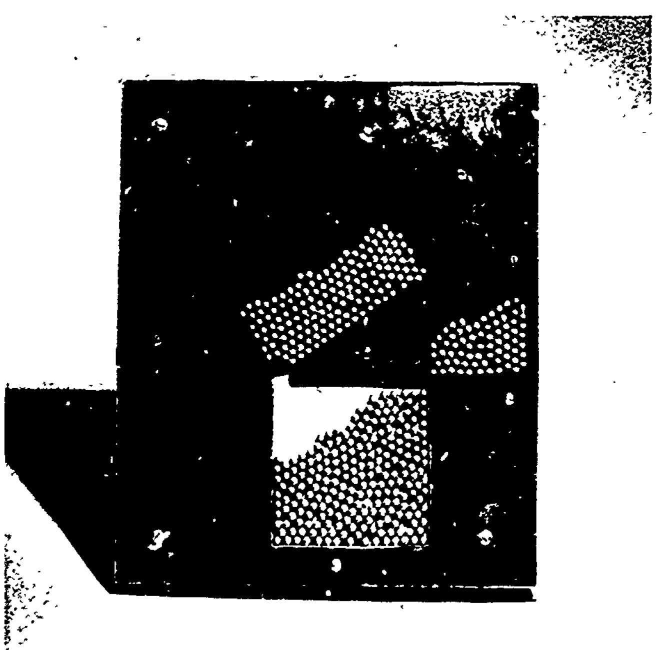


FIGURE 10.48

## MATHEMATICAL GAMES AND PUZZLES

A mathematical game is a mathematically based activity that involves a set of "legal" moves, operations, or rules, and usually some kind of competition. Most mathematical games are devised for two or more persons or teams, but many of these can also be played by one person; and some games are intended for one person, as is solitaire in cards.

Some mathematical games are puzzles. Kasner and Newman consider a puzzle to be "an ingenious game or problem."[4] The structure of a mathematical puzzle is characterized by some "internal relationship between component elements of the puzzle."[5] Creators of puzzles use clever ways of complicating a basic internal relationship by elaboration, restatement, addition of distracting details, and so on. The distinction between games and puzzles is not always clear, and the expression "games and puzzles" is often used in a collective sense to refer to both. Solving a puzzle means discovering the internal relationship that exists between the components of the puzzle or inventing a scheme that will expose the relationship.

Games and puzzles can make a valuable contribution to a mathematics program. For example, they can be used to:

1. Add interest to practice periods
2. Teach mathematical vocabulary
3. Teach mathematical ideas
4. Help students develop effective study habits
5. Provide motivation when beginning a new topic
6. Provide for individual differences
7. Promote the development of a positive attitude toward mathematics
8. Help students develop intuition in solving problems

4. Edward Kasner and James R. Newman, *Mathematics and the Imagination* (New York: Simon & Schuster, 1940), pp. 157–58.
5. Ibid.

9. Give students an opportunity to exercise imagination
10. Summarize or review a unit.

Exposure to games and puzzles should ordinarily be for brief periods. Care must be exercised that the excitement generated does not diffuse student attention. A game or a puzzle can, of course, be used as the basis of a lesson.

Games and puzzles can be created by teachers and students; however, a great many are available from commercial sources. A trip to any major dealer of games and toys will convince teachers and students of the wide variety that is available. A list of names of producers and distributors is given at the end of this chapter.

The binary system of numeration serves as the basis for a number of good games. One example is the well-known game of Nim. The strategy for winning with this game involves practice with mathematical operations.

Another game that involves the binary system is the Tower of Hanoi. This game consists of three pegs secured to a base with a number of discs of different diameters placed on one peg in order of size, with the smallest on top (Figure 10.49). To win the game one must transfer all discs from one peg to a second peg by moving a single disc at a time and without ever placing a larger disc on a smaller one. The game as proposed by M. Claus (Lucas) in 1883 involved eight discs, but fewer can also be used.[6] The manipulation involved can be simplified by using dots on a sheet of paper instead of pegs, and coins instead of discs.

FIGURE 10.49

An interesting example of a puzzle is one that involves arithmetic operations in which letters are used as numerals. The challenge in such a puzzle is to determine the digits that correspond to the letters. Below are several examples of this kind of puzzle:

$$\begin{array}{r} MEAT \\ MEAT \\ MEAT \\ MEAT \\ MEAT \\ FISH \\ \hline WEEK \end{array} \qquad \begin{array}{r} SEND \\ \underline{MORE} \\ MONEY \end{array}$$

$$(AB) \times (AB) = CDDD$$

$$\frac{EVE}{DID} = .TALK\,TALK\ldots$$

Magic squares are another kind of puzzle. Various construction methods can be found in books on recreational mathematics. Those with an odd number of entries are usually easy to construct. Students enjoy constructing magic squares for special occasions such as birthdays and anniversaries. Figure 10.50 shows an example of a

6. Walter William Rouse Ball, *Mathematical Recreations and Essays* (London: Macmillan & Co., 1905), p. 107.

| | | | |
|---|---|---|---|
| 1 | 14 | 4 | 15 |
| 8 | 11 | 5 | 10 |
| 13 | 2 | 16 | 3 |
| 12 | 7 | 9 | 6 |

FIGURE 10.50. *Extremely magic square*

magic square invented by Benjamin Franklin. In this square various combinations of four entries have a sum of 34. For example, the entries at the four corners add to 34. It is interesting to find just how many combinations of four entries have a sum of 34.

Certain problems from elementary number theory can be presented to students as puzzles. One such problem involves determining a number by having a student follow a few easy instructions. For example, have the student select a positive integer between 1 and 60. Then give instructions for the following:

1. Have him divide his number by 3 and tell you the remainder. Suppose this remainder is $a$.
2. Have him divide his number by 4 and tell you the remainder. Suppose this remainder is $b$.
3. Have him divide his number by 5 and tell you the remainder. Suppose this remainder is $c$.

Now divide $40a + 45b + 36c$ by 60. The remainder will be the number the student selected. This problem is an application of the Chinese remainder theorem. An explanation of the solution can be found in books on number theory.

Number theory problems involving Diophantine equations can also be presented as puzzles. Consider, for example, this problem: "Find a number, given that the sum of the number plus the sum of its digits is 73."

Problems involving identification of Pythagorean triples are still another source of puzzles. The following example is illustrative: "In what right triangle whose sides are a Pythagorean triple is the sum of the legs a perfect square?"

Puzzles that involve manipulation of objects have a strong appeal. A simple puzzle of this kind is one that requires the separation of a tangled set of wires into component parts. Various puzzles of this kind are on sale at toy stores. A fairly sophisticated wire puzzle is the Chinese ring puzzle. Solving this puzzle involves the use of the binary system of numeration.

Dissection puzzles have always enjoyed great popularity. These range from simple jigsaw type puzzles that are suitable for use at the primary level to dissection demonstrations of the Pythagorean theorem at the secondary level (Figure 10.51).

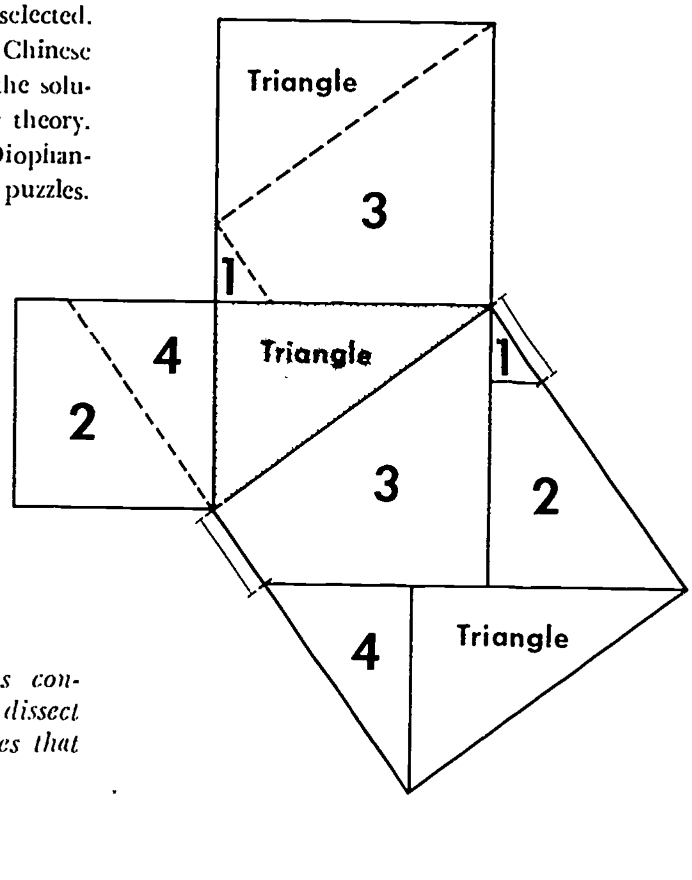



FIGURE 10.51
*Given a right triangle with squares constructed on its sides, it is possible to dissect the two smaller squares into five pieces that fit on the largest square.*

Separating a piece of cardboard that has the shape of a quadrilateral into four pieces by cutting along lines joining the midpoints of opposite sides gives us another example of a dissection puzzle (Figure 10.52). The challenge in this puzzle is to arrange the four pieces so that the resulting figure has the shape of a parallelogram.

Puzzles involving mosaics and tessellations are similar to dissection puzzles. These call for combining differently colored tiles to form designated figures or to cover the plane. The theory of tessellations may be found in a number of texts, and articles on tessellations have appeared in many periodicals.

The chessboard can be used as a basis for a number of good puzzles. There is, for example, the "Eight Queens" problem, which involves placing eight pieces on the board so that no two are in the same row, column, or diagonal. This problem can be solved by using determinants and matrices. There is also the problem of occupying every square of the board by making continuous legal moves with a knight.

Sources of games and puzzles are so varied that the observant teacher can constantly find new ones. Games and puzzles selected for use by a mathematics class should be intriguing enough to engage students' interest and just hard enough to provide a respectable challenge. A bibliography of games and puzzles is given at the end of this chapter.

A major problem in using games and puzzles in a mathematics class is the limitation of time. A skillful teacher will try to use after-school hours or free time during unscheduled modules. In the case of competitive games, using a timer helps keep games from taking too much time. Playing only parts of games is another way of making effective use of time that is available for competitive games.

## MATHEMATICS CONTESTS

Mathematics contests can be an integral part of a school's mathematics program. In some schools contests are conducted within a given class or subject area with winners moving on to compete in district or regional contests.

Frequently, school teams are formed for the purpose of challenging neighboring schools in a contest involving competitive mathematical games. In conducting contests like these it is im-

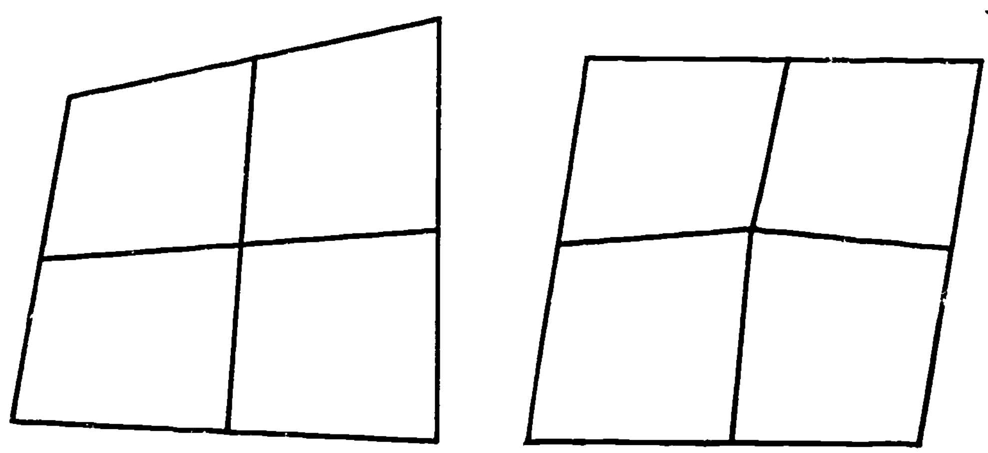


FIGURE 10.52. *The figure on the left shows a quadrilateral with the midpoints of opposite sides connected by segments. If the four smaller quadrilaterals are cut out, the pieces can be arranged so that the resulting figure has the shape of the parallelogram on the right.*

portant that rules and criteria be established beforehand and adhered to strictly. There must be a judge or team of judges to serve throughout the contest. Awards for winners must be announced and appropriate recognition given to all participants. Local school and commercial newspapers are generally willing to publish the results of a well-organized contest involving mathematical games.

A more sophisticated kind of contest is one that involves administration of cleverly constructed paper-and-pencil tests to large numbers of students meeting in well-controlled testing situations. Contests of this kind are sometimes sponsored by local school districts.

Perhaps the most prestigious examples of contests involving paper-and-pencil tests are the National High School Mathematics Contest in the United States and the Olympiads of eastern Europe.[7]

Contests involving paper-and-pencil tests are generally suitable only for the better students; however, other activities such as games, talks, and seminars can be included in a day of contests, and these usually have a somewhat wider appeal. It is possible to make contests part of a mathematics field day with something for all who are interested in mathematics, regardless of ability.

A teacher can do several things to help prepare students for a mathematics contest. For example, old tests can be made available. A well-stocked library on many different areas of mathematics is valuable, and constant encouragement is a must.

As in the case of projects, recognition should be given to all students who compete, whether they are winners or not. Contests can add a strong positive influence to a mathematics program.

## SUMMARY

The challenge to teachers of mathematics is to prepare citizens who will be able to identify and solve mathematical problems that may not have been encountered before. Since it is difficult, if not impossible, to predict the kinds of problems that future citizens will face, teachers must be concerned with the development of skills and attitudes in students that will give them confidence to cope with new problems. Mathematics projects, exhibits, games, puzzles, and contests can be used to help students develop such confidence.

7. Howell L. Gruver, *School Mathematics Contests: A Report* (Washington, D.C.: National Council of Teachers of Mathematics, 1968).

## REFERENCES FOR SIXTY-SEVEN PROJECT TOPICS

Listed below are sixty-seven topics for which references are included. References for the different topics follow the list of topics.

1. Ancient Computational Methods
2. Measurement of Time
3. Codes and Ciphers
4. Computers
5. Curve Stitching
6. Numeration Systems
7. Nomographs
8. Measurement and Approximation
9. Map Projections
10. Linkages
11. Magic Squares
12. Paper Folding
13. Paradoxes and Fallacies
14. Optical Illusions
15. Approximation of $\pi$
16. Pythagorean Theorem
17. Pythagorean Triples
18. Finite Geometries
19. Game Theory
20. Fourth Dimension
21. Algebraic Structures
22. Transfinite Numbers
23. Symbolic Logic
24. Non-Euclidean Geometry
25. Probability
26. Boolean Algebra and Switching Networks
27. Statistics and Statistical Inference
28. Topology
29. Mathematics and Art
30. Mathematics and Music
31. Linear Diophantine Equations
32. Duodecimal System
33. Linear Programing
34. Congruences
35. Inversive Geometry
36. Curve Fitting
37. Trachtenberg System of Rapid Calculation
38. Algorithms and Flow Charts
39. Abacus
40. Prime Numbers
41. Perfect Numbers
42. Figurate Numbers
43. Integers as Ordered Pairs of Natural Numbers
44. Markov Chains
45. Repeating Decimals
46. Golden Section
47. Hyperbolic Functions
48. Continued Fractions
49. Trisection of an Angle
50. Nine-Point Circle
51. Complex-Number Operations by Geometry
52. Snowflake Curve
53. A Mathematical System of $2 \times 2$ Matrices
54. Mass-Point Geometry
55. Lattices
56. Egyptian Computational Methods
57. Greek Computational Methods
58. History of Mathematical Symbols
59. Gaussian Integers
60. Curves of Constant Breadth
61. Women Mathematicians
62. History of the Slide Rule
63. The Four-Color Problem
64. The Königsberg Bridge Problem
65. Development of Arabic Numerals
66. Fibonacci Numbers
67. Sundials

## 1. ANCIENT COMPUTATIONAL METHODS

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Creative Publications
Cuisenaire Co. of America, Inc.
Ideal School Supply Co.
Educational Fun Games
Imout Arithmetic Drill Games
Kalah Game Co.
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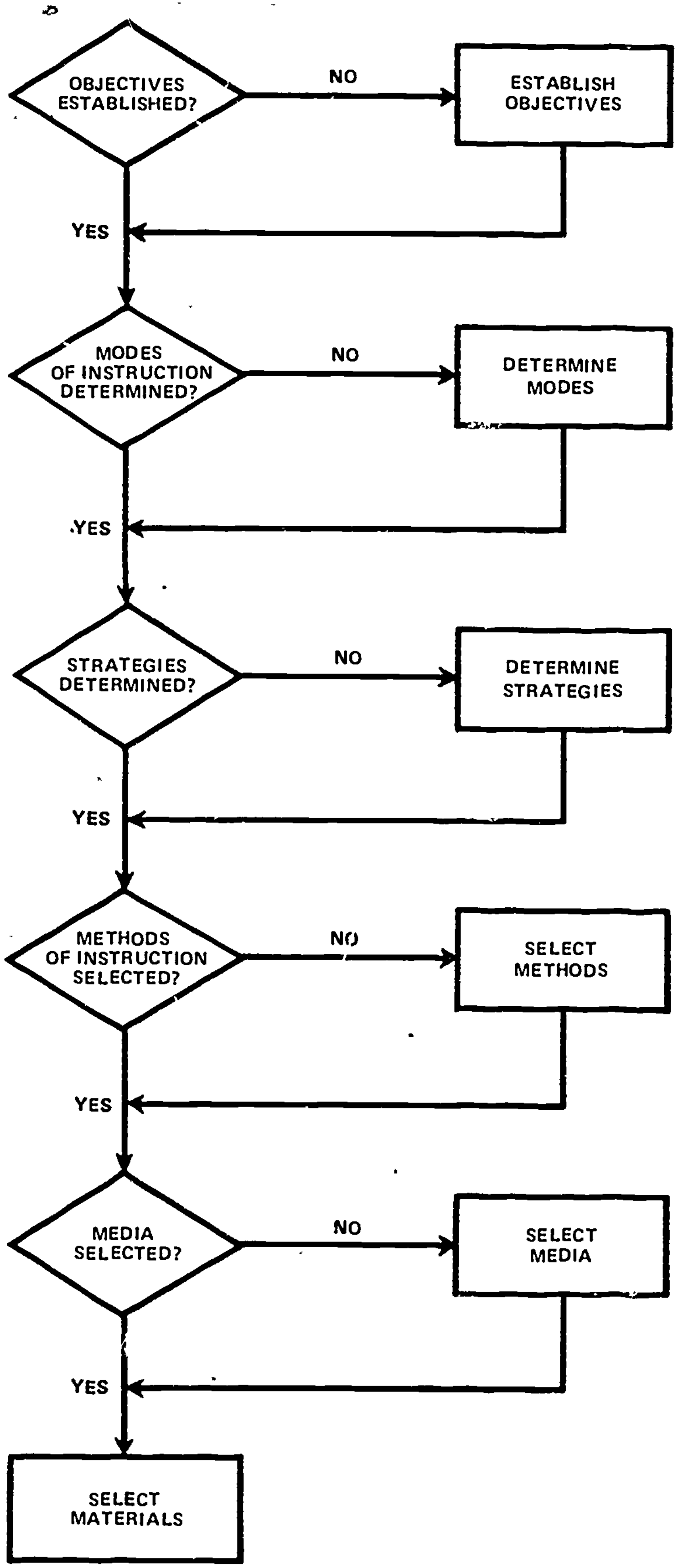
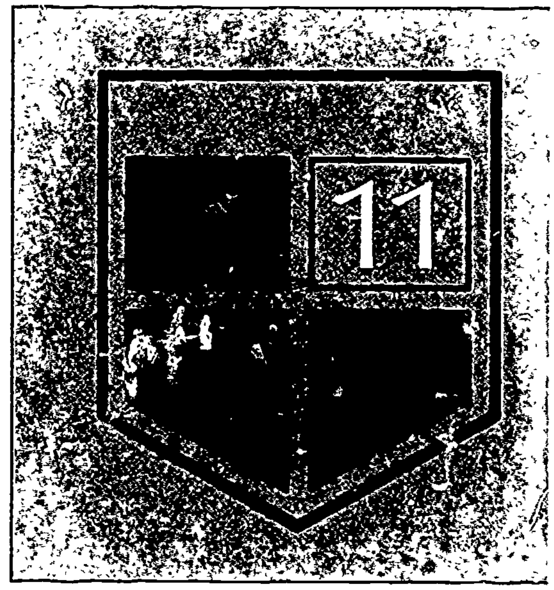



A SYSTEMS APPROACH TO MATERIALS SELECTION

CHAPTER 11

# A SYSTEMS APPROACH TO MATHEMATICS INSTRUCTION



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Chapter 11 begins with a brief introduction to the general concept of a systems approach to problem solving. Then the use of systems analysis in the solution of educational problems, particularly instructional problems in the mathematics classroom, is discussed. Applications of "systems thinking" to both the analysis of materials and classroom management are explored. Consideration is given to the various roles of the teacher in a clearly defined instructional system and to possible roles of an electronic computer in such a system. Finally, a detailed example of a systems analysis of a particular instructional problem is presented.

# 11. A SYSTEMS APPROACH TO MATHEMATICS INSTRUCTION

President John Garfield once spoke of the ideal instructional system as a boy on one end of a log with Mark Hopkins on the other.

In the course of time this simple system has grown complex. We have done most with the log, transforming it into the multi-resourced modern school and classroom. We have learned less about the boy—although, after having dealt with him for a long time in large numbers, we are now again attempting to reach him as an individual.

Most of us are not Mark Hopkinses, but modern technology and research have done much to make available to all of us the kinds of knowledge Mark Hopkins possessed about the boy, the learner, and about what is to be learned.

The simple boy-log-Mark Hopkins system has become monumentally complicated in today's classrooms. Although we may lack the instinctive genius of a Mark Hopkins, we, as teachers, may still learn much about optimizing the learning situation of our students by facing the fact that our modern teaching environment is a complex instructional system—and learning to act on this knowledge.

## WHAT IS A SYSTEMS APPROACH TO INSTRUCTION?

Today references to "systems," "systems analysis," and a "systems approach"—concepts that have become so common in the fields of business and government—are appearing in the field of education.

Our goal in this chapter is not the extension of the dialogue going on today about systems. Rather, it is to bring to the mathematics classroom teacher, in a discursive and descriptive rather than a technical way, an explanation of what is meant by a *systems approach* to instruction, particularly what a systems approach to his own classroom instruction might be like presently and in the immediate future.

### *What Is a System?*

The term *system* seems frequently to be used without any definition, under the assumption that it is a primitive concept that would be understood by all. Where there are attempts at definition, these vary considerably. In fact, there are several uses of the term *system*, some of which are so closely related that, in the same discussion, the term shifts from one referent to another.

We shall give one definition, then return to a discursive approach to this topic. The American Association of School Administrators has defined a system as an "array of resources designed and dedicated to achieve an objective according to some plan of action. It connotes order, unity, purpose, arrangement, and plan."[1] This definition is put forth neither as the best nor as the most appropriate nor as the one we shall eventually adopt. However, it seems a good starting point; for it is stated in familiar language and, because of its source, has considerable educational acceptability. We shall resist the temptation to introduce the plethora of other definitions that appear. Their relationships to the given definition vary considerably. Some are merely linguistic modifications; others differ in important ways. Instead, we shall describe and differentiate between two rather familiar uses of the word *system*. Then we shall explain what a systems approach to classroom instruction and systems thinking on the part of the classroom teacher mean to us.

Commonplace usage helps us to understand the systems concept if we note the use of the terms *solar system, railway system, Davies-Brickell system of school board management, Bell Telephone system, Chicago school system, human system,*

1. *EDP and the School Administrator*. (Washington, D.C.: American Association of School Administrators, 1967), p. 2.

*Goren system* (of bridge bidding), *system for breaking the bank at Monte Carlo, Dewey decimal system* (for library classification), and *double entry bookkeeping system.*

These systems fall into two quite distinct categories. The word *system* is being used in two related, but quite different, senses.

In the phrases *solar system, railway system, Bell Telephone system, Chicago school system, human system,* the system is an *existential* system —a complex, dynamic situation in the real world involving physical objects, interactions among these physical objects, and basic laws that govern these interactions. Among these existential systems, some (like the solar system) are natural, and some (like the Bell Telephone system) are not. Man-made systems have two characteristics not common to the natural systems. They are designed to achieve certain goals, usually stated in terms of behavioral outcomes. They also include a management component whose function is to determine operating policies and rules of procedure so as to achieve the objectives of the system in an optimal fashion.

An airline system provides an example of an existential system that is man-made. There are goals: the transfer of people and freight at economical cost levels, in adequate comfort and with various services, such as meals, so as to derive a profit. These goals are often stated in highly specific behavioral terms. Very specific arrival times, for example, are stated in timetables. Often specific goals in terms of minimum percentage of profit are established.

The airline system, in addition to the specific, behaviorally stated *objectives,* involves a variety of *things* (equipment and people, in this case) and various *operations* and *interactions* of these things (such as the operations of flying, ticket selling, landing, servicing). All these interactions are governed by laws and rules of procedure that regulate the manner in which the various resources are combined and the operations carried out for the achievement of the goals. Finally, there is the *management,* whose function it is to set up these rules of procedure and these operating policies to best accomplish the objectives of the system. They keep highly informed on the functioning of the system, study the various alternative courses of action, and undertake the decision making that produces a more effective pursuit of the objectives. In fact, the basic system is in a constant state of redesign as management attempts to put it into an optimal state by introducing new things (better planes) and new interactions that are more productive of the goal achievement (curbstone baggage-checking speeds up the handling of customers, makes schedules more effective, and produces customer satisfaction and hence greater use and greater profit). The basic laws, operating policies, and rules of procedure are often modified to the same end (stewardess-pilot coordination to avoid hijacking, changed scheduling to avoid stacking over busy airports at busy hours). This example of an airline system clearly fits the definition of a system as stated by the American Association of School Administrators.

This discussion of existential systems—whether man-made or natural—covers only certain of the systems mentioned in the introductory listing of commonplace uses of the word *system.* What can we say of systems such as the Goren system of bridge bidding, the system for breaking the bank at Monte Carlo, the Dewey decimal system for library classification, the Davies-Brickell system of school board management, or the double-entry bookkeeping system? Here the word *system* refers not so much to the underlying reality—the complex dynamic situation that certainly is present—as to a particular approach to the handling of that dynamic situation. The *system* is the particular organization or strategy for handling the interacting entities that are present. This use of the word is illustrated by the following example: Two people are faced with the same physical reality or dynamic situation. One of them says, "You go ahead and use *your* system; I am going to stick to *my own* system." With the same reality there are two systems, two means of approaching the problems connected with this basic reality, two ways of handling or managing the situation.

Systems described by this sense of the word are referred to as *management* systems.

There is an obvious relationship between *existential* systems and *management* systems. For the operation of one particular existential system there may be several management systems. The difference between these two uses of the word *system* and the interrelationship of these two types of systems can be seen in the example of the game of bridge. The game itself is an example of an existential system. There are entities—people and cards—interacting under codified laws that govern the play of the game, specify penalties for their infringement, and award points for various possible outcomes. Certainly bridge has very specific behavioral objectives, from the overall one of getting a higher score than the opponents to the more specific ones of taking enough tricks to fulfill a contract (or to set one!), reaching a slam contract if it is makable, attempting to make a particular finesse, and so forth. A bridge game is a contest between two *management groups*, each attempting to manage the existential system, the game itself, to its advantage.

The methods used by each management group illustrate the idea of a *management system*. One of the two sets of partners may be using the Goren system of bidding for best achieving the goals of the game. The other pair may use the Italian system, hoping for the same kind of results. Here the word *system* does not refer to the basic game itself or to the interplay between the people and cards. It refers to the management approach to the game. Thus it does make sense, within one and the same game of bridge, with its inflexible laws of procedure, to bring two different management systems to bear in order to try to optimize the achievement of the goals established within the existential system.

Because of the tight interplay of these two types of systems, one finds the word *system* being used in the same discussion, in almost succeeding sentences, first to refer to the existential system and then to refer to the management system. Though care should be exercised in such dual usage, it is rarely as confusing as it may sound because the context in which the term is used should tell the careful listener or reader the sense in which it is being used.

Many systems that exist in human affairs operate with their managers making a countless succession of decisions without ever truly being aware of the many alternatives available to them in their decision-making role or of the many tools for their use in the conscious pursuit of optimal decisions. A systems approach to management of existential systems is a way of thinking on the part of managers of such systems that makes them keenly aware of being involved in an existential system and alive to the fact that they can do much to optimize the pursuit of the objectives of the system by paying conscious attention to the system they are managing and to its potential by analyzing its functions and by entering into the decision-making process in a highly aware manner, using whatever modern tools are available.

## *Systems in Education*

The establishment of a management system for an already-functioning existential system is one thing. To start with a *task*—such as the education of the young—and to establish a management system for both the design *and* the operation of an existential system to carry out this task is, although not essentially different, surely broader in scope.

In the field of education, many apply a systems approach to this broad task of the education of the young. Their concern is with the specification of the general educational objectives of the culture and with the roles to be played by various structures in society in the attainment of these objectives by learners. They view not only the educational establishment but all other formal and informal institutions that contribute to the educational process—family, peer group, church group, communication media, and so forth—as subsystems of their overall system. In short, their existential system contains whatever in the culture contributes to the education of the young. Their

role as designers and managers of this system is to state in specific terms what the culture means by an "educated adult," to determine for various subsystems the roles they are to play, and to construct a management system to facilitate the efficient interaction of these subsystems to ensure that all required goals are achieved by learners.

Others take the educational establishment itself as their existential system. They tacitly or explicitly assume that the overall goals for this system have been assigned. They construct a management system to determine which agencies within the establishment should be given the responsibility for inducing the achievement of which goals. They also concern themselves with the development of the financial and administrative structures necessary to the operation of these agencies.

Still others concern themselves with the application of a systems approach to the management of a single school district or building. Finally, others apply the techniques of systems analysis at the classroom level.

Our concern in this chapter is with this last, admittedly limited, application of a systems approach. In focusing on this, we do not mean to ignore the importance of systems analyses of a broader nature. It is merely that our interest and (we assume) that of our readers centers at the classroom level. It is at this level that the classroom teacher acts as designer, manager, and component of an instructional system.

Whenever a teacher accepts an assignment to a class he accepts these design, management, and component roles. His choice is not to accept or reject them but rather whether he will exercise them consciously and effectively to provide optimum instruction or will let the existential system that is his responsibility operate somewhat "on its own" and hope for the best.

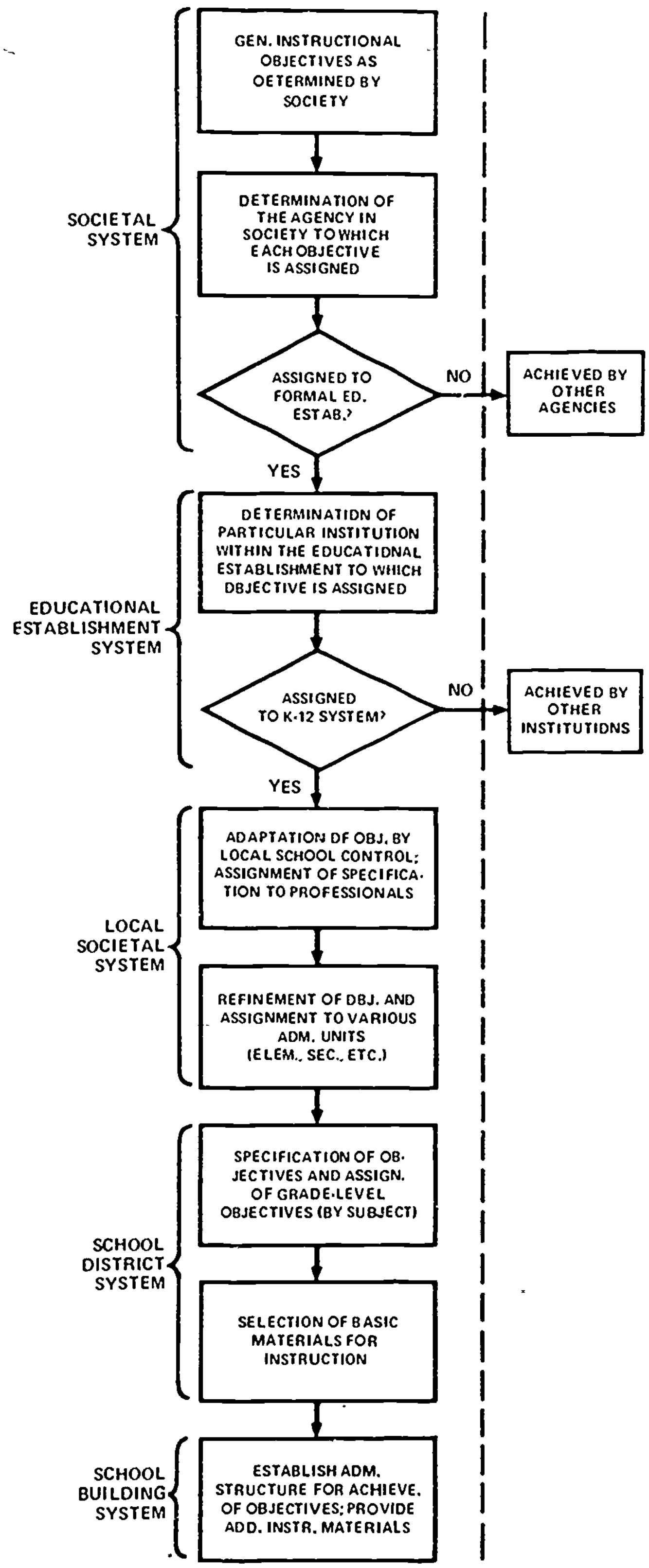



FIGURE 11.1. *Flow chart relating various levels of the education "system"*

A systems approach to classroom instruction will not ensure success. It does not provide a magical set of rules that will optimize opportunities for learning by students. It is nothing more nor less than an organized approach to decision making. By spelling out the stages in decision making it establishes a framework in which "best" decisions can be determined.

The teacher plays a role as systems designer at two levels. As part of curriculum committees he contributes to the assignment and statement of objectives to be achieved by learners in his classroom system. In this he applies the "first axiom" of systems-design:

1. Objectives to be obtained within the system must be stated specifically, in terms of behaviors to be exhibited by learners as a result of their presence in the system.

As part of materials-selection committees he applies two other "axioms" of systems design:

2. A system designed to achieve specific objectives must be designed with these objectives in mind, and the roles of components within the system must be specifically stated.

   That is, materials selection follows the establishment of instructional objectives. Any material selected for use should contribute directly to learner-achievement of some or all objectives. A system cannot be designed without both the learners and the objectives being considered. (Good materials? To do what? For whom?) Furthermore, consideration not only of the isolated value of materials but also of how they will fit with others as components of the *overall* system of instruction is an important aspect of systems thinking.

3. No systems design can be considered final.

   A system is a dynamic thing. The design must include provision for frequent monitoring of its operation and for modification of management or design to improve its functioning. Thus, materials must be selected and a plan established to provide this feedback.

Within the classroom itself the teacher as designer-manager analyzes feedback and modifies the original system design. The teacher component assumes responsibility for inducing the achievement of certain objectives by the learner. The experience in all of these roles is catalogued for use in his next assignment to curriculum or materials-selection committees.

Your classroom *is* an instructional system. You are involved in the design and management of that system. These facts are not influenced by whether or not you *operate* with conscious recognition of the system's existence and operation. Systems thinking is merely *awareness of the fact that the system exists and is functioning*. It is recognition of the fact that you, as teacher, are part—a managerial part—of the system. Systems thinking requires continual attentiveness to the specific objectives of the system. It includes an awareness of the wide variety of possible resources and of their relationship to the objectives. It is a habit of a mind that will think openly and creatively about the possible alternative pathways for pursuing objectives—a mind that is open to feedback and modification of plans to improve effectiveness. It is awareness that decision making is a constant process in the classroom, one that often takes place in an inertial, drifting manner but that should be a conscious process that overlooks no good alternative. It is an attitude that seeks to identify problems; to obtain the data to make decisions that are called for; and to bring to the implementation of these decisions the most appropriate of available resources in strategy, media, procedure, and software.

This is a hasty and informal description of a systems approach to classroom instruction. It is deliberately so—a general opening sketch in familiar terms to provide a commonality of background.

In what follows we shall add greater detail to this rough sketch. In the next section we shall discuss briefly various components available for use in instructional systems, and some of the unique qualities these components possess. Then

we return to the dynamic situation of the teacher-managed classroom for a more detailed look at the possibilities available today to a systems-oriented teacher. Next, we consider computer-managed instruction. Finally, we illustrate a systems approach to instruction, using a particular topic in mathematics as our example.

## ANALYSIS OF SYSTEM COMPONENTS

The concept of an instructional system requires that the system designer assign the accomplishment of specific instructional objectives to specific materials, devices, procedures, and people. Thus it is important that the designer have both a knowledge of what components are available for use and an understanding of the particular capabilities of each component. In previous chapters various possible components have been discussed in detail but in relative isolation. Here we shall view each in terms of its role in an overall system.

### *Human Components of the System*

A major role of the teacher in a classroom instructional system is, as it has always been, that of manager of the system. This role will be discussed later. In addition, the teacher is a component—a major teaching device—of the system. The presence of other components broadens rather than narrows the teacher's role as component. In a teacher-textbook system, without other components, the teacher must be personally involved in essentially all of the teaching-learning activities. This limits his role to that of lecturer, discussion leader, or monitor, working with full-class activities. The availability of additional components of instruction together with newer methods of class organization and scheduling broadens his role to include the following:

1. Lecturer, in team teaching situations
2. Discussion leader, or tutor, with small groups or individuals (supplementing team teaching lectures) during office hours in open scheduling structures or, in a self-contained classroom, with other components of the system supplying instruction for the remainder of the class
3. Evaluator, in a more detailed manner than in the past.

With a variety of modes of instruction available to induce the achievement of each objective and with improved tests and devices for monitoring student performance and diagnosing student problems, the role of the teacher as evaluator takes on added dimensions and importance. Instead of simply determining who learned and who did not *after the fact*, a well-designed system will provide the teacher-evaluator with the ability to select for each individual learner the strategy, methodology, procedures, and media of instruction that will *a priori* maximize his chances of learning. It will make possible close monitoring of his progress and precise diagnosis of learning problems if these arise. In short, the teacher-evaluator in a well-designed instructional system will have the opportunity to prescribe the "best" instruction for *each* learner—and periodically to reassess and modify the original prescription to assure the achievement of the goals of the instruction.

Team teaching and open scheduling structures provide the opportunity for several teachers to share the various roles described above in the instruction of a single (perhaps large) class of students. How the roles are assigned to individual teachers will, no doubt, depend upon the particular training, experience, interests, and competencies of those involved.

Much has been written about the unique contribution of teachers to the instructional process. We shall not repeat any of that here, nor shall we engage in the unending debate between "humanists" and others who claim this or that device will better perform essentially all of the functions usually assigned to human teachers. We shall only comment that whatever functions teachers are to carry out can surely be

better discharged by teachers operating within the framework of a carefully designed instructional system as described in the preceding section.

Teachers do not constitute the only human component of an instructional system. In many systems the human components include teacher aides assigned to a class on a regular basis. Their assistance in the management function as well as in the tutorial and evaluation functions described above provides even more opportunities for the teacher to make his best professional contribution to student learning.

The use of visiting lecturers from within the school or community as well as from colleges and universities adds still another human component to the system. Far beyond the entertainment value of such appearances is their educational value. These can be planned and timed either to make real contributions to the achievement of specific instructional objectives or to aid in providing a meaningful environment in which specific objectives can be achieved.

Finally, the learner himself is a human component of the system. Not only does he provide the internalized changes that *are* his learning, but also he contributes to his own learning and that of others as a questioner and discussant. In questioning and discussing he participates in both the presentation and evaluation roles. In certain class-conduct procedures he also contributes to classroom management. In small group work he may act as discussion leader. He may serve as tutor-partner with a less able student. In the learner-centered classroom he may select some of the materials he will use, decide which of several modes of instruction he will use, choose the strategy or medium of instruction he prefers, and make decisions as to which nonbasic objectives he will achieve or on which parts of basic instruction he needs more work.

## *Printed Medium Components*

There was a time when a discussion of printed materials used in a mathematics classroom would have begun—and *ended*—with comments on the textbook assigned for use in the course. Today that is rarely the case. This fact has been recognized by companies producing textbooks, for most now present "packages" (often called *systems*) including basic texts, together with many other materials in printed and other media. Some of these are designed for teacher use; others for use by individual students. We shall use these classifications—teacher use and student use—in discussing printed (and other) components of an instructional system.

*Printed Components for Teacher Use.* The textbook and accompanying teacher's manual comprise the primary printed component for teacher use in most instructional systems. However, the teacher's manual of today is usually a great departure from the "answer book" of a few years ago. It is likely that the manual will include answers, statements of objectives, comments on methods and strategies, some background mathematics, suggestions for enrichment activities, lists of supplementary materials, and so forth. It may, in fact, be essentially an outline for a complete instructional system of which the accompanying text is but one component.

In any case, the text usually defines the scope and sequence of the basic content to be presented. It serves as the basic "dictionary" of terms. It may tend to limit or even to determine the choices of instructional strategies that can be used. It does not determine the objectives of a course—teachers often teach "up, down, or out" from a text—but it will determine to what degree other components (including the human ones) must be used to achieve the course objectives.

Other teacher-use components in printed form are alternate texts; background books; pictures, diagrams, and charts for classroom display; spirit-duplicator masters (prepared or blank); and diagnostic and achievement tests. The ever-increasing mathematical sophistication of both elementary and secondary teachers has produced the eclectic attitude that makes alternate texts and background books about mathematics and

its applications important components of a system. The use of display materials gives previews of coming topics, presents constant reviews of important facts, and illustrates the wide variety of connections of mathematics to the "real world." The availability of well-designed tests is essential to the expanded evaluation that is a basic part of a modern instructional system.

This increased sophistication of teachers, together with the increasingly professional role of the teacher made possible by the availability of other components of the system, has expanded the role of the spirit duplicator from its traditional one as a printer of tests to include the printing of a wide variety of materials from commercially available or teacher-prepared masters. With tests to diagnose student needs, knowledge of what is necessary to satisfy these needs, and time to use this knowledge, the amount of supplementary materials for individual or class-wide use prepared and printed by teachers has increased until the spirit duplicator rivals chalk as an adjunct to instruction.

*Printed Components for Student Use.* Again, the primary printed component is the textbook. However, it is accompanied by teacher-prepared printed materials and by workbooks for additional drill. Programed texts provide extra instruction on basic topics, reteaching for remediation, and presentation and teaching of enrichment topics on an individual or small-group basis. Alternate texts give students the opportunity to use different strategies to achieve objectives. Books about mathematics, applications of mathematics, and mathematicians contribute to an environment for learning by placing what is to be learned in a context that is relevant to the learner.

Most printed (and other) materials for student use are designed to satisfy the needs of individual students by providing for differences in amount, time, strategy, or medium of instruction. It is such student materials, regardless of the medium in which they are presented, that are central to the concept of an instructional system—that the instruction must be sufficiently flexible to provide the opportunity for all to achieve the specific objectives of the instruction.

### Audio and Visual Components

Under this classification we include chalk and chalkboards, overhead projectors and transparencies, slides, filmstrips, films (16-mm and 8-mm "loops"), audiotapes, television, and combinations of these. It is in the consideration of these components that the financial restraints on the system designer become very important. Thus it is essential that the designer have an understanding of what each component can or cannot do to aid in making decisions about the balance between instructional gain and cost.

*Audio and Visual Components for Teacher Use.* Chalk and chalkboard are the traditional visual components of classroom instruction in mathematics. The extent of the role of the chalkboard in all phases of instruction could, we are sure, be established by interviews of either mathematics teachers or owners of dry-cleaning establishments. However, the overhead projector is becoming almost as common an accessory to instruction. Its role partially overlaps that of the chalkboard but also extends beyond it. In large classes its use as a replacement for the chalkboard provides improved viewability. The fact that the teacher can remain in one place, facing the class, may be important for acoustical reasons. In any class, the ability to turn back to any part of a long presentation (the part you've always *just erased* in board work) may be important.

Advance preparation of transparencies containing diagrams, results to be used, and so forth, for use in conjunction with either chalkboard or "continuous easel" clearly extends the role of the projector beyond that of chalkboard replacement. The preparation of projectuals containing well-constructed questions for periodic use during presentations as the basis for "go on—go back" decisions is an important aspect of the improved evaluation function of a well-designed system.

The overhead projector has tended to limit the role of slides and filmstrips. Seldom are these used to present diagrams or printed information. However, they remain the primary devices for the presentation of pictorial representations of static situations. As such they are classroom-extension devices, bringing into the classroom learning experiences not already present there. They can provide adequate substitutes for observations of real world situations. In comparison with printed display pictures, both slides and filmstrips have the advantages of easier viewability of detail and easier storage. Slides provide flexibility of sequencing, without distractions. Filmstrips are generally easier to use. Either can be used in conjunction with audiotape commentaries.

When the dynamics of changing situations rather than the details of static situations are to be illustrated, motion picture films replace filmstrips or slides. Sixteen-millimeter "full reel" films provide overviews and reviews and contribute to motivation for learning. In many cases they are used by teachers for reviews of content or methodology to aid them in preparing their presentations. They often contain complete developments of enrichment topics for use by a subset of the class while the teacher continues basic instruction with others.

Single-concept films (8-mm "loops") provide the initial confrontation with a problem important to all instruction. The simplicity of their use makes them both a teacher-use and student-use component.

A number of mathematical concepts have an essentially dynamic characteristic. Films make it possible to show the change from thinking of individual objects to thinking of a set of objects or from thinking of two sets to thinking of the one set that is their union. They make possible the visualization of the limiting procedure intrinsic to the determination of the area or circumference of a circle. Other components of a system can show discrete steps in these processes. Only film can show the changes that produce these steps.

A new dimension has been added to film-filmstrip use by the availability of "mixes" of these media. A single film may contain motion picture sequences with periodic automatic-stop sequences for filmstrip reviewing of important frames of the film.

Audiotape can be used by teachers to listen in on their own teaching and on class discussions, to aid in improvement of future presentations. Small group discussions can be taped for use in monitoring student learning. Drill tapes can be produced to provide oral drill without direct teacher involvement. Tapes of presentations and discussions can save hours of teacher conduct of make-up or remedial work.

Television has a role as a presentation device for centrally located films or filmstrips. One film library in a large school system with multichannel television transmission facilities provides immediate availability of this library to many classrooms. In addition, television can be used to provide the advantages of large group instruction without the necessity for physical togetherness of the group. Special interest topics can be presented to a few students in each of many classrooms. In large or regular group instruction, TV can give each student a "front row seat" to view details of visual presentations. In some computer-managed systems, television is a major presentation and response-accepting component.

*Audio and Visual Components for Student Use.* Several of the uses of such components by students to provide individualization of instruction were mentioned above. Student work at the chalkboard is a major day-to-day means of evaluating student progress in many classes. The availability of videotapes, films, filmstrips, audiotapes, and so forth containing key parts of presentations is a source of additional or make-up instruction for individuals or small groups of students. Remedial or enrichment packages containing audio, visual, and printed components are valuable adjuncts to basic instruction. Audiotape can be used by students to provide needed oral-stimulus drill. Beat-the-tape drills introduce time control when this is desirable.

In summary, well-constructed subsystems containing audio, visual, and other components can be used to produce for one or a few students approximate replications of what has occurred, or would occur, in teacher-conducted instruction.

### *Models, Manipulative Devices, and Games as Components*

The use of models and manipulative devices to introduce concepts and to form the basis for abstractions is common. They form part of both teacher demonstrations and student experimentation. Teacher-conducted games involving teams of students are a major method of bringing time control into learning. Games that simulate real-world situations are becoming important components of instruction in social studies and other areas. Little of this has been done in mathematics. However, enrichment games ("Equations," "Sets," "WFF 'N PROOF," etc.) can play an important part in the mathematics classroom.

### *Machines as Components*

Machines have an important role as presentation devices for many of the components discussed previously. Here, reference is to machines that are *themselves* the instructional component. In particular, we shall discuss calculating machines, computers, and machines designed specifically to provide or control feedback to teacher or student.

Use of calculating machines can be both a means and a goal of instruction. In some courses one of the course objectives is learning to use a desk calculator. In some, desk calculators are tools for use in problems requiring lengthy computations. The availability of calculators may allow the introduction of applied problems that are closer approximations of reality (only problems in mathematics books have "nice" answers; real world problems seldom do!).

Some research indicates that the use of calculators may allow weaker students the opportunity to circumvent algorithmic "blocks" and do some "real" mathematics. This, in turn, may motivate their learning of basic facts and algorithms.

A later section of this chapter contains a discussion of computers as managers of instructional systems or subsystems. As a component of a system a computer is, first of all, an efficient calculating machine. It can also be used to simulate experimentation by producing, on command, the quantitative aspects of experimental results. Its speed of computation allows simulating of dynamic processes by producing rapidly occurring discrete results. (For example, a computer programed to produce a ten-place printout of values of

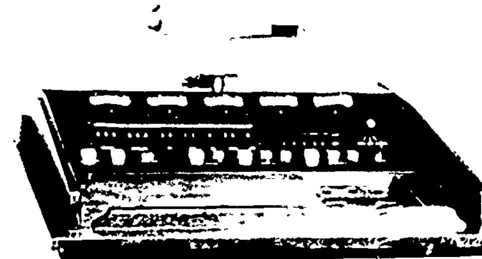
$$\left(1 + \frac{1}{n}\right)^n$$

for increasing positive integral values of $n$ gives an impressive example of the limit process.) Of course, learning computer programing and computer-oriented mathematics may be one of the objectives for some courses in mathematics.

In the traditional classroom, knowledge of student progress was provided both the teacher and the student informally, on a day-to-day basis; formally, by periodic testing. With the introduction of both individualized and large group instruction more sensitive formal feedback components have become necessary.

The early teaching machines, for use with individualized instruction, were designed to provide immediate controlled feedback to the learner and a record of learner progress that could be monitored by the teacher as desired. These rather primitive, usually manually operated devices were the forerunners of currently available sophisticated, electrically operated machines using filmstrips and audiotapes in the presentation of material and permitting written or oral responses.

Feedback components for large group instruction are also quite sophisticated. At the minimum, they allow lecturers to determine immediately the percent of a class choosing each response to a multiple-choice question displayed



*Courtesy of Raytheon Learning Systems Company*

FIGURE 11.2. *Combination automated and manual-operated lectern used to control multimedia presentations and to accumulate student-response data on a magnetic tape recorder*

from a transparency or slide. More elaborate devices permit the monitoring of the responses of individual students (a "light board" shows which students are giving which response), recording of individual responses on counters, or processing of responses for storage in the memory of a computer. Additional use of computers as feedback components will be discussed in the section on computer-managed systems.

## TEACHER-MANAGED INSTRUCTIONAL SYSTEMS

Now we return to the main theme of this chapter. With the first section as a general outline of systems thinking and the second as a catalog of systems components, we consider as our existential system a single mathematics class (at any grade level)--the physical environment, the learners, the content, and the teacher: a teacher who is intent on using a systems approach to maximize the efficiency of learning by his students.

What is the classroom teacher's role in the design and management of the classroom instructional system in which he normally finds himself? To answer this we must set the scene. We picture a typical school year and a teacher who has been assigned to teach mathematics to a set of students for that year. The organizational structure is the familiar self-contained class of approximately thirty students. A basic text has been selected for use. Enough money is available for some additional materials (films, filmstrips, models, etc.).

This instructional system is really a subsystem of a larger system. Hence we shall assume that the students come into this particular course with a past history of mathematics in which certain behavioral objectives, considered as required incoming behaviors for the present course, have been achieved. Further, the teacher has before him the goals of this particular subsystem—a particular set of terminal behaviors or outcomes to be achieved by the learners. While these are often described *en globo* as "teaching the content of fourth-grade arithmetic" or "teaching the third semester of algebra," we shall understand these phrases to mean much more than the rapid and thorough covering by the teacher of the content of the textbook assigned to this course. We view them as a shorthand description of the many specific behavioral objectives to be achieved by students in the course in question, with student success measured not by completion of the "work to be

covered" but by satisfactory acquisition of the behavioral objectives of this particular segment of the curriculum. Our assumptions, then, are that the students in this course have been adequately prepared, with, of course, some variation in the entering behaviors; that the teacher has a careful specification in behavioral terms of the objectives of the present segment of the curriculum; and that these are reasonable outcomes to expect in the time allotted for the particular class in question.

Faced with this new class, the classroom teacher begins to function as a systems manager on two levels. In the large, he must conceive of and plan the entire year's work. This is principally a design task. Then, at any particular point within the year, he must engage in the active management of the system at a particular point in time and space. He has to be engaged in inducing learning of a specific bit of mathematics.

## *The Teacher-Managed Instructional System in the Large*

The classroom teacher has a special design and management role to play with respect to his initial planning of the entire year's work. It is here that he must view the behavioral objectives for the entire course, plan the sequence in which these are to be attained, and make some rough time allotments to specific objectives (Figure 11.3). He must pay particular attention to the sequence in which the attainment of objectives is pursued because some objectives are not terminal objectives but are, rather, prerequisite behaviors for the attainment of others. Sequencing will depend on both the content itself and the particular teaching strategy he intends to use. He must make at least a rough estimate of satisfactory levels of performance for various goals. These levels may differ, depending on the particular objective.

This first approximation of his detailed plan for the entire year must include decisions concerning resource materials so that those that are not immediately available can be ordered to ensure their arrival for use at the appropriate time. He should carefully analyze his classroom—the arrangement and design of the room itself and the possible permanent acquisition and placement of an overhead projector, together with other auxiliary materials such as those described in previous chapters of this yearbook. Does he unconsciously tend to arrange his classroom for this particular class just as he did when he was teaching an entirely different subject at an entirely different level to an entirely different type of student? If so, this is decision by inertia. He should seriously consider whether there is anything in the nature of the new course that would suggest changes in classroom arrangement. Is there anything about the ability

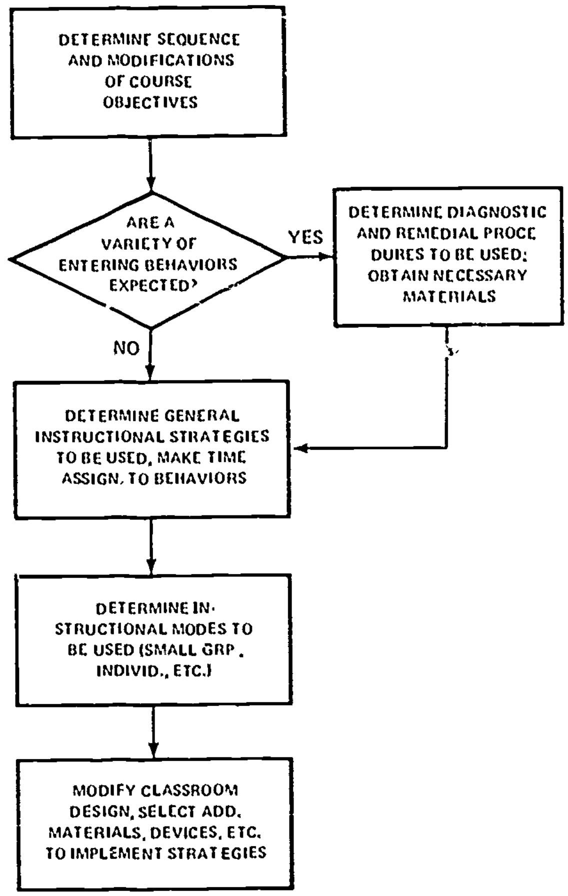



FIGURE 11.3. *Flow chart of a classroom system "in the large"*

levels or size of this class that would make the use of different patterns of classroom organization more feasible, that would make easier the use of enrichment materials, and so forth?

## *Classroom Systems Thinking in the Small*

Important as the teacher's role is in planning this course in the large, a major task of teaching lies in entering the classroom day by day to assist the students and manage the class as they attempt to learn some specific mathematics. This is the place where the teacher is constantly engaged in a decision-making role that vitally affects the degree to which his class achieves the desired objectives.

Now our setting becomes even more definite. The class is somewhere in the middle of the year's work. The teacher has induced adequate achievement of the behavioral goals of the course up to this point and is about to tackle a new bit of material—a new set of behavioral objectives to be achieved by learners.

As the systems manager, the teacher's attention first turns to the objectives of the new unit. These objectives are sufficiently spelled out in behavioral terms. Still, it is important for the teacher to be acutely *aware* of the behavioral specifications of this new unit. If, having completed work on factoring in elementary algebra, for example, students are about to embark on the solution of quadratic equations, the teacher must be sharply and specifically aware of the types of problems he will expect the students to be able to work on the completion of this unit, the various conceptual questions he will expect the students to be able to answer, the various procedures that he will expect the students to have mastered, and the various distinctions he will expect the students to be able to make.

Besides making himself aware of the specific behavioral objectives to be achieved in the new unit, the teacher must review the mathematics involved and be sure that he is thoroughly familiar with it and the way it is presented by the materials to be used. Next he looks at the mathematics structure again and determines which past behaviors are necessary entering behaviors for learning the new material.

*Diagnosis.* Having reviewed these factors—expected outcomes, the content to be taught, needed initial behaviors—he thinks of these with regard to his own particular class. His first managerial decision is diagnostic. If he is about to engage in solving quadratic equations by factoring, he will certainly need to know the competence of this particular class in the factoring of quadratics and their understanding of the real import of the following fact: For real numbers $a$ and $b$, $ab = 0$ if and only if $a = 0$ or $b = 0$.

If he has taught these topics to his class earlier in the year he may rely on his own judgment and recollection of the past to decide that they are sufficiently well grasped by the class to move ahead into the new material, with only passing reference to the fact above and with no real need to review factoring. If he is less sure of their skill in factoring, he might present a few problems at the board, asking students to work them and respond immediately. This is an example of an essential component of a systems approach—a process of feedback whereby throughout the instructional process information from the ongoing situation is read as frequently as possible and decisions are made and plans are changed in the light of this information, this feedback.

Several choices are possible at this point (see Figure 11.4). If the teacher reads the classroom response to these few problems in factoring as highly positive, he may proceed with the introduction of the new material. If the response is somewhat disappointing, he immediately makes the decision, well within his scope, to stop and reteach or reexplain the points of difficulty with the factoring process, confirming success by some additional problems.

If the results of this informal diagnosis are truly bad, the new topic should be postponed until remedial materials have been used. Although the teacher should be ready for this

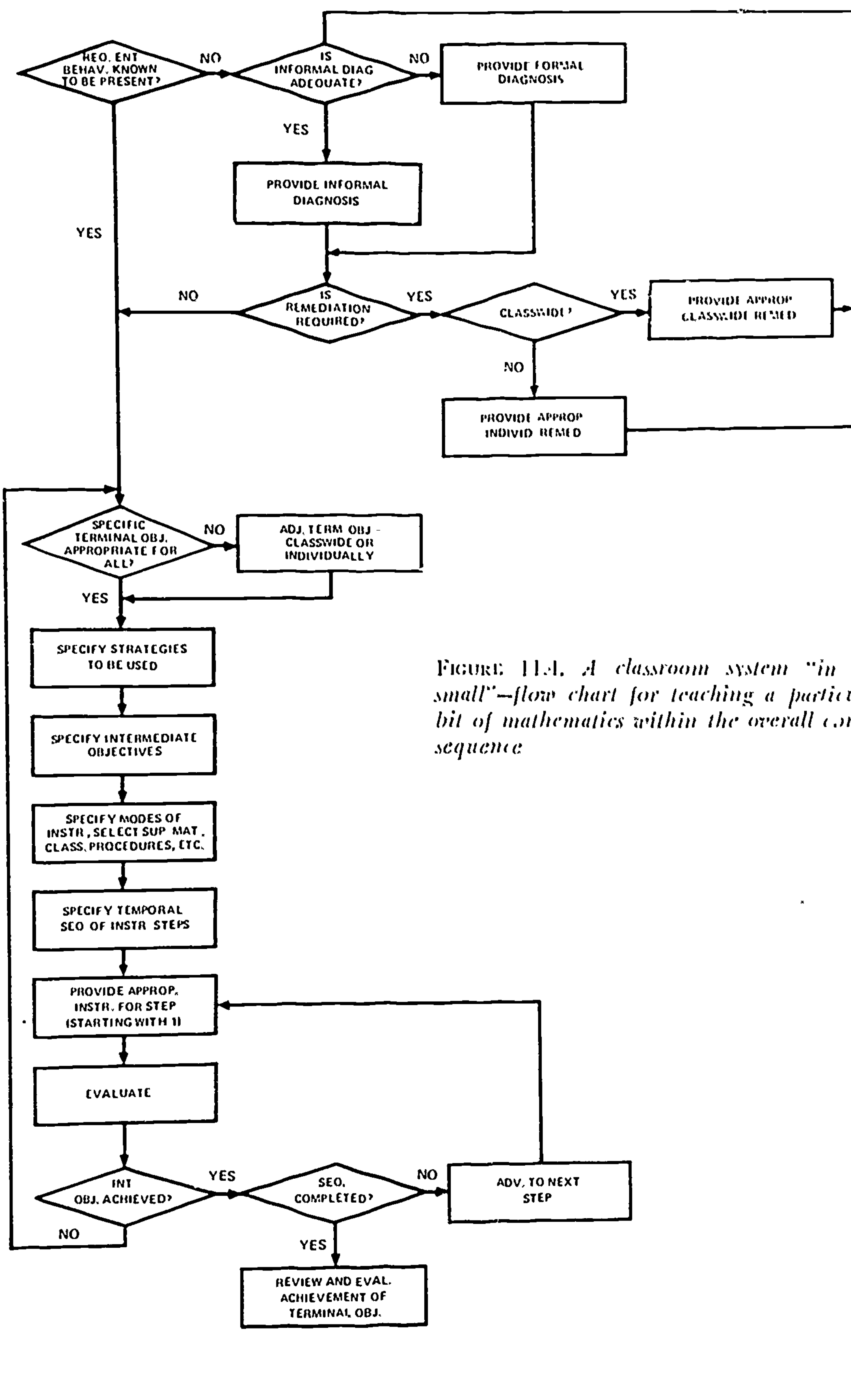



FIGURE 11.4. *A classroom system "in the small"—flow chart for teaching a particular bit of mathematics within the overall course sequence*

eventuality, it is to be hoped that it would not occur—that the teacher would not misjudge the preparation of the students to this extent. If he had been dubious about their factoring skills, he should have made an earlier decision to conduct a formal diagnosis with an in-depth test on the fundamental factoring types that must be used later in the new material. This diagnosis would reveal the true state of the factoring skills of his students and would lead him to additional decisions as to what kind of remedial work would have to be done at this point, how much would be needed, and by whom. One decision might lead to remedial work for the entire class. Another decision might be that it is indicated for only a few students and could be accomplished by pairing them with more capable students, by directing them into programed materials to be done at a separate time, and so forth.

The teacher, as a farsighted manager, might have arranged to have ready an overhead projector and several transparencies with examples and problems on factoring that could be used easily and efficiently in the classroom for both diagnostic purposes and, if the remediation indicated is relatively light, for remediation with the entire group.

So far we have seen that the teacher as systems manager must make a number of decisions and may bring to bear a variety of materials and media in the assessment and reaffirmation of the initial behaviors that are prerequisite to learning the new material. The teacher also has several decisions to make about the terminal behaviors expected of this particular class with respect to the new material. While the system has already specified the minimal behaviors expected of everyone in the class, the teacher may feel that this particular class is somewhat better than average or seems to have a greater facility with this particular kind of material and that it would be quite appropriate and feasible to set and pursue some behavioral objectives with the entire class that are beyond the minimal objectives set by the overall system. He may have found from his teaching of this course in the past that there are some behavioral objectives not present in the blueprin, he was given that are necessary at a later point in the course for his particular way of teaching other new material. This is an especially crucial kind of a decision for a teacher—what things, properly emphasized and achieved now, will make the going much easier later on? What things, if not properly acquired now, will cause great difficulty later?

*Teaching New Content.* Having made a careful inventory of necessary entering behaviors and having provided any remedial instruction that was necessary, the teacher is now ready to begin the new unit of instruction. Here he is faced with some of his major decisions. He already has determined the terminal behaviors he will expect from this particular class. Now he must select his overall instructional strategy for inducing the attainment of these objectives.

The teaching of a new topic can be viewed as proceeding in three stages, or phases.

First there is the introductory, or overview, phase. If done successfully, this produces involvement of the learners with what is to be learned. It is the most dramatic part of the teaching process, designed to provide impact, motivation, interest, and effect.

Many alternate strategies are possible here. Use of some kind of discovery method is quite common. For some topics a formal, almost lecture type of presentation, skillfully paced so that students can anticipate each step in a logical development, provides impact. Carefully selected examples leading to student-constructed generalizations, a problem solving approach abstraction from applications, and a "How would you define . . . ?" approach are other possibilities.

This introductory phase may be quite short, but it is very important. If it establishes interest in the topic and makes the learning of it seem personally important to each student, the remainder of the teaching and learning to be done will be made much easier.

Once the strategy to be used has been selected, a decision must be made as to the type of in-

struction (full group, small group, individualized) that best fits the strategy and instructional mode. Of course, general decisions on materials have been made as part of the overall course plan; but these must be refined, and often modified, when the actual instruction of a unit begins.

After the rather dramatic entry into the unit provided by the introductory phase comes the second, or developmental, phase of the instruction. The first phase was a rough sketch. This one is a carefully constructed design. Here the essential points of the topic are presented in an orderly sequence: results made reasonable by the introduction are established. The learner becomes aware of the full scope of the behaviors he is to acquire from the instruction.

The teacher is again confronted with decisions regarding strategies, classroom organization, and materials. As a systems manager he will certainly plan to obtain adequate feedback so that initial decisions can be modified. He will make choices among available materials, sometimes choosing a less-than-perfect medium of presentation to take advantage of a particularly good piece of software (a good filmstrip may be better than a mediocre film, although his knowledge of media tells him film is the superior medium for presentation of a particular topic).

Within this phase the teacher makes another key decision, of a somewhat different type. No one but the teacher can decide the degree of precision of language and the level of mathematical formalism (rigor) that are appropriate for his class. Of course, correct use of certain language and ability to reproduce or construct certain formal arguments may be behavioral objectives of the course. Beyond this, however, is the general level of formalism to be used in the development itself. Between the rote learning of facts and techniques and the logical development of completely rigorous mathematics is a broad spectrum of mathematically and pedagogically defensible positions. No one but the teacher himself can decide the optimum position for his particular class.

Following this orderly development of the topic comes phase three, that of acquisitional activities, with each learner engaging in those activities that will produce his achievement of the objectives expected of him. Here the teacher-manager's chief decisions are in regard to appropriate work materials and activities—problems, exercises, projects, readings, and so forth. These, in turn, must be paired with appropriate instructional modes.

Decisions concerning assignments are perhaps the most important the teacher makes. Of course, assignments must be reasonable in terms of number of exercises and expected length of work time. However, this is not sufficient. The easy decision, "Do the first fifteen problems," is seldom the best one. The work to be done must be specifically chosen to contribute to student achievement of the full range of objectives with respect to the material in the assignment. Surely the assignment must be tailored to the needs of the class and, it is hoped, to the needs of individuals in the class. Again, only knowledge gained from a sensitive feedback structure makes this possible.

Not only may it be necessary to make different assignments for different students, perhaps using different strategies and media, in order to obtain achievement of basic objectives, but it may also be necessary to modify terminal objectives for some students, accepting achievement of a minimal set of goals by some, adding enrichment objectives (in breadth or in depth) for others. Only careful planning will either reveal these needs or provide resources to satisfy them. A systems approach to instruction is designed to produce in the most efficient manner, maximum possible goal achievement *by each student*. Lack of provision of the necessary flexibility in all aspects of instruction near the end of the acquisitional phase can nullify the effects of a previously well-executed plan. Providing this flexibility will be a stern test for the best systems manager. It is here that materials selection is most vital, for it is here that individual differences must be taken thoroughly into account. The teacher who believes students learn *only* what, and when, he

is teaching will find his task at this stage difficult, if not impossible. The teacher-manager who understands the full range of possibilities for instruction by other components of the system, has taken the steps to acquire the necessary variety of materials, can operate successfully in several types of classroom organizational structures—perhaps simultaneously—and has developed the feedback structure that tells him what is needed, by whom and when, will receive at this time the payoff for all of his hard work. The payoff is knowing each student has been given an optimum learning opportunity.

Throughout the above discussion emphasis has been placed on the many decisions involved in any instructional process. These decisions do not arise *because* of a systems approach. They are inherently present. A systems analysis of instruction is intended only to point out their existence and to provide a framework within which orderly decision making can take place. It seems that three alternatives exist. One is to ignore the necessity for such decision making, that is, to allow decisions to occur by default (consciously or unconsciously deciding not to decide *is* a decision when the responsibility of making decisions is yours!). A second is to be aware of the need to make decisions but to neglect to obtain the information, acquire the materials, or generate a plan that will produce "best approximations of best decisions." The third is the systematic approach to decision making encompassed by the term *systems approach*.

*Evaluation*. The major evaluation of student performance at the end of the instruction on a unit is only a small part of the overall evaluation procedure. From the beginning of the introductory phase of the instruction the teacher, as classroom manager, must be acutely aware of what he intends to achieve with each instructional activity and of the degree of achievement he attains. Many teachers rely entirely on informal observation of student reactions for this evaluation. However, student reactions are not always good indicators. (Students are sometimes not aware that they have misconceptions.) A more reliable procedure is to include brief diagnostic questions or exercises of a single-concept nature regularly throughout the teaching process. Informal presentation of these on the board, a minute or two for thought, and a discussion of the results are all that is necessary. For example, after the topic of absolute value has been introduced, the definition explained, and some basic properties discussed, the entire class might be asked to write an answer to the question "If $x$ is less than zero, which of the following is the absolute value of $x$: $x$ or $-x$?" The distribution of responses and a discussion of the question can be very informative to both teacher and students.

Of course, one of the immediate benefits of such an evaluation or diagnostic question is that the students begin to know what they do not know. Then, instead of listening complacently for three or four more days of development only to find out at the end that they are completely lost when they thought they were doing well, they immediately are forced to recognize their deficiencies and focus clearly on the basic definitions before too much complex structure has been built on top of them. They are much more alert to the subtle difficulties and the types of responses they must be able to give.

All through the introduction and development stages of the material, the teacher should use a variety of data for evaluation—from facial expressions, to the questions of the students, to their responses to such diagnostic questions as just illustrated. Each bit of information causes him to slightly modify his approach: to repeat or dwell longer on certain concepts or types of exercises, and so forth. The more effective this ongoing diagnosis and remediation is, the less of a problem there will be with the larger evaluation at the end of the unit or topic.

The evaluation at the end of the unit must be carefully designed to measure the degree of attainment by the students of *all* objectives expected of them. These tests should be designed not only to rate the students and provide grades. They should also be diagnostic in nature. If

some students have not achieved objectives that are necessary for their progress in the next unit, these deficiencies must be removed before going on. The outcome of the evaluation should indicate the need, degree, and scope of necessary further instruction on the unit being tested. Good day-by-day evaluation should preclude a classwide disaster at this posttest stage. However, it is possible that some students have responded well to bits of the unit but exhibit only confusion when the bits are put together. As a systems manager, a teacher must be prepared for a wide range of results. Some or all of the techniques described for preinstruction remediation of deficiencies in entering behavior are appropriate here. Again, many decisions are necessary. Surely "They haven't achieved the objectives, but we'll go on anyway" is among the least acceptable.

### *The Teacher-Manager and the Motivational Components of the System*

As the next section will illustrate, there are ways in which a computer-managed instructional system can be superior to a teacher-managed one. For example, the computer's ability to keep complete, instantaneously available records of the progress of each student—his accomplishments, his difficulties—and to present rapidly to him a selection of well chosen special items that might remedy particular deficiencies cannot be matched in a teacher-managed classroom.

However, the human teacher has many unique abilities. One in particular deserves comment. This is his ability to interact with the student on a nonmathematical plane but in relationship to the learning situation. In other words, the teacher has motivational resources that the computer lacks. Motivation being what it is (the generator or starter for the self-activity that is the essence of learning), this is a tremendous advantage.

Such nonmathematical things as the teacher's personality, his interest in the students and respect for them, his sense of humor, his willingness to help, his sensitivity to nuances, his ability to recognize not only differences among students but differences among the same students on different days, and his ability to give highly specific encouragement to the poor student who does surprisingly well or to teasingly elicit a better response from a good student who has goofed are priceless advantages. The art of teaching extends far beyond the science of teaching, and it is here that the human teacher is irreplaceable.

In the area of motivation, one of the greatest elements is intrinsic: it lies in a student's sense of achievement, his feeling that he has successfully accomplished things up to this point and is well prepared and capable of accomplishments to come. (To the extent that the computer helps to individualize instruction and helps diagnose and bring specific remedies to the learning needs of each of the students, it also helps to motivate the student to continue on in an effective program.)

Nevertheless, the human condition being what it is, the extrinsic interactions of the human teacher with the human personalities of his students offer many opportunities for the human teacher to optimize the dynamic situation that is the classroom instructional system. The teacher can make various decisions about his interactions with the class that will definitely affect the learning process—decisions based on instinct and insight, the gestalt of the given moment, a feeling for aesthetic principles, and reactions to indefinable elements of the classroom atmosphere. These are the essential human elements of an otherwise highly structural systems approach to classroom instruction.

### *A Résumé*

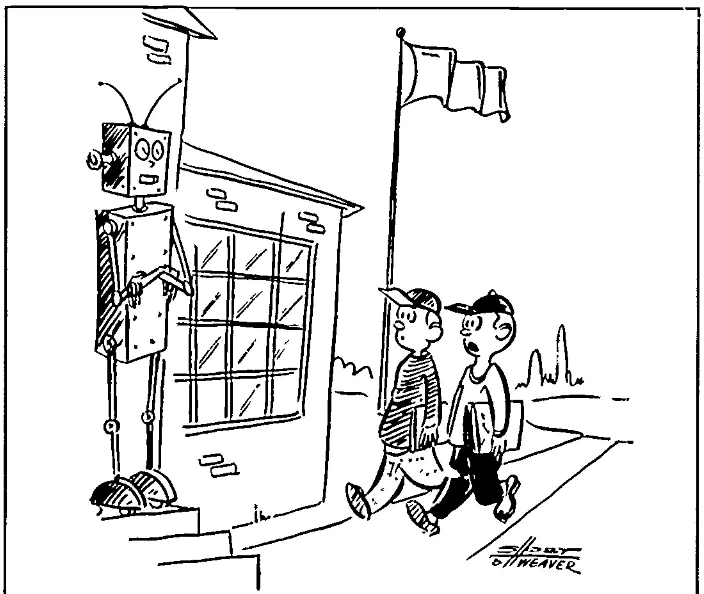
In brief, a classroom teacher who desires to act consciously and effectively in the role in which he is inevitably cast—that of the manager of a classroom instructional system—must know, do, and be many things.

1. He must know well the mathematics he is to teach.
2. He must be well aware of the specific behavioral objectives for the instruction.

3. He must be well acquainted with the particular group of students he has in this class.
4. He must decide on teaching strategies for the introduction of the new material, its development, and its acquisition.
5. He must decide on the most appropriate pattern of classroom organization corresponding to these strategies.
6. He must select and plan the use of the various media that will prove most effective for the topic in question and for this particular class.
7. He must make a further set of difficult decisions regarding the specific software for each medium that best promotes student learning.
8. He must make immediate decisions of diagnosis and evaluation, recycling, and engaging in remedial work to assure the achievement of the objectives pursued. He must choose the diagnostic materials and the remedial materials.
9. He must select complementary and enrichment materials for the individualization of instruction.
10. He must make decisions much less easily arrived at, much more elusive, about factors of motivation and classroom atmosphere that tend ultimately to promote a psychological climate in which attention to and pursuit of the desired objectives will be eagerly engaged in by the students.
11. He must evaluate the extent to which the objectives have been attained and make a value judgment as to the need for reteaching and the extent to which this should be done.

The qualities that these requirements demand of the teacher include the following: (1) an adequate knowledge of mathematics, (2) an understanding of and rapport with his students, (3) a great deal of flexibility and adaptability, and (4) an ability to come to a decision with some degree of rapidity as opposed to vacillation. He must be efficient in the use of time and resources, for both are limited; he must be sensitive and open so as to be receptive to the feedback provided him from the learning situation.

The role of a teacher as manager of a classroom instructional system is one of the most challenging and most human of roles. A teacher who learns to use a systems approach as he views his teaching assignment can make a great many decisions that will improve the learning situation for students and by that very fact increase the pleasures of teaching and the joys of accomplishment for the teacher himself. The world of work and occupation can become an art form and each day a series of episodes of challenge and excitement—episodes not without their difficulties and adventures but far removed from the feelings of aimless drifting or mechanical pacing that so frequently tend to creep into our teaching if we are not alert to the potential at our disposal—not used to systems thinking.



"I HAVE A FEELING THINGS ARE GOING TO BE DIFFERENT THIS YEAR."

*Reprinted from Educational Technology*

## COMPUTER-MANAGED INSTRUCTIONAL SYSTEMS

When one begins to write on the uses of computers in the management of instructional systems, he is immediately aware of the timeliness of comments in the field. A complete, detailed description of the state of things at the time of writing would be merely a historical account at the time of publication. The field is developing too rapidly for timely reporting in book form. However, an account of what has been and is being done indicates directions the field is taking with respect to both the role of the computer in the management of instruction and the plans of manufacturers for making computer-managed systems or subsystems available to schools. Thus it is possible to present a composite picture of what was, is, and most probably is to come.

It should be emphasized that our discussion deals *only* with the computer as manager of the actual instruction of students. Its classroom use as a sophisticated calculating machine, as well as its other uses as a component of a system, was discussed earlier in this chapter. Its other uses in schools—the usual business-machine functions, record storage, construction of schedules, and so forth—are relevant to our discussion only as they increase the probability of the availability of computer facilities for use in the management of instruction.

### *Evaluation Management*

The use of computers for test scoring is a component role that has become widely accepted, even expected. Now this almost traditional evaluation role is being extended in a number of ways. Several variations in the mode of response exist. Multiple-choice testing can be used with test consoles. With these, the student, after having "signed on" for identification purposes, communicates his answer directly to the computer by pressing appropriate buttons or inserting a stylus in appropriate holes. He may receive the results practically as soon as he completes the test. The printout may include his score, his rank in class, the class average, a list of questions answered incorrectly together with correct answers for them, his cumulative course average, and so forth. In addition, extensive analyses of test data may be printed out for the teacher. The performance of each student may become part of his course record, stored in the computer's memory for later use. The use of teletypewriters allows the procedure above to be adapted to constructed-response examinations of a short-answer type. Electronic pencils allow the computer to determine what position on a television screen is being indicated and hence to record responses to a wide variety of visually presented questions. At present, the evaluation of neither oral responses nor lengthy, constructed responses is possible. The acceptance of handwritten, rather than merely typed, construction seems desirable. Computer people say that these possibilities present difficult, but not insoluble, problems.

Just as a variety of response modes is possible, so is a wide variety of modes of presentation of questions in addition to the traditional printed tests and the use of teacher-controlled oral, transparency, or slide presentations of questions. These include the use of audiotape, alone or in conjunction with visual presentations by means of television, and the use of filmed or videotaped presentations of questions that include a dynamic element.

It is with the use of the computer to present questions as well as to record responses that the shift from its use as an evaluation *component* to at least some degree of evaluation *management* takes place. Its simplest management function is time control of stimulus presentation when not only correctness but also response time constitute the behavior being measured.

Far more complicated evaluation management functions can be carried out. By using the decision-making capacity of the computer the sequence of questions to be presented to the student can be determined *as he takes the test*, with his response to a single question or his response pattern on a block of questions used

to determine what he is asked to do next. This can provide much more evaluative information than simply the number of answers that are right or wrong. For example, it can show that a student knows his addition facts when they are presented in the normal (addition table) sequence but not when they are presented in random order. It makes possible the presentation of a "hard" addition problem: horizontal form, with carrying. If this is done correctly, the student is good at addition. If it is done incorrectly, the same problem (or a comparable one) may be presented in vertical form, then one involving other addition facts, then one with no carrying, and so forth. The response pattern on such a sequence gives a clear picture of exactly what the student knows and does not know about addition. This adds an important dimension to both achievement and diagnostic testing.

The decision-making function of the computer can be used to make individualized assignments for students on the basis of their response patterns on tests. On multiple-choice tests not only the identification of questions answered incorrectly but the particular incorrect answers selected can be used to branch students to various review or remedial topics. Patterns of responses to a series of constructed-response questions can form the basis for comparable assignment decisions. In theory, even better decisions could be made by having the computer compare the sequence of actual constructed responses with many such possible sequences stored in its memory. The practicality of this is limited by the large number of reasonable responses, which leads to a relatively long response time by the computer. When the number of acceptable responses is limited so that the comparison problem is manageable, decision making is based on little more information than that available from right-wrong patterns or from multiple-choice questions. (Six acceptable answers to each of twenty questions gives $6^{20}$ possible response patterns. There are "only" $2^{20}$ possible right-wrong patterns.)

Even more sophisticated computer roles in evaluation and evaluation-assignment management appear certain.

### *Management of Drill*

Computer management of drill and practice includes all of the management functions described for evaluation, together with several other important features. The means of communication between student and computer are those described above.

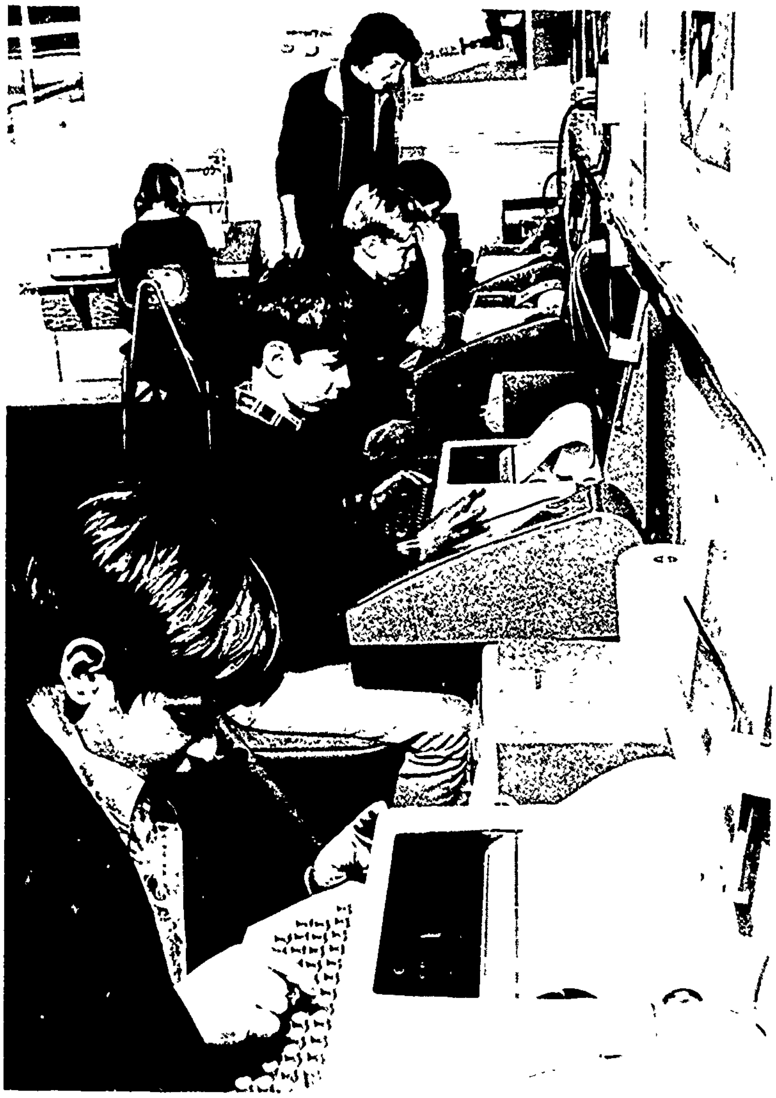
In a computer program for drill management the computer may make evaluation decisions to determine both the type of problem to be presented and the appropriate level of difficulty within a type. Some such decisions are:

> "He is performing above grade level in subtraction but poorly in multiplication. Give him more multiplication, less subtraction drill."

> "He misses multiplication problems in which nine appears as a factor. Give him drill on multiplication facts involving nine."

> "He does addition problems of this level of difficulty well when they are presented in vertical form, poorly in horizontal form. Lower the level of difficulty and bring him back through the lower levels using horizontal form."

Such programs store a record of each student's performance so that there is continuity to successive drill sessions, although they may be several days apart. They make decisions about lowering, staying, or raising the difficulty level on the basis of student performance. They may allow the student two or more attempts at the same problem. They may refuse to accept incorrect answers by locking all but the correct typewriter key. They may accept "Help" or "Show me" requests from students. Such requests may lead the computer to do the next step of a problem or to work in detail one or more comparable problems. They may provide answers to more complicated questions from students. If answers to students' questions are not in the computer's memory, it may record the



Courtesy of News and Publications Service, Stanford University

FIGURE 11.5. *The setting and the student in a computer-managed or computer-assisted instructional system (taken at Walter Hays High School, Palo Alto, one of a number of schools linked by teletype to the computer facilities at Stanford in project of computer-assisted instruction generated by Patrick Suppes)*

questions and, in turn, ask the computer programer to add the answers to its storage.

Such programs may accept students' requests for certain kinds of drill and may even author such drill by inserting different numbers in preprogramed drill patterns. They may put a time control on responses by refusing to accept the answer after a certain time or by giving it then. The response time of the student, as well as his response, may become a part of the student's record for use in future decision making. All computer-student interaction may be printed out for use by teachers and by computer programers to improve the system.

Before continuing with descriptions of what computers can do, it is important that we stop to assert that they can do nothing other than what they are told to do by the program they are running. They can present no material except that which they have been programed to present. When they present an easier problem, it is a problem the program writer determined to be easier. When they "make decisions" based on student performance, they merely act on the decision made *a priori* by the programer.

It is not the purpose of this interjection to comfort the reader by "putting the computer in its place." It is rather to emphasize the extreme importance of the software in any computer-managed system. This software includes both the computer software (the computer program that determines all of the computer's decisions and actions) and the instructional software (the actual instructional material presented to the student). Weaknesses in either of these software components will produce weaknesses in instruction. To quote a saying of computer people: "Garbage in, garbage out!" The quality of instruction from a computer-managed system cannot exceed the quality of the system's software—no matter how large, impressive, or expensive the computer.

## *A Complete Computer-Managed System*

It is possible to extend the computer's man-

agement function to include not only evaluation, assignment, and drill but also most other aspects of the instructional process. Such a computer-managed system might operate as follows: When a student enters the system he is extensively tested. This testing will establish a complete profile of the student. This will certainly include a precise analysis of his entering mathematical behavior. This may be done by establishing his position on several parallel K–12 strands representing the "big ideas" of school mathematics. It will include both concept development and skill development for each strand.

Part of this preliminary evaluation will provide data for the computer to use in making its initial decision as to the optimum instructional strategy (discovery, tell-reconstruct, etc.) to be used with that student. Perhaps data will be collected as the basis for deciding how his learning is best mediated (verbally, pictorially, abstractly, concretely, etc.). No doubt, information will be collected for use in decisions on what media should be used in presentations. Some measure of mathematical ability will be made as the basis for deciding on step size, pacing, and reasonable terminal objectives for the student.

This student profile will be stored in the computer's memory. As he proceeds through the initial sequence selected by the computer, it will constantly monitor his progress and use the information it obtains to modify his record in its memory. These modifications may lead to changes in its initial decisions on how best to instruct him.

Presentation to the learner will be by means of typewriter, television, and audiotape, and possibly by computer-constructed oral language, using electronic circuits to simulate human vocal sounds. The learner will communicate with the computer by typewriter and electronic pencil, using the pencil to point to symbols on a television screen or to move them. Eventually he will probably use ordinary handwriting and voice. Not all decisions concerning instruction will be made by the computer. The learner will be able to request repeats and reviews, to ask questions, and to select some of his own topics of instruction. He will instruct the computer to give him certain information or to do certain computations for him. He will use the computer to simulate experiments and real world situations from which discoveries can be made.

Not all of a student's learning will be directly monitored by the computer. He will be instructed to do certain things with traditional components of instruction—manipulative devices, books, teachers, fellow students—and then to return to the computer for a "discussion" of these activities.

He will, no doubt, be scolded, complimented, coaxed, encouraged—perhaps even "loved"—by his complicated electronic tutor. He will certainly, computer people claim, be taught much more, with much greater efficiency than has ever before been possible. They will, they say, have built both a better log *and* a better Mark Hopkins and will have made both available to all learners.

## *Plans for Making CMI Available*

Three problems of the availability to schools of computer-managed instruction (CMI) require consideration. These are:

1. The availability of the *hardware* (computers, response stations, etc.) to individual schools
2. The availability of good *computer software* (computer programs that manage the instruction)
3. The availability of good *instructional software* (the instructional material to be presented to the student).

The first of these is an economic problem. Three possible solutions are often proposed. One is that of shared-time systems. In such a system a school or school district rents response stations and buys time on a large computer owned and operated by the computer manufacturer or other organization separate from the school itself. Such a computer installation can manage a wide variety of instruction for many

students in many geographical locations. The cost to each school depends on the actual amount of computer time it uses for instructional and other purposes (bookkeeping, scheduling, etc.). Another possibility is the rental or purchase of a computer of moderate size by a large school district or several small ones for their exclusive use. Also, some manufacturers produce small but sophisticated computers. They believe their computers and the accompanying response systems to be priced within the budget of an individual school building. Such systems generally rely heavily on out-of-computer storage of both computer and instructional software, using the internal computer memory mainly for student profiles and general management programs.

The production of computer software (computer programs to manage instruction) has presented an interesting problem. It has required the development within the computer industry and other research establishments of people with knowledge of both what computers can do and how students learn.

Similarly, the development of instructional software (materials for student use) has required the involvement of people with knowledge of computers, theories of learning, and the content to be presented.

Furthermore, both software problems have been confronted in an almost paradoxical situation. Computer facilities will not be generally available until software exists, but their general availability is necessary in order for large numbers of people to become involved in software development.

The necessity for the availability of a computer facility for software development means that most of this material is being produced "in house" by computer manufacturers and related companies or by research projects associated with large school systems or universities. Some hardware manufacturers have purchased traditional textbook publishing companies, intending to "put their books on the computer." This approach has not, in general, been successful. It seems certain that development of software will continue to be slow until enough of it is available to warrant significant in-school utilization. Then the increasing availability of computer facilities and the involvement with computer-managed systems of an increasing number of people in the educational community will produce a steady, adequate supply of software. At first, both computer software and instructional software will be rented or bought with the computer. Eventually, no doubt, instructional software (especially) will be developed by individual teachers or groups of teachers for use in their own classes and for publication by computer manufacturers and software publishing companies, as textbooks and other traditional components of instruction are now developed and published.

As with any new instructional tool, the quality of material is likely to be uneven for some time. It is important that potential consumers of CMI material be aware of this. They must look beyond the impressive hardware. They should ask questions such as the following:

"Is the content that is presented correct and accurate?"

"What are the objectives of this instruction? Are these adequate for our students?"

"Are data available to show that this instruction *does* produce achievement of these objectives with students such as ours?"

"What is the per-student cost of this instruction? Is this cost justified by improved learning or increased efficiency or other factors?"

Computer-managed instruction is nothing more than a way of managing instruction. The quality and value of the instruction can be determined only by the answers to questions such as these.

## A SYSTEMS ANALYSIS OF AN ITEM OF INSTRUCTION

In this section we shall illustrate the various aspects of a systems approach to instruction by applying this technique to a specific content

item, intersection of sets. We shall give what amounts to the initial system-planning phase of the instruction related to this topic, including:

1. Specification of entering behaviors
2. Specification of terminal behaviors
3. Initial system design, including determination of instructional strategies and assignment of functions to specific components of the system.

That is, we shall show what might be either the first draft of an outline prepared by editors or authors for a company producing systems materials or the first step in classroom system planning by a teacher whose general goal is to teach intersection of sets to his class. It is not our purpose to defend the specific behaviors we assume or expect, nor do we claim our decisions on strategy or media are the only—or even the best—ones. We intend only to illustrate by a specific example the ideas discussed earlier in this chapter.

### *Assumed Entering Behaviors*

Students entering this instruction must be able to do the following things:

1. Exhibit the ability to think of a collection of objects as a single entity by using collective nouns and singular verbs to describe collections. (The team is ______. The class is ______.)
2. Construct sentences including the term *set* and a sufficient qualifying description to indicate that certain given objects are to be thought of as parts of a single collection. (The set of boys in this room. The set containing Mars, your pencil, and my dog.)
3. Identify the terms *element of* and *member of* by responding correctly to questions such as these: Is John a member (element) of the set of boys in this class? Is France a member (element) of the set of countries in North America?
4. Given objects and a verbal description of a set, correctly sort the objects to separate members of the set from nonmembers.
5. Construct sentences including the terms *element of* and *member of*, names of objects, and verbal descriptions of the sets to indicate that the named objects are or are not members of the described sets.
6. Read symbols such as {___, ___, . . .} as "the set whose elements are ___, ___, . . ." and as "the set of ___." That is, recognize the use of braces enclosing symbols to indicate a set whose elements are represented by the enclosed symbols.
7. Construct a braces-notation symbol for a set, given a verbal description determining it.
8. Recognize symbols enclosed by a simple closed curve as a name for the set whose elements are represented by the enclosed symbols. Give a verbal description or braces notation for the set so represented, and conversely.
9. Read $A = \{1, 2\}$ as "Set A is the set whose elements are 1, 2." That is, recognize the use of capital letters as names for sets and the use of the equality symbol to connect names for the same set.
10. Respond "The empty set" or "The null set" to questions such as this: What set has as its elements all people more than thirty feet tall? Construct verbal descriptions of the empty set.
11. Write $\emptyset$ or { } in response to a request to write a symbol that represents the empty set. Read $A = \emptyset$ as "Set $A$ is the empty set."
12. Given two sets, identify each as a subset or not a subset of the other.
13. Given a set, determine some or all subsets of it, including identification of both the empty set and the entire set as subsets.
14. Given a set, construct one or more sets of which it is a subset.
15. Use the symbols $\subset$ and $\supset$ with either braces notation or letter names for two sets to correctly identify one as a subset of the other.

16. Identify the use of + and · with numerals (as in 6 + 3, 5 · 4) as expressions for the sum and product of the numbers represented. Respond correctly to their use as instructions to determine simpler symbols for the sum and product ($6 + 3 = ?$ $5 \cdot 4 = ?$ Answers: 9, 20).
17. Construct real number instances of each of the field properties (commutative properties of addition and multiplication, associative properties, etc.). State which property is illustrated by a given instance.

## *Expected Terminal Behaviors*

As a result of the instruction, students are expected to be able to do the following:

1. Given two sets represented by enclosure notation, identify the set of common members as the *intersection* of the sets. Describe the intersection of the two sets as "the set whose members are ______ . ______ . . . . . ."
2. Recognize and construct real-world examples to justify the use of the term *intersection* in mathematics.
3. Given two sets represented by enclosure notation, use braces notation to represent their intersection.
4. Given two sets represented by braces notation, construct enclosure-notation representations of them, showing appropriate elements in their intersection. Construct a braces-notation representation of this intersection.
5. Recognize $A \cap B$ as a symbol for the intersection of set $A$ and set $B$. When the elements of $A$ and $B$ are given, recognize $A \cap B = ?$ as an instruction to construct a representation for this intersection—for example, $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$, $A \cap B = \{2, 3\}$. Similarly, recognize $\{a, b\} \cap \{a, c, d\}$ as a symbol for the intersection of the represented sets and $\{a, b\} \cap \{a, c, d\} = ?$ as an instruction to produce $\{a\}$.
6. Identify this dual use of symbols as analogous to the uses of + and · in arithmetic.
7. Respond "The set of all elements in both set $A$ and set $B$" when asked to define the term *the intersection of set A and set B* (or $A \cap B$).
8. Respond correctly to examples of any of the special cases $A \cap B = \emptyset$, $A \cap B = B$, $A \cap B = A$, with the given sets represented by braces or enclosure notation.
9. Identify $A \cap B = B$ as equivalent to $B \subset A$ and $A \supset B$. Identify $A \cap B = A$ as equivalent to $A \subset B$ and $B \supset A$. Correctly respond to and construct examples of these.
10. Identify "$A$ and $B$ are disjoint sets" as equivalent to $A \cap B = \emptyset$. Correctly respond to and construct examples of this.
11. Translate verbal descriptions of the intersection of two sets that include the term *intersection* into a description not containing this term and vice versa, as in the following examples: The intersection of the set of all girls and the set of all redheaded people is the set of all redheaded girls. The set of all men over forty years of age is the intersection of the set of all men and the set of all people over forty years of age.
12. Distinguish verbally between $A \cap B$ and $B \cap A$. Give a heuristic argument to show that for all sets $A$ and $B$, $A \cap B = B \cap A$. Identify this as the commutative property of set intersection. Recognize and construct instances of this property.
13. Identify the use of parentheses by distinguishing (verbally) between
$$(A \cap B) \cap C \text{ and } A \cap (B \cap C).$$
Give a heuristic argument to show that for all sets $A$, $B$, and $C$,
$$(A \cap B) \cap C = A \cap (B \cap C).$$
Identify this as the associative property of set intersection. Recognize and construct instances of this property.

## *The Design of a System*

In the design that follows we have listed instructional activities in the order of their proposed use. Within each activity we have listed the contribution of that activity to objective achievement, materials to be used in the activity, and a plan for the instruction itself. Detailed descriptions are given of some materials to provide a rationale for their selection. While no attempt has been made to select exotic components, neither has the overall cost of the system been considered. In later revisions of this preliminary design, trade-offs of cost against instructional efficiency would be considered.

The instruction provided is structured according to the three-phase procedure described in the section headed "Teacher-Managed Instructional Systems." We have adopted a guided discovery strategy with minimal verbal mediation of learning. It is assumed that this system will be used within a group-progress administrative structure.

### ACTIVITY ONE: PREINSTRUCTION ORIENTATION

*Purpose*

To provide an informal introduction to the topic and a motivation for the mathematical use of the term *intersection*

*Materials*

Prints (posters) showing real world situations illustrating the everyday use of the term *intersection:* intersections of streets, roads, walls at the corner of a room; the "area of possible conflict" of two dogs tied to different posts so that their circular ranges overlap; and so forth

Pictures of intersections of sets of points—two intervals on a line, two rays on the same line, two lines in a plane, and so forth—and of sets of objects represented by enclosure notation

In all pictures the intersections should be shown in the *same* distinguishing color to focus attention on the sameness in otherwise different situations.

*Use*

These pictures should be posted several days before formal instruction begins. Their eventual use in formal instruction is described later.

### ACTIVITY TWO: DIAGNOSIS

*Purpose*

To obtain an inventory of the abilities of the students with respect to the lists of expected entering behaviors and expected terminal behaviors

*Materials*

Diagnostic test measuring all expected entering behaviors and at least the key terminal behaviors expected

For less formal evaluation, questions covering key entering behaviors may be presented on transparencies for use in class discussions.

*Use*

If this is an isolated topic on sets, formal evaluation of entering student behaviors is recommended. If it is part of a unit on sets, informal evaluation may be sufficient.

### ACTIVITY THREE: ADJUSTMENT FOR A VARIETY OF ENTERING BEHAVIORS

*Purpose*

To ensure that *all* students begin instruction on the topic with *all* necessary entering behaviors

To preplan modifications in the instructional sequence for those students (if any) who have already acquired some expected terminal behaviors

*Materials*

One-reel 16-mm film covering developmental aspects of all entering behaviors

The film medium is particularly appropriate for illustrating the dynamics of the change from thinking about individual objects to thinking of a set with these objects as members.

Printed work sheets for student activity in the acquisitional phase of all entering behaviors

Transparencies containing the major points of the film to be used

*Use*

If the results of the diagnostic test are a class-wide disaster, as might be the case if this is the first instruction on sets in a course, the unit on intersection should be deferred until a complete unit on basic set concepts has been taught. If

such disastrous results occur for a small subset of the class, possible decisions are the following:

- Use the film and work sheets as self-instructional material. Provide opportunities for work in and outside of class. Possibly limit the set of terminal objectives for these students.
- Use the film and work sheets with these students on a small group instruction basis outside of class or in class while others work outside this unit of instruction.
- Use the film or transparencies for classwide review. Use the work sheets for in-class instruction, with more able students serving as group leaders for small group work.

It is to be hoped that such disasters will not occur. However, the diagnostic test may reveal severe, but isolated, deficiencies among some of the students. Use of the film or transparencies, class discussion, and work on appropriate work sheets individually or in small groups with comparable weaknesses should provide sufficient remediation.

Minor deficiencies may be removed by use of only the film or transparencies and class discussion, or the removal of deficiencies may be accomplished by troubleshooting at appropriate places in the instructional sequence.

If diagnostic testing reveals prior acquisition of some terminal objectives by some or all students, modification of the sequence that follows will be necessary for them. Such modification for individual students can take the form either of adding further (enrichment) terminal objectives with individualized instructional activities to induce their attainment or of work outside the unit of instruction.

## ACTIVITY FOUR: STEP 1 OF BASIC INSTRUCTION

*Purpose*

To induce achievement of the following terminal objectives:

1. Identification of the intersection of given sets
2. Rationale for the use of the term *intersection*
3. Use of braces notation to represent the intersection of two sets

*Materials*

Prints (posters), as in Activity One

Transparencies, with more examples like those on the prints, especially with sets represented by enclosure notation

*Use*

Teacher-led, discovery-oriented discussion, using the already-familiar prints:

"What is illustrated by all of these? What might we call these 'common parts'? Why? Is ___ in both set *A* and set *B*? Is it in the *intersection* of these sets?" Students construct sentences such as "The intersection of set ___ and set ___ is the set whose members are ___, ___, . . . ." and "The intersection of these sets contains ___, ___, . . . ."

Transparencies are used for feedback and for additional instruction, as needed.

## ACTIVITY FIVE: STEP 2 OF BASIC INSTRUCTION

*Purpose*

To induce achievement of the next on the list of terminal objectives:

1. Given braces notation for two sets, construct enclosure notation for them and braces notation for their intersection.

*Materials*

Three-minute 8-mm silent film

Transparencies

Work sheets

*Use*

The film, used for the introduction, shows the following:

- Two sets represented by braces notation
- Overlapping closed curves
- Symbols for elements that "slide into" closed curves, with elements of the intersection sliding into superposition in the overlapping part
- Braces notation for the intersection

The film continues, with several examples with nonempty intersections and with neither set a subset of the other. Transparencies with com-

parable content can be used to replace the film and for feedback and additional instruction, if needed. Work sheets contain drill:

"Given braces notation for two sets, construct enclosure notation for them and braces notation for their intersection and, finally, braces notation for intersections without the intermediate (enclosure notation) step."

### ACTIVITY SIX: STEP 3 OF BASIC INSTRUCTION

*Purpose*

To induce achievement of the next three on the list of terminal objectives:

5. Recognize $A \cap B$ as both a symbol for the intersection of $A$ and $B$ and an instruction to construct a "simple" representation for the intersection ($A \cap B = ?$). Make similar use of $\cap$ with braces notation for sets.
6. Identify this dual use as analogous to the uses of + and • in arithmetic.
7. Give a formal definition of the *intersection of set A and set B.*

*Materials*

Three-minute 8-mm silent film
Transparencies
Work sheets

*Use*

The film introduces the concepts in the following sequence:

- An example such as
  $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g, h\}$
  appears.
- The following sentence appears:
  The intersection of $A$ and $B$ is {__, __, __}.
  Time is given for the students to respond orally, then the symbols appear.
- Immediately below this the following appears:
  $A \cap B = \{a, c, e\}$.

This sequence of sentences is repeated in another example, with time for student response allowed for the third sentence as well as the second sentence. Then examples are given with the second sentence omitted.

Transparencies (or chalkboard pictures) of comparable examples provide the basis for discussion and feedback:

"Suppose we don't have a list of elements of sets $A$ and $B$. Can we, nevertheless, represent their intersection?" Students answer, "Yes, $A \cap B$." The uses of $A \cap B$ as an instruction and as a symbol for a result are discussed. "Have you seen such double usage before?" leads to analogy with $3 + 4$ and $5 \cdot 6$ in arithmetic. "Can you describe $A \cap B$ without knowing the elements of $A$ and $B$?" leads to the statement of the formal definition of $A \cap B$ as the *set of all elements in both set A and set B.*

Work sheets requiring repeated exhibition of all behaviors above provide the acquisitional phase of this activity.

### ACTIVITY SEVEN: EVALUATION

*Purpose*

To determine achievement of terminal objectives 1 through 7

*Materials*

Achievement test on those objectives

*Use*

These objectives encompass the *basic* ideas and notation related to the intersection of sets. Remaining objectives involve special cases and interesting consequences of this concept. With some classes or some students in a class, the remaining objectives (especially 11, 12, and 13) may be considered enrichment. In any case, evaluation and necessary additional instruction for any students needing it should precede (or possibly replace) further new instruction.

### ACTIVITY EIGHT: STEP 4 OF BASIC INSTRUCTION

*Purpose*

To induce achievement of the following objectives:

8. Respond correctly to examples of:
   $A \cap B = \emptyset$, $A \cap B = B$, $A \cap B = A$.
9. Identify $A \cap B = B$ as equivalent to $B \subset A$ and $A \supset B$, and identify
   $$A \cap B = A$$
   as equivalent to $A \subset B$ and $B \supset A$.

10. Identify "$A$ and $B$ are disjoint sets" as equivalent to $A \cap B = \emptyset$.

*Materials*

Transparencies with overlays

Work sheets

*Use*

The three cases in objective 8 are illustrated using transparencies with overlays. For example, $A \cap B = \emptyset$ is illustrated by showing set $A$, represented by enclosure notation, with a set $B$ such that $A \cap B = \emptyset$ shown on an overlay. The overlay contains the question "$A \cap B = ?$"

The cases $A \cap B = A$ *and* $A \cap B = B$ are comparably illustrated. After these have been shown without comment and the questions answered by students, a teacher-led discussion centers on these questions: "How 'big' can the intersection of two sets be? How 'small' can it be?" These lead to the observations in objective 9 and the observation and terminology in objective 10.

Work sheets illustrate all special cases, as well as nonspecial cases for discrimination. Sets are represented by enclosure notation and braces notation and are determined by verbal descriptions. Students determine intersections, respond "$A \supset B$," '$B \supset A$," or "$A$ and $B$ are disjoint sets" when sets and intersections are given, and construct sets $A$ and $B$, given that $A \cap B = A$, $B$, or $\emptyset$.

## ACTIVITY NINE: STEP 5 OF BASIC INSTRUCTION

*Purpose*

To induce achievement of the following terminal objective:

11. Translation of verbal descriptions containing the term *intersection* into other verbal descriptions, and conversely.

*Materials*

Transparencies

Audiotapes

Work sheets

*Use*

A "restatement game" provides the basic instruction. The teacher presents on transparencies (or chalkboard) verbal descriptions of two sets and asks for a verbal description of their intersection which does not contain the term *intersection*. After this description has been given, the intersection of this intersection and a third given set is constructed. The example below illustrates the idea.

Given two sets:

The set of all students in this school

The set of all redheaded people

Intersection of the two given sets:

The set of all redheaded students in this school

Given a third set:

The set of all football players

Intersection of the intersection of the first two sets and the third set:

The set of all redheaded football players in this school

The same example is repeated by starting with the second and third sets and then introducing the first set, and finally by starting with the first and third sets and then introducing the second set. Students are asked if the last set obtained is the same in all cases. The answer yes is an informal introduction to the final two objectives.

Now the process is reversed. A set whose elements are described by several characteristics is given, and the students construct sets whose intersection is the given set. Discussion of the range of possible answers for each example is instructive—and it's fun! This game can be extended by having one student or a team of students construct descriptions for others to translate. Audiotapes with time delays for student responses, or work sheets, can provide challenging examples for all students and easier drill for those less able.

## ACTIVITY TEN: STEP 6 OF BASIC INSTRUCTION

*Purpose*

To induce the achievement of the final terminal objectives:

12. Distinguish between $A \cap B$ and $B \cap A$.

Argue that $A \cap B = B \cap A$. Observe similarity with $a + b = b + a$ and $a \cdot b = b \cdot a$ for numbers. Name this the *commutative property*, motivated by analogy with number properties.

13. Identify use of parentheses and argue that
$$(A \cap B) \cap C = A \cap (B \cap C).$$
Name this property by analogy with
$$(a + b) + c = a + (b + c)$$
and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for numbers.

*Materials*

Four- to six-minute 8-mm silent film

Transparencies, with overlays

Work sheets

*Use*

The film provides the introduction:

- Enclosure notation for a set (labeled set $A$) appears.
- Enclosure notation for a nondisjoint set $B$ fades in. Labels for $B$ and $A \cap B$ appear.
- Previous figure moves to left of the picture. Same set $B$ appears at the right. Then set $A$ fades in and label $B \cap A$ appears. Finally $A \cap B = B \cap A$ appears.

Next a comparable sequence for $(A \cap B) \cap C$ and $A \cap (B \cap C)$ is shown. Teacher-led discussion, using transparencies with overlays, reconstructs the sequence from the film and provides other examples for discussion. Transparencies or chalkboard review of properties of operations on numbers leads to comparison and naming of these properties of set intersection. Work sheet drill provides the acquisitional phase.

ACTIVITY ELEVEN: EVALUATION

*Purpose*

To determine student performance with respect to all terminal objectives

To determine what reteaching, if any, is necessary

*Materials*

Achievement test

*Use*

The test is used both as a measure of achievement and as an indicator of possible need for remedial instruction. Whether such instruction is done at this time or later is determined by the position and role of this unit in the overall course structure.

## *Continuing Toward a "Best Design"*

It should be reemphasized that this is a preliminary design for a system. The next phase would be production (or acquisition) of individual components, testing of these in isolation and in sequence, and revision of either individual components or of the entire design until the required behaviors are induced with the intended audience. Throughout this process both instructional and economic factors are considered. A systems approach to instruction includes both the preplanning illustrated by our preliminary design and the determination to accept and act upon feedback until a "best design" is achieved that will, within given restraints, produce optimum instruction for all students who are to learn within the system.

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# APPENDIX

# APPENDIX

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Educational Games and Aids
330 W. 42d St.
New York, N.Y. 10036

Media-Graphics, Inc.
14918 Minnetonka Blvd.
Minnetonka, Minn. 55343

Meredith Corp.
Education Div.
440 Park Ave.
New York, N.Y. 10016

Meridian Co.
1976 Innes
San Francisco, Calif. 94124

Midwest Publications, Inc.
P.O. Box 307
Birmingham, Mich. 48012

Minnesota Mining and
Manufacturing Company
3M Center
St. Paul, Minn. 55101

Models of Industry, Inc.
2100 5th St.
Berkeley, Calif. 94710

Modern Learning Aids
Div. of Modern Talking Picture
Service, Inc.
1212 Avenue of the Americas
New York, N.Y. 10036

Money Management Institute
Household Finance Corp.
Prudential Plaza
Chicago, Ill. 60601
Montgomery Ward Co.
1400 University Ave.
St. Paul, Minn. 55101
J and F Morse
112 W. Lake Rd.
Penn Yan, N.Y. 14527
Movie-Mite Corp.
1001 E. Jefferson Ave.
Detroit, Mich. 48207
Mover Div.
Vilas Industries Ltd.
25 Milvan Dr.
Weston, Ont.
Canada
Mundus Co.
P.O. Box 356
Riverdale, Md. 20840

—N—

Nasco
901 Janesville Ave.
Ft. Atkinson, Wis. 53538
National Council of Teachers of Mathematics
1201 16th St., N.W.
Washington, D.C. 20036
National Film Board of Canada
P.O. Box 6100
Montreal 3, Que.
Canada
National Instructional Films
99 S. Main St.
Spring Valley, N.Y. 10977
National School Supply and Equipment
79 W. Monroe St.
Chicago, Ill. 60603
Nilty Div.
St. Regis Paper Co.
3300 Pinson Valley Pkwy.
Birmingham, Ala. 35217
Northern Signal Co.
Saukville, Wis. 53080
Novation, Inc.
253 State St.
St. Paul, Minn. 55107
Number Films
17350 Gresham St.
Northridge, Calif. 91324
A. J. Nystrom and Co.
3333 Elston Ave.
Chicago, Ill. 60618

—O—

Optisonics Corp.
Montgomeryville Industrial Center
Montgomeryville, Pa. 18936
F. A. Owen Publishing Co.
Dansville, N.Y. 14437
Oxford University Press
417 5th Ave.
New York, N.Y. 10016
Ozalid Div.
General Aniline and Film Corp.
Johnson City, N.Y. 13790

—P—

Pacific Coast Publishers
4085 Campbell Ave.
Menlo Park, Calif. 94025
Paillard, Inc.
1900 Lower Rd.
Linden, N.J. 07036
Palfrey's Educational Aids
7715 E. Garvey Ave.
Rosemead, Calif. 91770
Pantheon Books, Inc.
22 E. 51st St.
New York, N.Y. 10022
Paxton Equipment and Supply
7401 S. Pulaski Rd.
Chicago, Ill. 60629
Pentagon Devices Corp.
21 Harriet Dr.
Syosset, N.Y. 11791
Philograph Publications Ltd.
Fulham High Street 69-79
London, S.W. 6
England
Physics Research Laboratories
P.O. Box 555
Hempstead, L.I., N.Y. 11551
Pickett Industries
436 E. Gutierrez
Santa Barbara, Calif. 93102
The Playway Games
C. N. McRae
Unadilla, N.Y. 13474
Plaxall, Inc.
5-26 46th Ave.
Long Island City, N.Y.
Playskool Manufacturing Co.
3720 N. Kedzie Ave.
Chicago, Ill. 60618
Plenum Publishing Corp.
227 W. 17th St.
New York, N.Y. 10011
The Plymouth Press
1232 W. 79th St.
Chicago, Ill. 60620
Popular Science Publishing Co.
Audio Visual Div.
355 Lexington Ave.
New York, N.Y. 10017
Frederick Post Co.
333 Sibley St.
St. Paul, Minn. 55101
Potter's Photographic Applications Co.
150 Herricks Rd.
Mineola, N.Y. 11501
The Charles T. Powner Co.
P.O. Box 796
Chicago, Ill. 60690
Prentice Hall
Educational Book Div.
Englewood Cliffs, N.J. 07632
Prindle, Weber and Schmidt, Inc.
53 State St.
Boston, Mass. 02109
Products of the Behavioral Sciences
1140 Dell Ave.
Campbell, Calif. 95008
Projection Optics Co., Inc.
271 11th Ave.
E. Orange, N.J. 07018
Psychological Service
1402 Stanford St.
Chase, Maryland 21027

—Q—

Quantum Corp.
1885 Marshall Ave.
St. Paul, Minn. 55104

—R—

Raytheon Learning Systems Co.
U.S. Highway 12 East
Michigan City, Ind. 46360
R.C.A.
Commercial Electronic Systems Div.
Bldg. 15-5
Front and Cooper Sts.
Camden, N.J. 08102
Renewal Products, Inc.
Old Country Rd.
Mineola, New York 11501
Responsive Environments Corp.
Learning Materials Div.
200 Sylvan Ave.
Englewood Cliffs, N.J. 07632
Rolatape, Inc.
1301 Olympic Blvd.
Santa Monica, California 90400

—S—

St. Paul Book and Stationery Co.
1233 W. County Rd. "E"
St. Paul, Minn. 55112

St. Regis Paper Company
Nalty Div.
3300 Pinson Valley Pkwy.
Birmingham, Ala. 35217

Sargent-Welch Scientific Co.
7300 N. Linder Ave.
Skokie, Ill. 60076

Schacht Associates, Inc.
1175 E. 156th St.
Bronx, N.Y. 10471

Schacht Rubber
Manufacturing Co.
238 Polk St.
Huntington, Ind. 46750

School Material Co.
1801 S. Michigan Ave.
Chicago, Ill. 60616

School Products Co. Inc.
312 E. 23d St.
New York, N.Y. 10010

School Service Co.
647 S. La Brea Ave.
Los Angeles, Calif. 90036

The Schur Co.
1 Emerson Pl.
Boston, Mass. 02114

Science Productions
Box 113
Oconomowoc, Wis. 53066

Science Related Materials
P.O. Box 1009
Evanston, Ill. 60204

Science Research Associates, Inc.
259 E. Erie St.
Chicago, Ill. 60611

Science Seminars, Inc.
P.O. Box 5019
Portland, Oreg. 97213

Sciences Materials Center
59 4th Ave.
New York, N.Y. 10003

Scientific Educational
Products Corp.
30 E. 42d St.
New York, N.Y. 10017

Scott, Foresman and Co.
1900 E. Lake Ave.
Glenview, Ill. 60025

Scritchfield Manufacturing Co.
1783 Logan Ave.
San Diego, Calif. 92113

Sears, Roebuck and Co.
Consumer Information Services,
Dept. 703
7401 Skokie Blvd.
Skokie, Ill. 60076

Selective Educational Equipment
(SEE), Inc.
3 Bridge St.
Newton, Mass. 02195

Self Teaching Flashers
1102 S. 54th St.
Lincoln, Nebr. 68506

Service and Snailes
Distributors Ltd.
9 Mayfair Mews
Toronto 189, Ont.
Canada

E. H. Sheldon Equipment Co.
Muskegon, Mich. 49443

Sifo Company
834 N. 7th St.
Minneapolis, Minn. 55411

Sigma Enterprises, Inc.
Box 15485
Denver, Colo. 80215

Simon and Schuster, Inc.
1 W. 39th St.
New York, N.Y. 10018

The L. W. Singer Co., Inc.
201 E. 50th St.
New York, N.Y. 10022

Skool-Aids Corp.
8702 Bessemer
Cleveland, Ohio 44127

Smith Corona Marchant
410 Park Ave.
New York, N.Y. 10022

Smith System Manufacturing Co.
56 S.E. Emerald
Minneapolis, Minn. 55414

Society for Visual Education, Inc.
1345 Diversey Pkwy.
Chicago, Ill. 60614

Soma Blocks
Parker Bros., Inc.
P.O. Box 900
Salem, Mass. 01970

Speed-Up Geometry Ruler Co.
14 Osborne Ave.
Baltimore, Maryland 21228

Spindler and Sauppe
1329 Grand Central Ave.
Glendale, Calif. 91201

Squibb-Taylor, Inc.
P.O. Box 20158
10807 Harry Hines Blvd.
Dallas, Tex. 75220

Standard Education Society, Inc
130 N. Wells St.
Chicago, Ill. 60606

Standard Projector and
Equipment Co.
1911 Pickwick Ave.
Glenview, Ill. 60025

The L. S. Starrett Co.
1-165 Crescent Street
Athol, Mass. 01331

STAS Instructional Materials, Inc
1225 8th St.
Berkeley, Calif. 94710

Steck-Vaughn Co.
P.O. Box 2028
Austin, Tex. 78767

Sterling Plastics Co.
Sheffield St.
Mountainside, N.J. 07092

Stipes Publishing Co.
10 Chester St.
Champaign, Ill. 61820

Summit Industries
P.O. Box 415
Highland Park, Ill. 60036

Howard Sutton Company, Inc.
17 Warren St.
New York, N.Y. 10007

Systems for Education, Inc.
612 N. Michigan Ave.
Chicago, Ill. 60611

—T—

Tal-Cap Inc.
831 N. 7th St.
Minneapolis, Minn. 55411

Teacher's Aids
1683 South 700 West
Woods Cross, Utah 84087

Teachers Publishing Corporation
(Publishers of *Grade Teacher*
Magazine)
23 Leroy Ave.
Darien, Conn. 06820

Teaching Aids
159 W. Kinzie St.
Chicago, Ill. 60610

Technicolor Inc.
Commercial and Educational Div.
299 Kalmus Dr.
Costa Mesa, Calif. 92627

Tecnifax Corp.
195 Appleton St.
Holyoke, Mass. 01040

3D Magna-Graph Corp.
Box 261
Park Ridge, Ill. 60068

Touch, Inc.
P.O. Box 1711
Albany, N.Y. 12201

Fern Tripp
Dinuba, Calif. 93618

TRETEX
Div. of Kelch Corp.
6111 W. Mequon Rd.
Mequon, Wis. 53092
TUF (See Avalon Hill Co.)
Charles E. Tuttle Co., Inc.
28 S. Main St.
Rutland, Vt. 05701
Tweedy Transparencies
208 Hollywood Ave.
E. Orange, N.J. 07018

—U—

United Chemical and School Supply Co.
2115 Como Ave. SE
Minneapolis, Minn. 55414
Universal Education and Visual Arts
221 Park Ave. South
New York, N.Y. 10003

—V—

Victor Animatograph Corp.
Div. of Kalart Victor Corp.
Hultenius St.
Plainville, Conn. 06062
Viewlex, Inc.
1 Broadway Ave.
Holbrook, N.Y. 11741
Viking Co.
113 S. Edgemont St.
Los Angeles, Calif. 90004
Vision Incorporated
1501 N. Ditman Ave.
Los Angeles, Calif. 90063
Visual Products Division
Minnesota Mining and Manufacturing Co.
3M Center
St. Paul, Minn. 55101
The Vis-X Co.
P.O. Box 11107
Los Angeles, Calif. 90011

—W—

Wabash Instrument Corp.
806 Manchester Ave.
Wabash, Ind. 46992
J. Weston Walch Publisher
Box 1075, Main Post Office
Portland, Maine 04101
Walker Educational Book Corp.
720 5th Ave.
New York, N.Y. 10019
Walker Products
1530 Campus Dr.
Berkeley, Calif. 94708
Wang Laboratories, Inc.
836 North St.
Tewksbury, Mass. 01876
Weber Costello Co.
1900 N. Narragansett
Chicago, Ill. 60639
Webster Div.
McGraw-Hill Book Company
Manchester Rd.
Manchester, Mo. 63011
Webster Paper and Supply Co.
Albany, N.Y. 12201
Weems and Plath, Inc.
48 Maryland Ave.
Annapolis, Md. 21401
Welch Scientific Co.
*See* Sargent-Welch Scientific Co.
Western Publishing Co. Inc.
School Library Dept.
850 3d Ave.
New York, N.Y. 10022
Western Publishing Education
1220 Mound Ave.
Racine, Wisconsin 53401
WFF 'N PROOF
Box 71
New Haven, Conn. 06501
W. H. M. Co.
P.O. Box 2012
Sioux City, Iowa 51101
John Wiley and Sons, Inc.
605 3d Ave.
New York, N.Y. 10016
H. Wilson Corp.
555 W. Taft Dr.
S. Holland, Ill. 60173
World Wide Games
Delaware, Ohio 43015
L. M. Wright Co.
686 E. Mariposa St.
Altadena, Calif. 91001
Wyle Laboratories
128 Maryland St.
El Segundo, Calif. 90245

—X—

Xerox Corporation
Curriculum Programs
600 Madison Ave.
New York, N.Y. 10022

—Y—

Yoder Instruments
E. Palestine, Ohio 44413

—Z—

Zinni Educational Materials
1715 Chestnut St.
Philadelphia, Pa. 19103